Approximations for zeros of Hermite functions

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Abstract. We present a convergent asymptotic formula for the zeros of the Hermite functions as $\lambda \to \infty$. It is based on an integral formula due to the authors for the derivative of such a zero with respect to λ. We compare our result with those for zeros of Hermite polynomials given by P. E. Ricci.

1. Introduction

By "Hermite functions", we mean solutions of the differential equation

$$y'' - 2ty' + 2\lambda y = 0. \tag{1.1}$$

The Hermite function $H_\lambda(t)$ can be defined (see, e.g., [**9**]) by

$$H_\lambda(t) = -\frac{\sin \pi\lambda\, \Gamma(1+\lambda)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left((n-\lambda)/2\right)}{\Gamma(n+1)}(-2t)^n, \tag{1.2}$$

or, in terms of the confluent hypergeometric functions, by [**4**]

$$H_\lambda(t) = \frac{2^\lambda}{\sqrt{\pi}} \left[\cos\frac{\lambda\pi}{2}\, \Gamma(\frac{\lambda}{2}+\frac{1}{2})\ {}_1F_1(-\frac{\lambda}{2},\frac{1}{2};\, t^2) + 2t\sin\frac{\lambda\pi}{2}\, \Gamma(\frac{\lambda}{2}+1)\ {}_1F_1(-\frac{\lambda}{2}+\frac{1}{2},\frac{3}{2};\, t^2)\right]. \tag{1.3}$$

In case λ is a nonnegative integer, formula (1.2) is to be understood in a limiting sense so that $H_\lambda(t)$ reduces to the Hermite polynomials (with the notation of, e.g., [**17**]). Thus $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12t$, etc.

We note also that

$$H_\lambda(t) = 2^\lambda\, \Psi(-\frac{\lambda}{2},\frac{1}{2};\, t^2),$$

with the notation of [**8**, p. 257] for confluent hypergeometric functions. In the polynomial case ($\lambda = n$), the zeros of $H_\lambda(t)$ are real and located symmetrically with respect to the origin. Here we study the real zeros of $H_\lambda(t)$ or, more generally,

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of any Hermite function, as functions of λ, in particular as $\lambda \to +\infty$. We use the notation $h_k(\lambda),\ k = 1, 2, \dots$ for the zeros, in *decreasing* order of $H_\lambda(t)$.

As shown in [**7**], when $n < \lambda \le n+1$, with n a nonnegative integer, $H_\lambda(t)$ has $n+1$ real zeros which increase with λ. As λ increases through each nonnegative integer n a new leftmost zero appears at $-\infty$ while the right-most zero passes through the largest zero of $H_n(t)$. See Figure 1 which provides graphs of the real zeros of $H_\lambda(t)$ (solid lines) as functions of λ. The small circles mark the zeros of Hermite polynomials. (This figure was produced using MAPLE and GNU plotutils, the GNU plotting utilities package.)

It is well known [**17**, (6.32.5)] that, in the polynomial case ($\lambda = n$), for the largest zero, we have

$$h_1(n) = \sqrt{2n+1} - 6^{-1/3}(2n+1)^{-1/6}(i_1 + \epsilon), \tag{1.4}$$

where $\epsilon \to 0$, as $n \to \infty$. Here $i_1 = 3.372134408\dots$ is the smallest positive zero of the Airy function $A(x)$, or in a more familiar notation, of $\mathrm{Ai}(-3^{-1/3}x)$.

This can also be written, with $\Lambda = \sqrt{2n+1}$, as

$$h_1(n) = \Lambda - a_1\Lambda^{-1/3}[1 + O(1)], \qquad \lambda \to \infty, \tag{1.5}$$

where $a_1 = 1.855757081\dots$ is the smallest positive zero of $\mathrm{Ai}(-2^{1/3}x)$.

This result has been extended by P. E. Ricci [**16**] to give several additional terms of an asymptotic expansion. Ricci's results were extended to the zeros of a certain generalization of the Hermite polynomials, with weight function $x^{2\mu}\exp(-x^2)$, where μ is a nonnegative integer, by S. Noschese [**14**].

Our main result will extend (1.5) to a convergent, and hence asymptotic, series expansion whose first five terms are given by

$$h_1(n) = \Lambda - a_1\Lambda^{-1/3} - \frac{1}{10}a_1^2\Lambda^{-5/3} + \left[\frac{9}{280} - \frac{11}{350}a_1^3\right]\Lambda^{-3} + \left[\frac{277}{12600}a_1 - \frac{823}{63000}a_1^4\right]\Lambda^{-13/3} + \cdots \tag{1.6}$$

In fact, with little additional trouble, we can obtain a similar expansion for any real zero of any Hermite function.

2. General solution of the Hermite equation

For each fixed λ, $H_\lambda(t)$ is a solution of (1.1) which grows relatively slowly as $t \to +\infty$. Following [**4**], we consider also a solution of (1.1) which is linearly independent of $H_\lambda(z)$:

$$G_\lambda(t) = \frac{2^\lambda}{\sqrt{\pi}}\left[-\sin\frac{\lambda\pi}{2}\,\Gamma(\frac{\lambda+1}{2})\ {}_1F_1(\frac{-\lambda}{2}, \frac{1}{2}; t^2) + 2t\cos\frac{\lambda\pi}{2}\,\Gamma(\frac{\lambda+2}{2})\ {}_1F_1(\frac{-\lambda+1}{2}, \frac{3}{2}; t^2)\right]. \tag{2.1}$$

The functions $e^{-t^2/2}H_\lambda(t)$ and $e^{-t^2/2}G_\lambda(t)$, which have the same zeros as $H_\lambda(t)$ and $G_\lambda(t)$ are linearly independent solutions of the modified Hermite equation

$$y'' + (2\lambda + 1 - t^2)y = 0. \tag{2.2}$$

From [**7**, §5], the Wronskian of $e^{-t^2/2}H_\lambda(t)$ and $e^{-t^2/2}G_\lambda(t)$ is given by

$$W = \pi^{-1/2}2^{\lambda+1}\Gamma(\lambda+1). \tag{2.3}$$

We are interested in the zeros of solutions of (1.1) or (2.2), that is, of linear combinations of H_λ and G_λ.

Solutions of (2.2) are also known as parabolic cylinder functions. As pointed out by C. Malyshev [**12**, (N2)], the functions $H_\lambda(x)$ and $G_\lambda(x)$ are related to parabolic cylinder functions by

$$\exp(-x^2/2)H_\lambda(x) = 2^{\lambda/2}D_\lambda(x\sqrt{2}), \tag{2.4}$$

and

$$\exp(-x^2/2)G_\lambda(x) = \frac{2^{\lambda/2}}{\sin\pi\lambda}\left(\cos\pi\lambda\; D_\lambda(x\sqrt{2}) - D_\lambda(-x\sqrt{2})\right). \tag{2.5}$$

3. A power series for the zeros

The function

$$y(t) = e^{-t^2/2}[\cos\alpha\; H_\lambda(t) - \sin\alpha\; G_\lambda(t)] \tag{3.1}$$

satisfies the differential equation

$$y'' + (2\lambda + 1 - t^2)y = 0, \tag{3.2}$$

and hence, if we write

$$\mu = (2\Lambda)^{-1/3}, \qquad \Lambda = \sqrt{2\lambda+1}, \qquad C_\lambda = \pi^{-1/3}2^{-\lambda/2-1/4}\lambda^{1/2}, \tag{3.3}$$

we find, after some simplification, that

$$Y(\lambda,t) = C_\lambda\, y\left(\frac{1}{2\mu^3} - \mu t\right)$$

satisfies the differential equation

$$\frac{d^2Y}{dt^2} + (t - \mu^4 t^2)Y = 0. \tag{3.4}$$

In view of the asymptotic information in [**11**, p. 292], we have, for

$$x = (2\lambda+1)^{1/2} - 2^{-1/2}\lambda^{-1/6}t,$$

$$e^{-x^2/2}H_\lambda(x) = 2^\lambda\Gamma(\lambda/1 + 1/2)(\lambda+1/2)^{1/6}\,\mathrm{Ai}(-t) + O(\lambda^{-2/3}), \quad \lambda\to\infty, \tag{3.5}$$

and

$$e^{-x^2/2}G_\lambda(x) = 2^\lambda\Gamma(\lambda/1 + 1/2)(\lambda+1/2)^{1/6}\,\mathrm{Bi}(-t) + O(\lambda^{-2/3}), \quad \lambda\to\infty. \tag{3.6}$$

This asymptotic information shows that $Y(\lambda,t)$ satisfies the initial conditions

$$Y(0) = \cos\alpha\;\mathrm{Ai}(0) - \sin\alpha\;\mathrm{Bi}(0), \qquad Y'(0) = \cos\alpha\;\mathrm{Ai}'(0) - \sin\alpha\;\mathrm{Bi}'(0). \tag{3.7}$$

The initial conditions (3.7) are independent of μ and the coefficient term $t-\mu^4t^2$ in (3.4) is an entire function of μ for each fixed t. Hence, from [**3**, p. 37], for fixed t, the solutions of (3.4),(3.7) are entire functions of μ. Thus a zero $z(\mu)$ of a solution of a nontrivial solution of (3.4),(3.7) is analytic in μ in a neighbourhood of $\mu = 0$:

$$z(\mu) = \sum_{k=1}^{\infty} c_k\mu^{k-1}, \qquad |\mu| < R, \tag{3.8}$$

for some $R > 0$, where c_1 is the corresponding zero of $\cos\alpha\ \mathrm{Ai}(-x) - \sin\alpha\ \mathrm{Bi}(-x)$. In other words, if $h(\lambda)$ is a zero of a solution of (2.2), then

$$h(\lambda) = \Lambda + \Lambda^{-1/3} \sum_{k=1}^{\infty} a_k \Lambda^{-4(k-1)/3}, \tag{3.9}$$

where the series converges for $\Lambda > M$, for some $M > 0$.

This is also an asymptotic series

$$h(\lambda) \sim \Lambda + \Lambda^{-1/3} \sum_{k=1}^{\infty} a_k \Lambda^{-4(k-1)/3}, \qquad \Lambda \to \infty, \tag{3.10}$$

in the sense that, for each positive integer N,

$$h(\lambda) = \Lambda + \Lambda^{-1/3} \sum_{k=1}^{N} a_k \Lambda^{-4(k-1)/3} + O(\Lambda^{-4N/3}), \qquad \Lambda \to \infty. \tag{3.11}$$

4. Continuous ranking of zeros

It is useful to consider the function

$$p_\lambda(x) = \frac{2^{-\lambda-1}\sqrt{\pi}}{\Gamma(\lambda+1)}\, e^{-x^2} [H_\lambda^2(x) + G_\lambda^2(x)]. \tag{4.1}$$

It was shown by Durand [**4**] that

$$p_\lambda(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(2\lambda+1)\tau + x^2 \tanh\tau} \frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}}, \qquad \lambda > -1. \tag{4.2}$$

In [**7**], we proved a formula,

$$\frac{dh}{d\lambda} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-(2\lambda+1)\tau} \phi(h\sqrt{\tanh\tau}) \frac{d\tau}{\sqrt{\sinh\tau\ \cosh\tau}}, \qquad \lambda > -1, \tag{4.3}$$

for the derivative of a zero of a solution of (2.2) with respect to λ. Here

$$\phi(x) = e^{x^2} \mathrm{erfc}(x), \tag{4.4}$$

where erfc is the complementary error function:

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{4.5}$$

The zeros $h(\lambda, k, \alpha)$ of a solution of

$$\cos\alpha\ H_\lambda(x) - \sin\alpha\ G_\lambda(x), \qquad 0 \le \alpha < \pi,$$

appear to depend on three variables, λ, α and the *rank* k (first, second, etc.) of a zero. But in fact α and k can be subsumed in a single variable κ if we define $h(\lambda,\kappa)$ by

$$\int_{h(\lambda,\kappa)}^\infty \frac{du}{p_\lambda(u)} = \kappa\pi. \tag{4.6}$$

See [**13**] for details. When $\kappa = 1, 2, \ldots$, we get the zeros, in decreasing order of H_λ and when $\kappa + \frac{1}{2} = 1, 2, \ldots$, we get the zeros, in decreasing order of G_λ. More generally, When $\kappa + \alpha/\pi = 1, 2, \ldots$, we get the zeros, in decreasing order of $\cos\alpha\, H_\lambda(x) - \sin\alpha\, G_\lambda(x)$.

For fixed λ is clear from (4.6) that $h(\lambda,\kappa)$, decreases as κ (> 0) increases.

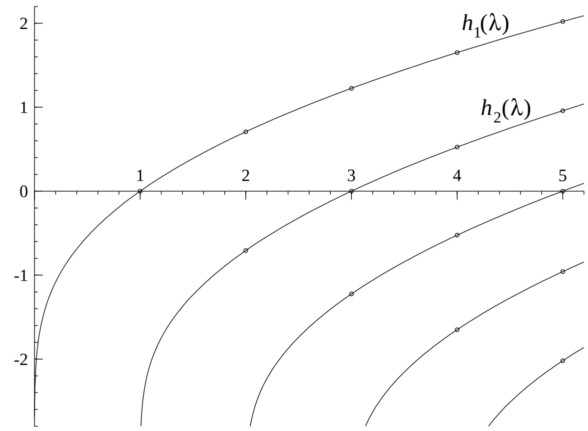

FIGURE 1

It is of interest to consider when $h(\lambda, \kappa) = 0$. For example, for an odd integral value $2n + 1$ of λ we are dealing with the Hermite polynomial $H_{2n+1}(x)$ and the $(n + 1)^{\text{st}}$ zero is at the origin, that is, $h(2n + 1, n + 1) = 0$. For general α, we have from (1.2) and (2.1),

$$\cos\alpha\ H_\lambda(0) - \sin\alpha\ G_\lambda(0) = \frac{2^\lambda}{\pi}\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)\cos(\alpha - \frac{\lambda\pi}{2}),$$

from which it follows that $h(2\kappa - 1, \kappa) = 0$.

In view of the notation introduced in (4.6), the curves of Figure 1, starting from the top, may be re-labelled $h(\lambda, 1),\ h(\lambda, 2), \ldots,$ where $h(\lambda, 1)$ is the largest zero of $H_\lambda(x)$, $h(\lambda, 2)$ is the next largest, etc.

The zeros of $G_\lambda(x)$ could be added to Figure 1, as curves lying about halfway between the curves representing the zeros of $H_\lambda(x)$. In fact, if we consider the zeros of all Hermite functions, their graphs would fill the entire half-plane $\lambda > -1$ in Figure 1.

From (4.6), and the consequence of (4.2) that $p_\lambda(u)$ is even in u, we get

$$\int_{h(\lambda,2\kappa)}^{\infty} \frac{du}{p_\lambda(u)} = 2\int_{h(\lambda,\kappa)}^{\infty} \frac{du}{p_\lambda(u)}, \tag{4.7}$$

so

$$\lim_{\lambda\to\lambda_0^+} h(\lambda, \kappa) = 0, \tag{4.8}$$

if and only if

$$\lim_{\lambda\to\lambda_0^+} h(\lambda, 2\kappa) = -\infty. \tag{4.9}$$

We note that $h(\lambda, \kappa)$ satisfies the differential equation (4.3) on $(\kappa - 1, \infty)$, that $h(\lambda, 2\lambda - 1) = 0$ and that

$$\lim_{\lambda\to\kappa-1} h(\lambda, \kappa) = -\infty.$$

This suggests that, for each fixed $\kappa > 0$, the function $h(\kappa, \lambda)$ could be defined as a solution of the initial value problem

$$\frac{dh}{d\lambda} = \Phi(\lambda, h), \qquad h(\kappa, 2\kappa - 1) = 0, \tag{4.10}$$

on the interval $\kappa - 1 < \lambda < \infty$, where

$$\Phi(\lambda, h) = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-(2\lambda+1)\tau} \phi(h\sqrt{\tanh\tau}) \frac{d\tau}{\sqrt{\sinh\tau\,\cosh\tau}}, \tag{4.11}$$

and $\phi(x)$ is as in (4.4).

The formula

$$\phi'(x) = -\frac{2}{\sqrt{\pi}} + 2x\phi(x),$$

and the asymptotic formula [**1**, 7.1.23] for $\operatorname{erfc}(x)$,

$$\phi(x) \sim \frac{1}{\sqrt{\pi}x}, \qquad x \to +\infty, \tag{4.12}$$

show that

$$-\frac{2}{\sqrt{\pi}} \le \phi'(x) < 0, \qquad 0 \le x < \infty. \tag{4.13}$$

This can be used to show that $\Phi(\lambda, h)$ is continuous and satisfies a Lipschitz condition in h in any bounded region of the half plane $\lambda > -1$, so that the usual existence and uniqueness theorems apply to (4.10). This is analogous to similar results for zeros of Bessel functions due to Á. Elbert and A. Laforgia; see [**5**, §1.1] and references.

5. Computing the coefficients

We now turn to the task of finding some of the coefficients in the convergent asymptotic expansion of a zero. For $h(k, \alpha)$, the kth zero, in decreasing order, of a Hermite function

$$\cos\alpha\ H_\lambda(x) - \sin\alpha\ G_\lambda(x),$$

the expansion will involve $a = a(k, \alpha)$ where $2^{-1/3}a$ is the kth positive zero of

$$\cos\alpha\ \operatorname{Ai}(-x) - \sin\alpha\ \operatorname{Bi}(-x).$$

We find that the first five terms are given by

$$\begin{aligned} h(k,\alpha) = \Lambda - a\Lambda^{-1/3} - \frac{1}{10}\,a^2\Lambda^{-5/3} + \left[\frac{9}{280} - \frac{11}{350}\,a^3\right]\Lambda^{-3} \\ + \left[\frac{277}{12600}\,a - \frac{823}{63000}\,a^4\right]\Lambda^{-13/3} + \cdots \end{aligned} \tag{5.1}$$

Our derivation of (5.1) is similar to that used by Á. Elbert and A. Laforgia in a discussion of zeros of Bessel functions [**6**].

The following Lemma enables us to evaluate certain double integrals which arise:

Lemma 1. *Let*

$$A_{j,k}(\Lambda) = \int_0^\infty \int_0^\infty e^{-\Lambda^2(u+t)^2} u^j t^k \,dt\,du, \tag{5.2}$$

where j and k are nonnegative integers. Then

$$A_{j,k}(\Lambda) = \Lambda^{-2-j-k}\,\frac{((j+k)/2)!\,j!\,k!}{2(j+k+1)!}.$$

Proof. Clearly $A_{j,k}(\Lambda) = \Lambda^{-2-j-k}A_{j,k}(1)$. Making the change of variable $\theta = u+t$ in the inner integral and interchanging the orders of integration, we get

$$A_{jk}(1) = \int_0^\infty \int_0^\theta e^{-\theta^2}(\theta-t)^j t^k dt\,d\theta, \tag{5.3}$$

or

$$A_{jk}(1) = \int_0^\infty e^{-\theta^2}\theta^{j+k+1}d\theta \int_0^1 (1-v)^j v^k dv = \frac{1}{2}\int_0^\infty e^{-w}w^{(j+k)/2}dw\int_0^1 (1-v)^j v^k dv. \tag{5.4}$$

But both of the integrals in the product here are well known gamma and beta functions leading to the desired result. □

As discussed in the previous section, we have, for $h = h(k,\alpha)$,

$$\frac{dh}{d\lambda} = \Phi(\lambda, h) \tag{5.5}$$

where

$$\Phi(\lambda,h) = \int_0^\infty \int_0^\infty e^{-(2\lambda+1)\tau - t^2 - 2th(\lambda)\sqrt{\tanh\tau}}\,\frac{dt\,d\tau}{\sqrt{\sinh\tau\cosh\tau}}. \tag{5.6}$$

The changes of variable $t = \Lambda v,\ \tau = u^2$ give

$$\Phi(\lambda,h) = \int_0^\infty \int_0^\infty e^{-\Lambda^2(u+v)^2} f(u,v,\Lambda,k)\,du\,dv \tag{5.7}$$

where

$$f(u,v,\Lambda,k) = \frac{2\Lambda u \exp\left(2\Lambda^2 uv - 2\Lambda(\Lambda+k)v\sqrt{\tanh(u^2)}\right)}{\sqrt{\sinh(2u^2)/2}} \tag{5.8}$$

and

$$k = h - \Lambda = a_1\Lambda^{-1/3} + a_2\Lambda^{-5/3} + a_3\Lambda^{-3} + a_4\Lambda^{-13/3} + \cdots \tag{5.9}$$

Alternatively, this may be written

$$\Phi(\lambda,h) = \int_0^\infty \int_0^\infty e^{-\Lambda^2(u^2+v^2)} g(u,v,\Lambda,k)\,du\,dv \tag{5.10}$$

where

$$g(u,v,\Lambda,k) = \frac{2\Lambda u \exp\left(-2\Lambda(\Lambda+k)v\sqrt{\tanh(u^2)}\right)}{\sqrt{\sinh(2u^2)/2}}. \tag{5.11}$$

The function f is analytic in u for $|u| < (\pi/2)^{1/2}$ and analytic in v for all v. Hence (see, e.g., [**2**, p. 33, Theorem 2]) it is analytic in the two variables in the region

$$R = \{(u,v)\colon\ |u| < (\pi/2)^{1/2}\}. \tag{5.12}$$

Hence it is expandable as a double Taylor series in u and v valid in R:

$$\begin{aligned} f(u,v,\Lambda,k) &= 2\Lambda - 4\Lambda^2 kuv + 4\Lambda^3 k^2u^2v^2 - \frac{2}{3}\Lambda u^4 + 2\Lambda^2(\Lambda/3+k)u^5v \\ &\quad - \frac{8}{3}\Lambda^4k^3u^3v^3 + \frac{1}{5}\Lambda u^8 - \frac{4}{3}\Lambda^4 ku^6v^2 - \frac{8}{3}\Lambda^3k^2u^6v^2 + \frac{4}{3}\Lambda^5k^4u^4v^4 \\ &\quad - \frac{13}{30}\Lambda^3u^9v - \frac{5}{6}\Lambda^2ku^9v + \frac{20}{9}\Lambda^4k^3u^7v^3 + \frac{4}{3}\Lambda^5k^2u^7v^3 - \frac{8}{15}\Lambda^6k^5u^5v^5 + \cdots, \end{aligned} \tag{5.13}$$

where we have included all the nonzero terms involving u^iv^j for $0 \le i+j < 12$. This expansion was obtained using MAPLE. Although the expansion does not converge throughout the range of integration $\{(u,v)\colon 0 \le u < \infty,\ 0 \le v < \infty\}$ we can justify term-by-term integration to get an asymptotic sequence by the method used to prove Watson's lemma. See, e.g., [**15**, pp. 71–72]. The idea is that, in the double integrals in (5.2), (5.7) and (5.10), as $\Lambda \to \infty$, we may safely ignore the contribution from all of the range except for a neighbourhood of $(u,v) = (0,0)$. Doing this and using Lemma 1, we find that

$$\begin{aligned} \Phi(\lambda,h) &= \Lambda^{-1} - \frac{1}{3}\Lambda^{-2}k + \frac{2}{15}\Lambda^{-3}k^2 - \frac{2}{15}\Lambda^{-5} + \frac{1}{7}\Lambda^{-6}(\Lambda/3+k) - \frac{2}{35}\Lambda^{-4}k^3 \\ &\quad + \frac{4}{15}\Lambda^{-9} - \frac{4}{63}\Lambda^{-6}k - \frac{8}{63}\Lambda^{-7}k^2 + \frac{8}{315}\Lambda^{-5}k^4 + \frac{13}{55}\Lambda^{-9} - \frac{5}{11}\Lambda^{-10}k \\ &\quad + \frac{10}{99}\Lambda^{-8}k^3 + \frac{2}{33}\Lambda^{-7}k^2 - \frac{8}{693}\Lambda^{-6}k^5 + O(\Lambda^{-7}). \end{aligned} \tag{5.14}$$

Substituting (5.9) into (5.14) leads to

$$\begin{aligned} \Phi(\lambda,h) &= \Lambda^{-1} - \frac{1}{3}a_1\Lambda^{-7/3} + \left(-\frac{1}{3}a_2 + \frac{2}{15}a_1^2\right)\Lambda^{-11/3} \\ &\quad + \left(-\frac{1}{3}a_3 + \frac{4}{15}a_1a_2 - \frac{3}{35} - \frac{2}{35}a_1^3\right)\Lambda^{-5} \\ &\quad + \left(-\frac{1}{3}a_4 + \frac{4}{15}a_1a_3 + \frac{2}{15}a_2^2 + \frac{8}{315}a_1^4 + \frac{5}{63}a_1 - \frac{6}{35}a_1^2a_2\right)\Lambda^{-19/3} + O(\Lambda^{-23/3}). \end{aligned} \tag{5.15}$$

On the other hand, by differentiating (3.9), we get

$$\frac{dh}{d\lambda} = \Lambda^{-1} - \frac{1}{3}a_1\Lambda^{-7/3} - \frac{5}{3}a_2\Lambda^{-11/3} - 3a_3\Lambda^{-5} - \frac{13}{3}a_4\Lambda^{-19/3} + \cdots,\ \Lambda > M. \tag{5.16}$$

The first two coefficients in (5.16) and (5.15) agree. Comparing the coefficients of $\Lambda^{-11/3}$, Λ^{-5}, and $\Lambda^{-19/3}$ leads to

$$a_2 = -\frac{1}{10}\,a^2, \qquad a_3 = \frac{9}{280} - \frac{11}{350}\,a^3, \qquad a_4 = \frac{277}{12600}\,a - \frac{823}{63000}\,a^4, \tag{5.17}$$

so (5.1) holds. It is clear that the method can be extended to get further coefficients.

6. Numerical check and comparison with Ricci's result

As a numerical check, we note that the largest zero of the Hermite polynomial $H_{20}(x)$ is approximately $5.387480890\ldots$. Using three, four and five terms in our approximation gives the steadily improving results 5.38816, 5.387523 and 5.387483.

Ricci's four-term formula [**16**, (5.22)] gives a value of approximately 5.1917. Ricci's method is quite different from that employed here. In terms of our notation, he gets an expansion of $h_1(\lambda)$ where the powers of Λ are $1, -\frac{1}{3}, -\frac{8}{3}, -\frac{15}{3}, \ldots$. One of his claims [**16**, (5.20), etc.] is that (in our notation)

$$\frac{h_1(\lambda)}{\Lambda} = 1 + a_1\Lambda^{-4/3} + O(\Lambda^{-11/3}). \tag{6.1}$$

This would imply the boundedness of

$$\Lambda^{11/3}\left[\frac{h_1(\lambda)}{\Lambda} - 1 - a_1\Lambda^{-4/3}\right] \tag{6.2}$$

as $\Lambda \to \infty$. But, according to (1.6), this expression $\sim a_2\Lambda$ as $\Lambda \to \infty$, so Ricci's result cannot be correct.

The more general results of S. Noschese [**14**] generalize those of Ricci [**16**] and use the same techniques. However, it is difficult to compare the final results [**14**, (57),(58)] with those obtained here since they involve constants w_0, w_1 defined by integrals involving Hermite polynomials.

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