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Differential Equations and Zeros of Orthogonal Polynomials[1]

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Abstract

This is a survey of some methods for finding inequalities, monotonicity properties and approximations for zeros of orthogonal polynomials and related functions. The methods are based on the use of the ordinary differential equations satisfied by the functions.

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1 Introduction

The purpose of this paper is to describe some of the classical methods for getting information about zeros of orthogonal polynomials from the differential equations satisfied by the polynomials. We do not discuss other methods (such as those based on recurrence relations and weight functions) for finding such information. We are interested especially in bounds and approximations for the zeros, spacing of the zeros and monotonicity of the zeros with respect to a parameter. We give special importance to methods based on the Sturm comparison theorem. Our results are applied to some of the classical polynomials.

2 Notation and basic results

We apply the results which we discuss mainly to the Jacobi polynomials and to their special cases. There are similar results for the Laguerre and Hermite polynomials. See the results and references in [52] and [39]. We recall that the Jacobi polynomials $P_n^{\alpha,\beta}(x),\ \alpha > -1,\ \beta > -1$ are orthogonal on $[-1, 1]$ with the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$. They satisfy the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{1}$$

For some applications, it is more useful to use one of the transformations of this differential equation [52, §4.24]

$$\frac{d^2u}{dx^2} + \left\{\frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{n(n+\alpha+\beta+1) + (\alpha+1)(\beta+1)/2}{1-x^2}\right\}u = 0, \tag{2}$$

where

$$u = (1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}P_n^{\alpha,\beta}(x),$$

and

$$\frac{d^2v}{d\theta^2} + \left\{\frac{1/4-\alpha^2}{4\sin^2\theta/2} + \frac{1/4-\beta^2}{4\cos^2\theta/2} + \left(n + \frac{\alpha+\beta+1}{2}\right)^2\right\}v = 0, \tag{3}$$

where

$$v = \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2} P_n^{\alpha,\beta}(\cos\theta).$$

In the special case $\alpha = \beta = \lambda - 1/2$ we have the ultraspherical or Gegenbauer polynomials $P_n^\lambda(x)$ which are orthogonal on $[-1, 1]$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$. We recall that [52, p.81] $(1-x^2)^{\lambda/2+1/4}P_n^\lambda(x)$ satisfies the differential equation

$$y'' + \left\{\frac{(n+\lambda)^2}{1-x^2} + \frac{1/2+\lambda-\lambda^2+x^2/4}{(1-x^2)^2}\right\}y = 0. \tag{4}$$

The Legendre polynomials $P_n(x)$ correspond to the particular case $\alpha = \beta = 0$ of the Jacobi polynomials or to the case $\lambda = 1/2$ of the ultraspherical polynomials.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are orthogonal on $(0, \infty)$ with respect to the weight function $e^{-x}x^{\alpha}$ and they satisfy the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0$$

while the Hermite polynomials $H_n(x)$ are orthogonal on (∞, ∞) with respect to the weight function e^{-x^2} and satisfy the differential equation

$$y'' - 2xy' + 2ny = 0.$$

We shall need also to discuss the Bessel function $J_\nu(x)$ which satisfies the differential equation

$$x^2y'' + xy' + (\nu^2 - x^2)y = 0.$$

For $\nu > -1$, all the zeros of $J_\nu(x)$ are real [54, p.483]. We denote the positive zeros in increasing order by $j_{\nu k},\ \ k = 1, 2, \ldots$.

We shall make occasional use of the symbol Δ for forward differences, i.e., $\Delta x_k = x_{k+1} - x_k,\ \Delta^{n+1}x_k = \Delta(\Delta^n x_k),\ n = 1, 2, \ldots$.

3 Monotonic variation of zeros of Jacobi Polynomials

Suppose that y is a polynomial satisfying a differential equation

$$y'' + P(x)y' + Q(x)y = 0, \tag{5}$$

where P and Q are meromorphic and that y has simple zeros $x_1, \ldots, x_n$ none of coincides with a singularity of P or Q. Then

$$\sum_{k=1}^{n}{}' \frac{1}{x_j - x_k} = -\frac{1}{2}P(x_j), \quad j = 1, \ldots, n \tag{6}$$

where, here and in what follows, the prime indicates that the singular term ($k = j$ here) in the sum must be omitted. This is most easily proved by putting

$$y = \prod (x - x_k) = (x - x_j)y_j$$

where

$$y_j = \prod_{k \neq j} (x - x_k)$$

Then $y' = (x - x_j)y_j' + y_j, \;\; y'' = (x - x_j)y_j'' + 2y_j'$, so

$$\frac{y''(x_j)}{y'(x_j)} = 2\frac{y'(x_j)}{y(x_j)}. \tag{7}$$

On the other hand, by logarithmic differentiation of y_j, we see that the required sum is $y'(x_j)/y(x_j)$ which is equal to $\frac{1}{2}y''(x_j)/y'(x_j)$ by (7) so the desired result follows from the differential equation (5).

The sums (6) were given by Stieltjes [48] the cases of the classical orthogonal polynomials. Thus for example, for the Hermite polynomials, we get

$$\sum_{k=1}^{n}{}' \frac{1}{x_j - x_k} = x_j, \;\; j = 1, \ldots, n \tag{8}$$

while for the Jacobi polynomials, we have

$$\sum_{k=1}^{n}{}' \frac{1}{x_j - x_k} + \frac{1}{2}\frac{\alpha+1}{x_j - 1} + \frac{1}{2}\frac{\beta+1}{x_j+1} = 0, \;\; j = 1, \ldots, n. \tag{9}$$

Such equations may be used to get various kinds of information about the zeros in question. At the most elementary level, we can see from (8), for example, that if the zeros are listed in increasing order, then $(x_n - x_k)^{-1} < x_n, \;\; k = 1, \ldots, n-1$. Stieltjes has shown how to use (9) to show that the zeros of the Jacobi polynomials decrease (increase) as $\alpha(\beta)$ increases. This is done by differentiating the equations (9) getting

$$\sum_{k=1}^{n}{}' \frac{1}{(x_j - x_k)^2}\left(\frac{\partial x_j}{\partial \alpha} - \frac{\partial x_k}{\partial \alpha}\right) + \frac{1}{2}\frac{\alpha+1}{(x_j-1)^2}\frac{\partial x_j}{\partial \alpha} + \frac{1}{2}\frac{\beta+1}{(x_j+1)^2}\frac{\partial x_j}{\partial \alpha} - \frac{1}{2}\frac{1}{x_j - 1} = 0,$$

or

$$\sum_{k=1}^{n} a_{jk}\frac{\partial x_j}{\partial \alpha} = \frac{1}{2}\frac{1}{x_j - 1}, \;\; j = 1, \ldots, n \tag{10}$$

where

$$a_{jj} = \sum_{k=1}^{n}{}' \frac{1}{(x_j - x_k)^2} + \frac{1}{2}\frac{\alpha+1}{(x_j-1)^2} + \frac{1}{2}\frac{\beta+1}{(x_j+1)^2}$$

and

$$a_{jk} = a_{kj} = -(x_j - x_k)^{-2}, \;\; k \neq j.$$

It can easily be checked that the matrix $A = [a_{jk}]$ in (10) is positive definite. In fact, if $u = (u_1, \ldots, u_n)^T$ is any nontrivial column vector, we have

$$2u^T Au = \sum_{j=1}^{n}\left\{\frac{\alpha+1}{(x_j-1)^2} + \frac{\beta+1}{(x_j+1)^2}\right\}u_j^2 + \sum\left(\frac{u_j - u_k}{x_j - x_k}\right)^2,$$

where the second sum on the right is over all j and k with $j \neq k$. The diagonal terms of A are positive and its off-diagonal terms are negative. We now use a result from linear algebra, which asserts that in this case all the entries in A^{-1} are positive. Hence, from (10), we see that each of the $\partial x_j/\partial\alpha$ is negative so that each zero decreases as α increases. It may be shown similarly that $\partial x_j/\partial\beta$ is positive and hence that each zero increases as β increases. The sum in (6) is the first, and that in (3) the second of a series of sums of the form

$$\sum_{k=1}^{n}{}' \frac{1}{(x_j - x_k)^m}, \quad k = 1, 2, \ldots$$

formulas for which may be obtained from the differential equation (5). See [1], [3], [9] and [44] for further information.

4 Sum rules and their use

"Sum rules" are formulas for sum of powers of zeros of a polynomial or other function. See [1], [8] and [9]. In a certain sense their use is a very old method going back to Euler, who used an ingenious method to find what we would call the smallest zeros of the Bessel function $J_0(x)$. Euler's method, which applies to certain transcendental functions as well as to polynomials, could be described as follows:

Suppose that an entire function $f(z)$, with $f(0) = 1$, has the power series expansion

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \tag{11}$$

and a product representation

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \tag{12}$$

Then the sums $S_m = \sum_{k=1}^{\infty} z_k^{-m}$ can be expressed in terms of the coefficients a_k in the following way. Logarithmic differentiation of (12) leads to

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{\infty} \frac{1}{z - z_k},$$

whence

$$S_1 = -a_1.$$

Further differentiation of (12) leads to

$$\frac{f''(z)}{f(z)} = \left(\frac{f'(z)}{f(z)}\right)^2 - \sum_{k=1}^{\infty} \frac{1}{(z - z_k)^2},$$

so we obtain

$$S_2 = -2a_2 - a_1 S_1 = -2a_2 + a_1^2$$

This process can be repeated. The general result is

$$S_n = -na_n + \sum_{i=1}^{n-1} a_i S_{n-i}. \tag{13}$$

Now in case all the zeros are positive, with

$$0 < x_1 < x_2 < \ldots,$$

we can see that

$$S_m^{-1/m} < x_1 < S_m/S_{m+1}, \quad m = 1, 2, \ldots. \tag{14}$$

so that we can get approximations of increasing accuracy for x_1. A modification of the method can then be used to find approximations to x_2, etc.

Actually what we are doing here is part of the general problem of reconstructing a function from its moments. That is, if we know all the moments

$$\sum_{k=1}^{n} x_k^r, \quad r = 1, 2, \ldots,$$

we should be able to reconstruct all the x_k; in other words the sums of powers tell us everything we need to know about the zeros. For this reason, Buendia, Dehesa and Galvez [8], extending the work of [9], have considered the problem of finding such sums for zeros of polynomial solutions of differential equations of arbitrary order. This is important in view of results which assert the existence of such equations for arbitrary sets of orthogonal polynomials [20], [5] and the explicit results which have been obtained in this direction [36]

5 Electrostatic interpretation of zeros

The x-zeros of the Jacobi polynomial may be thought of as the positions of equilibrium of $n(\geq 2)$ unit electrical charges in the interval $(-1, 1)$ in the field, with logarithmic potential, generated by charges $(\alpha+1)/2$ and $(\beta+1)/2$ placed at 1 and -1. The physical interpretation makes it clear that the zeros move to the left when α is increased and to the right when β is increased. This gives an easy proof of the result of Stieltjes discussed above; [48], [52, Theorem 6.21.1]. How do we get this result from differential equations? First, since the energy of a pair of charges p at t_1 and q at t_2 is given by $-pq \log |t_1 - t_2|$, we see that the position of equilibrium corresponds to the maximum of

$$T(x_1, \ldots, x_n) = \prod_{k=1}^{n} (1 - x_k)^{(\alpha+1)/2} (1 + x_k)^{(\beta+1)/2} \prod_{k<m} |x_k - x_m|. \tag{15}$$

It is not difficult to show that the maximum position must be unique [52, §6.7]. At the maximum, we must have $\partial T/\partial x_k = 0, \quad k = 1, \ldots, n$ which leads to the set of equations (9). The zeros of Jacobi polynomials satisfy these equations but do these equations *characterize* the zeros? To see that they do, we introduce the polynomial $f(x) = (x - x_1)\ldots(x - x_n)$ and get from (9)

$$\frac{1}{2}\frac{f''(x_k)}{f'(x_k)} + \frac{1}{2}\frac{\alpha+1}{x_j - 1} + \frac{1}{2}\frac{\beta+1}{x_j+1} = 0, \quad j = 1, \ldots, n$$

or

$$(1 - x_k^2)f''(x_k) + [\beta - \alpha - (\alpha + \beta + 2)x_k]f'(x_k) = 0.$$

This means that $(1-x^2)f''(x)+\{\beta-\alpha-(\alpha+\beta+2)x\}f'(x)$ is a polynomial of degree at most n which vanishes for all the zeros of $f(x)$. Thus it is a constant multiple of $f(x)$. By comparing the coefficients of x^n we get the constant factor to be $-n(n+\alpha+\beta+1)$. Hence we see that f is a polynomial solution of (1), so it is a constant multiple of $P_n^{\alpha,\beta}(x)$.

These ideas have been extended to the case of complex zeros by Hendriksen and van Rossum [21]. They consider point charges of strength $(a + 1)/2$ at the origin and $(c - a)/2$ at the point a^{-1}, $(a > 0)$ on the real axis. It is supposed that c is not zero or a negative integer. As $a \to \infty$ the point charges become $+\infty$ and $-\infty$ respectively while their sum remains constant $(c + 1)/2$.The field force $G(z)$ at a point z is given by

$$G(z) = \frac{(c+1)\bar{z} - 1}{2\bar{z}^2}.$$

It is then easy to show that if n positive unit charges are in equilibrium in this field they must coincide with the zeros of a polynomial solution of the differential equation

$$z^2y'' + [(c+1)z - 1]y' - n(c+n)y = 0 \tag{16}$$

i.e., they must be the zeros of ${}_2F_0(-n, c+n, x)$.Thus we have an interpretation of the zeros of the Bessel polynomials [19].

Hendriksen and van Rossum [21] go on to consider the situation of m positive point charges of strength q placed at $r\omega_k$ where $\omega_k, \quad k = 1, \ldots, m-1$ are the mth roots of unity and a nonnegative charge of strength p at the origin.They show that $n(> m)$ positive free unit charges will be in equilibrium in the resultant field if and only if they coincide with the zeros of a polynomial solution of a certain differential equation.Several special cases of this result are then considered.

Much other work has been done on the characterization of orthogonal polynomials as extremal solutions of certain problems; see [38] for some references.

6 The Sturm comparison theorem

The Sturm comparison, dating back to [49], in of its most simple forms, may be stated as follows. See [6], [11], [23].

Theorem 1 *Let $y(x), Y(x)$ be nontrivial solutions on an interval $I = (x_0, X_0)$ of the equations*

$$y'' + f(x)y = 0, \tag{17}$$

$$y'' + F(x)y = 0, \tag{18}$$

where $f(x), F(x)$ are continuous on I. Let $y(x)$ have consecutive zeros at a and b in I and let $F(x) \geq f(x), \quad a \leq x \leq b$. Then $Y(x)$ has a zero at a point c such that $a < c < b$, except in the trivial case where $F(x) = f(x)$ in $[a, b]$ and $Y(x)$ is a constant multiple of $y(x)$.

Proof. We observe that if $Y(x)$ has no such zero we may assume, without loss of generality, that $y(x) > 0, \quad Y(x) > 0, \quad a < x < b$, and hence $y'(a) > 0, \quad y'(b) < 0$. Multiplying the equations $y'' + fy = 0, \quad Y'' + FY = 0$, by $Y, \quad y$ respectively, and subtracting, we get

$$[y'Y - yY']' = [F - f]yY \tag{19}$$

Integrating from a to b, we have

$$[y'(x)Y(x) - y(x)Y'(x)] \Big|_{x=a}^{x=b} = \int_a^b [F(t) - f(t)]y(t)Y(t)dt. \tag{20}$$

Using $y(a) = y(b) = 0$, this becomes

$$y'(b)Y(b) - y'(a)Y(a) = \int_a^b [F(t) - f(t)]y(t)Y(t)dt \tag{21}$$

Now the right-hand side here is ≥ 0 with equality only if $F(x) \equiv f(x), \quad a \leq x \leq b$. The left-hand side is ≤ 0 (use $y'(a) > 0, \quad y'(b) < 0), \quad Y(b) \geq 0, \quad Y(a) \geq 0)$ with equality only if $Y(b) = Y(a) = 0$. Thus we have a contradiction, except in the trivial case when the equations are the same *and* one solution is a constant multiple of the other. This completes the proof of the Theorem.

Theorem 2 *Let the hypotheses of Theorem 1 be satisfied. Suppose we do not have the trivial case described in Theorem 1, that $Y(a) = 0$, and that $y'(a+) = Y'(a+) > 0$. Then $Y(x) < y(x), \quad a < x \leq c$.*

Proof. We simply note that under the given hypotheses, it follows from (19) that $Y(x)/y(x)$ decreases from 1 to 0 as x increases from a to c.

When $F(x) \geq f(x)$ on an interval, we say that (18) is a *Sturm majorant* of (17) on the interval.

There is also a version of Theorem 1 on the interval (x_0, b) where b is the first zero of y on (x_0, X_1). It runs as follows:

Theorem 3.*Let $y(x), Y(x)$ be nontrivial solutions of the equations* (17), (18) *on an interval $I = (x_0, X_0)$ where $f(x), F(x)$ are continuous on I. Let $y(x)$ have its first zero at b in I, let*

$$\lim_{x \to x_0+} [y'(x)Y(x) - y(x)Y'(x)] \geq 0. \tag{22}$$

and let $F(x) \geq f(x), \quad x_0 < x \leq b$. Then $Y(x)$ has a zero at a point c such that $x_0 < c < b$, except in the trivial case where $F(x) = f(x)$ in $(x_0, b]$ and $Y(x)$ is a constant multiple of $y(x)$.

Proof. The proof is similar to that of Theorem 1. If $Y(x)$ has no such zero we may assume, without loss of generality, that $y(x) > 0, \quad Y(x) > 0, \quad x_0 < x < b$, and hence $y'(b) < 0$. As in the proof of Theorem 1, we are led to

$$y'(b)Y(b) - \lim_{x \to x_0+} [y'(x)Y(x) - y(x)Y'(x)] = \lim_{x \to x_0+} \int_x^b [F(t) - f(t)]y(t)Y(t)dt. \tag{23}$$

Now the right-hand side here is ≥ 0 (possibly $+\infty$) with equality only if $F(x) \equiv f(x), \quad x_0 < x \leq b$. The left- hand side is ≤ 0 (use $y'(b) < 0, Y(b) \geq 0$ and (22)) with equality only if $Y(b) = 0$. Thus we have a contradiction, except in the trivial case when the equations are the same *and* one solution is a constant multiple of the other.

One easy consequence of Theorem 2 is an application to the shapes of the successive arches of the graph of a nontrivial solution of a *single* differential equation. This result, as far as its application to zeros is concerned was already in Sturm's paper [49, p.173]; it is called the "Sturm convexity theorem" in [6, p.272]. It was discussed also by E. Hille [22] and by G. Szegö [51] but finds its most detailed exposition in a paper by E. Makai [42].

Theorem 4. *We suppose that y is a nontrivial solution of* (17) *where f is continuous and decreasing on I and that y has consecutive zeros at $x_1, \ x_2, \ x_3$ on I. Then*

$$x_2 - x_1 < x_3 - x_2 \tag{24}$$

i.e., the sequence of zeros of y is **convex**. *Similarly if f increases, the sequence is concave.*

Proof. The inequality (24) follows from the more general result that if we rotate the arch of the graph of y between x_1 and x_2 through 180° "about x_2" the resulting arch will be contained entirely within the arch joining x_2 to x_3. (See Fig. 1.) To prove this last result, we need to show that

$$y(x) > y(2x_2 - x), \quad x_2 < x < x_2 + d \tag{25}$$

where $d = x_2 - x_1$, and we suppose, without loss of generality, that $y(x) < 0, \quad x_1 < x <$

x_2. Then we use the fact that $\psi(x) = -y(2x_2 - x)$ satisfies $\psi''(x) + f(2x_2 - x)\psi(x) = 0$, which is a Sturm majorant for (17), on taking account of the decreasing nature of f, so that the desired result follows from Theorem 2.

In most of the applications, we are dealing with not a single zero but a sequence of zeros. In this case the most appropriate form for the Sturm comparison theorem is as follows:

Theorem 5. *Let the hypotheses of Theorem 1 or of Theorem 3 hold except that the inequality $F(x) \geq f(x)$ is assumed to hold on the whole interval I. Let $x_k, X_k, \;\; k = 1, 2, \ldots$ be the k-th zeros of y and Y respectively on I. Then $X_k < x_k, \;\; k = 1, 2, \ldots$*

Proof. It is simply a matter of using Theorem 1 or Theorem 3 for the first zero and then using Theorem 1 for the remaining ones.

7 Higher monotonicity

Given that Theorem 3 shows the convexity of the zeros based on the monotonicity of the coefficient function in the equation (17) we may ask about *higher monotonicity*, that is whether some more information about the coefficients or special solution of (17) might imply information about higher differences of the zeros. Questions of this kind were first raised and answered in a paper [40] by L. Lorch and P. Szego and several theorems were proved by them and by later writers. Lorch and Szego show, among other things, that if $y_1(x), \;\; y_2(x)$ are linearly independent solutions of (17) and if $p(x) = y_1^2(x) + y_2^2(x)$ satisfies

$$p^{(j)}(x) \geq 0, \;\; j = 0, \ldots, N \tag{26}$$

then

$$\Delta^{j+1} x_k \geq 0, \;\; j = 1, \ldots, N \tag{27}$$

where $\{x_k\}$ is the sequence of zeros of an arbitrary solution of (17). The most satisfactory application to orthogonal polynomials is due to L. Durand [10] who was able to prove the formula

$$e^{-x^2}[H_\nu^2(x) + G_\nu^2(x)] = 2^{\nu+1}\frac{1}{\pi}\int_0^\infty e^{(-2\lambda+1)t + x^2 \tanh t}(\cosh t \sinh t)^{-\frac{1}{2}} dt, \tag{28}$$

where $H_n(x)$ is the Hermite polynomial and $G_n(x)$ is a suitable linearly independent solution of the Hermite equation, or more suitably $e^{-x^2/2}H_\nu(x), e^{-x^2/2}G_\nu(x)$ are linearly independent solutions of

$$y'' + (2\nu + 1 - x^2)y = 0.$$

Now (28) shows that the Lorch-Szego condition (26) holds in the present case with $N = \infty$, so we find that if $x_k,\ k = 1, \ldots, [n/2]$ are the positive zeros in increasing order of the Hermite polynomial $H_n(x)$, then

$$\Delta^{j+1} x_k \geq 0, \ \ j = 1, \ldots, [n/2].$$

8 Elementary inequalities

Although the Sturm comparison theorem is over 150 years old [41], its value in providing information about the zeros of special functions has not always been fully appreciated. It is true that Sturm himself used his result to show that the spacing of the zeros of Bessel functions $C_\nu(x)$ increases or decreases to π according to whether $|\nu|$ is less than or greater than $\frac{1}{2}$. Also M. B. Porter (see [7]) used the Sturm theorem to show that the positive zeros of $J_\nu(x)$ increase as $\nu(\geq 0)$ increases. However, somewhat stronger results may be obtained using other methods; see [54], [45]. It was perhaps this situation which led G.N. Watson [54, §15.82] to write in 1922 concerning these methods (for zeros of Bessel functions) that: "The results hitherto obtained in this manner are of some interest, though they are not of a particularly deep character...". As R. Askey points out in the notes accompanying [51] in G. Szegö's collected works, Szegö's paper [51], based on Sturm methods, was "the start of many very significant applications of Sturm's comparison theorem" in that it led to results on zeros of Legendre polynomials which are sharper than those obtained by other methods.

We recall that all the x-zeros of the Legendre polynomial $P_n(x)$ lie in the interval $(-1, 1)$ or, equivalently, the θ-zeros of $P_n(\cos\theta)$ all lie in the interval $(0, \pi)$. Denoting these zeros in increasing order by $\theta_k,\ k = 1, \ldots, n$ the earlier results were

$$\frac{k - 1/2}{n + 1/2}\pi < \theta_k < \frac{k}{n + 1/2}\pi \tag{29}$$

due to Bruns and the stronger inequalities

$$\frac{k - 1/2}{n}\pi < \theta_k < \frac{k}{n + 1}\pi \tag{30}$$

due to Markoff and Stieltjes. Now to use the Sturm comparison theorem we note that $z = (\sin\theta)^{(1/2)} P_n(\cos\theta)$, which has the same zeros θ_k on the interval in question, satisfies the differential equation

$$z'' + \{(n + 1/2)^2 + (2\sin\theta)^{-2}\}z = 0 \tag{31}$$

This is a Sturm majorant for the equation

$$z'' + \{(n + 1/2)^2\}z = 0 \tag{32}$$

with solution $z = \sin(n + 1/2)(x - x_0)$ for arbitrary x_0. This shows that there is a θ-zero of $P_n(\cos\theta)$ in every interval of length $\pi/(n + 1/2)$, in particular, that

$$\frac{k-1}{n+1/2}\pi < \theta_k < \frac{k}{n+1/2}\pi, \quad k = 1, \ldots, n \tag{33}$$

Though the proof is simple, this result is not very encouraging in that the lower bound is weaker than that in (29) and (30) and the upper bound is the same as that in (29) and weaker than that in (30). However, Szegö remarks that the lower bound may be improved to give that in (29) by using the symmetric property $\theta_k = \pi - \theta_{n+1-k}$ and the upper bound in (33).

Next Szegö shows that this can be improved to (30) by making an ingenious use of the convexity of the sequence $\theta_k, \ \ k = 1, \ldots, [n/2] + 1$, which follows from Theorem 4.

Now this shows that one can get the Bruns and Markoff-Stieltjes Stieltjes results from the Sturm theorem "plus ingenuity". The advantage of the Sturm method is that it can be applied just as easily to the case of Jacobi polynomials $P_n^{\alpha,\beta}(x), \ \ -\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$. It yields

$$\frac{k + (\alpha+\beta-1)/2}{n + (\alpha+\beta+1)/2}\pi < \theta_k < \frac{k}{n + (\alpha+\beta+1)/2}\pi, \tag{34}$$

by comparing the normalized Jacobi equation with an appropriate trigonometric equation.

9 Non-elementary inequalities: zeros of Bessel functions

The above inequalities were obtained by comparing the appropriate differential equation with the trigonometric equation. Clearly, better, though more complicated, results can be obtained if we compare with other equations. Returning to the Legendre case, Szegö [51] used the fact that (31) is a Sturm majorant of

$$z'' + \{(n + 1/2)^2 + (2\theta)^{-2}\}z = 0 \tag{35}$$

which has the solution $z = \theta^{(1/2)} J_0[(n + 1/2)\theta]$, so there is at least one θ_k in every interval $(j_{0,k-1}, \ j_{0,k}), \ k = 1, \ldots, n$. But since $j_{0,n}/(n + 1/2) < \pi$, there is exactly one zero in each such interval so we get

$$j_{0,k-1}/(n + 1/2) < \theta_k < j_{0,k}/(n + 1/2), \ k = 1, \ldots, n. \tag{36}$$

The upper estimate here is quite strong. The lower one is not particularly strong; a better one is obtained by comparing (31) with

$$z'' + \{(n + 1/2)^2 + K/4 + (2\theta)^{-2}\}z = 0 \tag{37}$$

which is satisfied by a suitably transformed Bessel function.

10 Scaling and transformation in the Sturm theorem

The idea of scaling is very simple. If $y(x)$ satisfies (17) then $z(x) = y(\lambda x)$ satisfies

$$z'' + \lambda^2 f(\lambda x) = 0 \tag{38}$$

and we may apply all of the previous results to find information about the zeros x_k/λ of $y(\lambda x)$. Laforgia [28] exploited this method to show, for example, that for the zeros $x_{n,k}^{(\lambda)}$ of the ultraspherical polynomials we have $\lambda x_{n,k}^{(\lambda)}$ increases as λ increases, $0 \le \lambda \le 1$. This result was refined in [47] and in [4], where it was shown that $[2n^2 + 1 + 2\lambda(2n + 1)]^{1/2} x_{n,k}^{(\lambda)}$ increases as λ increases, $-1/2 \le \lambda \le 3/2$ and it was pointed out that there seems to be a certain limitation in the Sturm method. In particular it does not seem to be possible to show by this method, as conjectured by Ismail and Letessier [25], that $\lambda^{1/2} x_{n,k}^{(\lambda)}$ increases as λ increases, $0 < \lambda < \infty$. Ismail and Letessier prove this for the largest zero, using recurrence relations.

L.Gatteschi and his co-workers (see [15], [16], [17], [18] and references) have contributed greatly to the refinement of the applications of the Sturm theorem. In [17], Gatteschi makes ingenious use of the theorem to provide very sharp upper and lower bounds for the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos\theta)$, in case $-\frac{1}{2} \le \alpha, \beta \le \frac{1}{2}$. He shows that an asymptotic formula, involving zeros of Bessel functions, due to Frenzen and Wong ([13], [14]) in fact provides a lower bound for these zeros (and also an upper bound, using $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$). Specifically, Gatteschi shows that

$$z'' + F(\theta) z = 0 \tag{39}$$

is a Sturm majorant for (3), where

$$F(\theta) = \frac{1}{2}\frac{f'''}{f'} - \frac{3}{4}\left(\frac{f''}{f'}\right)^2 + \left(\frac{1}{4}\alpha^2\right)\left(\frac{f'}{f}\right)^2 + f'^2$$

and

$$f(\theta) = N\theta + \frac{1}{4N}\left[\left(\frac{1}{4}\alpha^2\right)\left(\frac{2}{\theta} - \cot\frac{\theta}{2}\right) + \left(\frac{1}{4} - \beta^2\right)\tan\frac{\theta}{2}\right].$$

This leads to the following lower bound for the zeros.

$$\theta_{n,k} \ge \frac{j_{\alpha,k}}{N} - \frac{1}{4N^2}\left[\left(\frac{1}{4} - \alpha^2\right)\left(\frac{2}{t} - \cot\frac{t}{2}\right) + \left(\frac{1}{4} - \beta^2\right)\tan\frac{t}{2}\right]. \tag{40}$$

where $N = n + (\alpha + \beta + 1)/2$ and $t = j_{\alpha,k}/N$.

The lower bound in (40) becomes an upper bound when t is replaced by $\tau = j_{\alpha,k}/\nu$, where

$$\nu = \left(N^2 + \frac{1 - \alpha^2 - 3\beta^2}{12} \right)^{\frac{1}{2}}. \quad (41)$$

Gatteschi shows also that between any pair of zeros of a Jacobi polynomial there occurs at least one root of a certain transcendental equation involving elementary functions. In the case of the kth zero, $\theta_{n,k}(\alpha), k = 1, 2, \ldots, [n/2]$, of the ultraspherical polynomial $P_n^{(\alpha,\alpha)}(\cos\theta)$, this leads to the inequalities

$$\phi_{n,k}(\alpha) \leq \theta_{n,k}(\alpha) \leq \phi_{n,k}(\alpha) + N^{-2}(\frac{1}{8} - \alpha^2/2) \cot \phi_{n,k}(\alpha), \quad (42)$$

where $N = n + \alpha + \frac{1}{2}$ and $\phi_{n,k}(\alpha) = (k + \alpha/2 - 1/4)\pi/N$. Gatteschi [17] makes comparisons with known bounds and gives numerical examples to illustrate the sharpness of his inequalities. For example for the smallest positive zero of $P_{10}^{(1/3,0)}(\cos\theta)$ he finds the lower bound 0.27202843 and the upper bound 0.27215052. The "exact " value is 0.27202854.

Many other authors have contributed to the use if Sturm methods. We mention [12], [29], [30] and the expository [31], for further information.

11 The method of Szegö and Turán

There are certain rather obvious ways in which the Sturm theorem gives information about the *spacing* of the zeros of orthogonal polynomials, or generally solutions of (17). For example, if $m^2 \leq f(x) \leq M^2$ in the interval of interest, then $y'' + M^2 y = 0$ is a majorant for (17), which is in turn a majorant for $y'' + m^2 y = 0$. Thus if $x_{n,k}, \ x_{n,k+1}$ are successive zeros of an n-th degree polynomial solution of (17), then

$$\pi/m \leq x_{n,k+1} - x_{n,k} \leq \pi/M \quad (43)$$

In the case of orthogonal polynomials, as we increase n we increase the number of zeros, while leaving the interval on which they lie appreciably unchanged. Hence, we expect that the spacing between the zeros decreases. A first result of this kind for the ultraspherical polynomials is due to Szegö and Turán [53]; it was extended to certain Jacobi polynomials in [2].

Theorem 6. *Let the hypotheses of Theorem 5 hold and let f be nonincreasing on I. Then $\Delta X_k < \Delta x_k$.*

Proof. We use the fact that $y(x+\delta)$ satisfies

$$y'' + f(x+\delta)y = 0 \tag{44}$$

and it has consecutive zeros at $x_1 - \delta, \ldots, x_m - \delta$. For a fixed k we choose $\delta = x_k - X_k$, where $\delta > 0$ on account of Theorem 5. This makes the kth zero of Y coincide with the kth zero of $y(x+\delta)$.

Since

$$f(x+\delta) \leq f(x) \leq F(x), \quad a < x < x_m - \delta, \tag{45}$$

we see by Theorem 5 that the next larger zero of Y occurs before the next zero of $y(x+\delta)$, that is $x_{k+1} - \delta > X_{k+1}$, giving the desired result. The result are immediately applicable to the ultraspherical polynomials giving a slightly strengthened version of the Szegö-Turán result.

Recall that $(1-x^2)^{\lambda/2+1/4} P_n^\lambda(x)$ satisfies the differential equation

$$y'' + q_n(x)y = 0 \tag{46}$$

where

$$q_n(x) = \left\{ \frac{(n+\lambda)^2}{1-x^2} + \frac{1/2 + \lambda - \lambda^2 + x^2/4}{(1-x^2)^2} \right\} \tag{47}$$

Now $q_n(x)$ is a decreasing function of x for $-1/2 \leq \lambda \leq 3/2$ so that we get

$$x(n+1,k) - x(n+1,k+1) < x(n,k) - x(n,k+1), k = 1, \ldots, [n/2] \tag{48}$$

for $-1/2 \leq \lambda \leq 3/2$, where $x(n,k)$ be the kth zero in decreasing order of $P_n^\lambda(x)$.

We may also combine the results of this section with the scaling idea of the previous section to get results of the kind described in [2].

12 Limitations of the Sturm method

In view of our remarks above about how there are some inherent limitations in the Sturm method, the question arises whether there are any improved version of the theorem which might be more effective. First of all, there are far-reaching generalizations of the Sturm theorem, to higher order equations, to partial differential equations, and so on; see [27], [46], and [50], for example. Then there are theorems in which the setting is that of second order ordinary linear differential equations but in which the hypotheses of the classical Sturm comparison theorem are weakened; [32], [33], [34] and, for an expository account, [37]. These latter theorems, though having weaker hypotheses, are not easy to apply since it is difficult to check their hypotheses. We mention also the indirect use of the Sturm theorem as in [43]; see also [31]. The idea here is that there are problems such as that of the decrease of $j_{\nu k}/\nu$, for $\nu > 0$ for which the direct use of the Sturm theorem seems impossible but for which an indirect use is effective.

Of course, it must be said that Sturm methods or indeed methods based on differential equations generally are not the only ones which can be used to study zeros of orthogonal polynomials. We mention, in particular that a method [35] based on the differential equation approach to Bessel functions has been found effective when applied to the recurrence relations satisfied by Bessel functions [26] and by orthogonal polynomials [24]

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