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CONTINUOUS RANKING OF ZEROS OF SPECIAL FUNCTIONS

MARTIN E. MULDOON

Abstract. We reexamine and continue the work of J. Vosmanskyý [23] on the concept of continuous ranking of zeros of certain special functions from the point of view of the transformation theory of second order linear differential equations. This leads to results on higher monotonicity of such zeros with respect to the rank and to the evaluation of some definite integrals. The applications are to Airy, Bessel and Hermite functions.

1. Introduction

The transformation theory of second order linear differential equations, including the concept of first phase, has been considered by many authors; see [2], [19]. The concept of continuous ranking of zeros of Bessel functions occurs in the work of Á. Elbert and A. Laforgia on zeros of Bessel functions starting in [8]; see [6, p. 67] for a summary and a recognition that the concept appeared already in the work of F. W. J. Olver [20, 21]. J. Vosmanský [23] made the connection between the transformation theory and continuous ranking concepts and gave applications to Bessel functions. Our purpose here is to reexamine the connection between these ideas and give further applications. We are thus able to obtain some new results on variation of zeros of special functions with respect to order; they include higher monotonicity properties of such zeros with respect to the rank and the evaluation of some definite integrals. The applications are to Airy, Bessel and Hermite functions.

2. Continuous ranking of zeros

We consider the differential equation

$$y'' + f(x)y = 0,\ a < x < b, \tag{2.1}$$

where f is continuous on (a, b), and the equation is nonoscillatory [13, p. 351] at a $(-\infty \le a < b \le \infty)$. Under these conditions, the equation has a principal solution at a [13, p. 355], i.e., a solution $y_1(x)$ such that for every solution $y(x)$ linearly independent of y_1, we have

$$\lim_{x\to a^+} y_1(x)/y(x) = 0. \tag{2.2}$$

Let y_2 be a solution of (2.1) such that the Wronskian

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \equiv 1, \tag{2.3}$$

and let

$$p(x) = y_1^2(x) + y_2^2(x). \tag{2.4}$$

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We suppose throughout that, for a fixed c, $a < c < b$,

$$\int_a^c \frac{du}{p(u)} = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^c \frac{du}{p(u)} < \infty. \tag{2.5}$$

We use the notation

$$y(x, \alpha) = \cos\alpha\, y_1(x) - \sin\alpha\, y_2(x) \tag{2.6}$$

for a general solution of (2.1). Note that $W(y_1, y_2)$ and $p(x)$ remain unchanged when y_1, y_2 are replaced by $y(x, \alpha)$, $y(x, \alpha - \pi/2)$.

The change of variables $y(x) = [p(x)]^{1/2} u(t)$, $x'(t) = p(x)$ [17, Lemma 2.3] transforms (2.1) into the trigonometric equation $u''(t) + u(t) = 0$ with general solution $u(t) = A \sin(t + B)$. Hence the general solution of (2.1) is given by

$$y(x) = A[p(x)]^{1/2} \sin\left(\int_a^x \frac{dt}{p(t)} + B\right). \tag{2.7}$$

We may redefine

$$y_1(x) = \sqrt{p(x)} \sin\left(\int_a^x \frac{du}{p(u)}\right),$$
$$y_2(x) = -\sqrt{p(x)} \cos\left(\int_a^x \frac{du}{p(u)}\right).$$

The zeros of $y_1(x)$ on (a, b) are the (finitely or infinitely many) numbers x_k for which

$$\int_a^{x_k} \frac{dt}{p(t)} = k\pi, \quad k = 1, 2, \ldots. \tag{2.8}$$

Definition 2.1. We define a function $x(\kappa)$ of the continuous variable κ by

$$\int_a^{x(\kappa)} \frac{dt}{p(t)} = \kappa\pi, \ 0 \le \kappa < K \tag{2.9}$$

where

$$x(K) = \int_a^b \frac{dt}{p(t)} \le \infty.$$

For positive integer values of κ, $x(\kappa)$ is a zero of y_1. For each nonintegral value of κ, $x(\kappa)$ is a zero of some solution of (2.1) other than y_1. In fact, for $0 < \alpha < \pi$, the solution

$$y(x, \alpha) = \cos\alpha y_1(x) - \sin\alpha y_2(x) = [p(x)]^{1/2} \sin\left(\int_a^x \frac{dt}{p(t)} + \alpha\right) \tag{2.10}$$

has its zeros where

$$\int_a^x \frac{dt}{p(t)} = (k - \alpha/\pi)\pi,$$

i.e., at the points $x(k - \alpha/\pi)$, $k = 0, 1, 2, \ldots$.

One can discuss the variation of the positive zeros of $y(x, \alpha)$ with respect to the variables α or k (the rank of a zero). However, α and k are not really independent; they may be subsumed in a single variable $\kappa = k - \alpha/\pi$. This was done by Elbert and Laforgia [8] in the case of Bessel functions. Their work is also described in [6] and [15]. To see that their approach is equivalent to Definition 1.1, we consider that the zeros $x_k(\alpha)$ of $y(x, \alpha)$, $0 < \alpha < \pi$ are the roots of the equation

$$y_2(x)/y_1(x) = \cot\alpha. \tag{2.11}$$

The graph of the left-hand side of (2.11) consists of branches which increase from $-\infty$ to $+\infty$ in (a, x_1) and in the intervals (x_k, x_{k+1}), $k = 1, 2, \dots,$ between the zeros of $y_1(x)$. This is most easily seen by using the relation

$$\frac{d}{dx}\frac{y_2(x)}{y_1(x)} = \frac{y_1(x)y_2^{'}(x) - y_2(x)y_1^{'}(x)}{y_1^2(x)} = \frac{1}{y_1^2(x)},$$

where the last equation follows from (2.3). As α decreases from π to 0, $\cot\alpha$ increases from $-\infty$ to ∞. Thus each zero of $y(x, \alpha)$ increases from one zero x_k of $y_1(x)$ to the next larger one x_{k+1}. At the same time a new smallest zero appears and increases from a to x_1. Thus it makes sense to define $x(\kappa)$ for any real $\kappa \geq 0$, by $x(0) = a$ and $x(\kappa) = x_k(\alpha)$ where $k = \lceil\kappa\rceil$ is the largest integer *less than* $\kappa + 1$ and $\alpha = \pi(k - \kappa)$. Thus $x(\kappa)$ is a continuous increasing function of κ on $[0, \infty)$. This agrees with what we found following Definition 1.1. The positive zeros of $y_1(x)$ correspond to $x(\kappa)$, $k = 1, 2, \dots$ and those of $y_2(x)$ correspond to $x(k - 1/2)$, $k = 1, 2, \dots.$

The same idea occurs in work of O. Borůvka [2, p. 34], who defines the (first) phase α of (2.1) with respect to $y_1, -y_2$ as a continuous function $\alpha \in C^0[a, b)$ given by

$$\tan(\alpha(x)) = -\frac{y_1(x)}{y_2(x)}, \tag{2.12}$$

for $y_2(x) \neq 0$. In fact, as shown in [2], $\alpha \in C^3[a, b)$, $\alpha'(x) > 0$ on $[a, b)$, $y_1(x) = [\alpha'(x)]^{-1/2}\sin\alpha(x), y_2(x) = -[\alpha'(x)]^{-1/2}\cos\alpha(x)$, and

$$p(x) = y_1^2(x) + y_2^2(x) = (\alpha'(x))^{-1}. \tag{2.13}$$

In a way equivalent to (2.9), we define the continuous function $x(\kappa)$, $\kappa > 0$ by

$$x(\kappa) = \alpha^{-1}(\pi\kappa). \tag{2.14}$$

Clearly $x(n)$ is a zero of $y_1(t)$ and $x(n - \frac{1}{2})$ is a zero of $y_2(t)$ for $n = 1, 2, \dots.$ More generally, if $\alpha = \pi(k - \kappa)$, where $k = \lceil\kappa\rceil$, then $x(\kappa)$ is the kth zero of

$$\cos\alpha\, y_1(t) - \sin\alpha\, y_2(t)$$

on (a, b).

The geometric interpretation of the first phase approach is as follows. The parametric curve $(y_1(t), y_2(t))$ has a spiral graph intersected by the straight line through the origin with slope $\cot\alpha$. It we fix α, the t-values at the points of intersection of the curve and the line are the zeros of $\cos\alpha\ y_1 - \sin\alpha\ y_2$. In particular, the parameters of the points of intersection of the curve with the axes are the zeros of y_1 and y_2. As we move along the curve, we pass through a continuum of points whose parameters are the values of $x(\kappa)$.

Our definition of continuous rank is essentially that given by J. Vosmanský [23], except that we have fewer restrictions on the interval (a, b) (it can be finite or infinite) and there is no need to restrict y_1, y_2 explicitly to be what Vosmanský calls *principal pairs*. Our title is inspired by that of [16].

3. Airy functions

The function $Ai(-x)$ [1, §10.4] is a principal solution at $-\infty$ of the Airy equation

$$y'' + xy = 0. \tag{3.1}$$

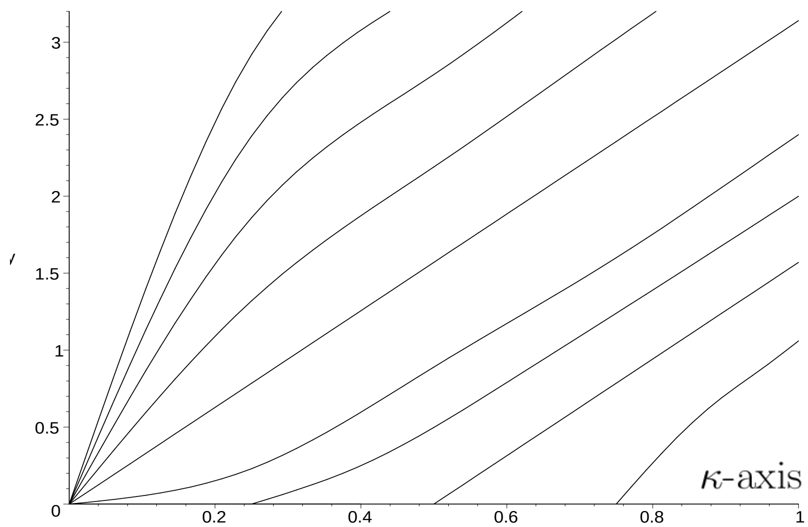

FIGURE 1. Approximate graphs of $j_{\nu\kappa}$, as a function of κ, for, from below, $\nu = -0.75,\ -0.5,\ -0.25,\ 0,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5$.

We may take $\pi^{1/2}Ai(-x),\ -\pi^{1/2}Bi(-x)$ as a pair of solutions with Wronskian 1. $Ai(-x)$ has no negative zeros and an infinite set of positive zeros, the smallest being $\alpha = 2.3381\ldots$. From (2.9), we see that

$$\int_{-\infty}^{\alpha} \frac{du}{Ai^2(-u) + Bi^2(-u)} = \pi^2. \tag{3.2}$$

We also have, using (2.13),

$$\frac{1}{\pi}\int_{-\infty}^{0} \frac{du}{Ai^2(-u) + Bi^2(-u)} = -\arctan \left.\frac{Bi(-x)}{Ai(-x)}\right|_{-\infty}^{0}. \tag{3.3}$$

which, in view of $Bi(0) = \sqrt{3}Ai(0)$ [1, p.446] and $Bi(x)/Ai(x) \to \infty$ as $x \to \infty$ [1, pp. 448–449], takes the more appealing form

$$\int_{0}^{\infty} \frac{du}{Ai^2(u) + Bi^2(u)} = \frac{\pi^2}{6}. \tag{3.4}$$

4. Bessel functions

With the usual notation for the Bessel functions, $y_1(x) = x^{1/2}J_\nu(x)$ is a principal solution at 0 and $y_2(x) = x^{1/2}Y_\nu(x)$ a linearly independent solution of the differential equation

$$y'' + \left[1 + \frac{1/4 - \nu^2}{x^2}\right] y = 0,\ 0 < x < \infty. \tag{4.1}$$

For $\nu \geq 0$, the condition (2.5) follows from the behaviour of the Bessel functions in the neighbourhood of 0. $J_\nu(x)$ has infinitely many real zeros whose behaviour with respect to ν is illustrated by a diagram in [24, p.510]. In [8] and later work of Elbert and Laforgia, the notation $j_{\nu\kappa}$ is used for what we call $x(\kappa)$ in the general situation described in §2. Figure 1 gives some graphs of the zeros $j_{\nu\kappa}$ as functions of κ; each curve represents a fixed value of ν. (These and the other figures were produced with the aid of MAPLE, but are based on a small number of values so should be regarded only as indication the general trends of the graphs.) Note that we get straight lines for $\nu = \pm\frac{1}{2}$. The figure also confirms the results of Elbert and Laforgia [9, Theorem 2.2] that $j_{\nu\kappa}$ is a convex function of κ for $0 \le \nu < \frac{1}{2}$ and a concave function of κ for $\nu > \frac{1}{2}$. Note that $j(-\frac{1}{2}, \kappa) = (\kappa - \frac{1}{2})\pi$ and that the values of $j_{\nu\kappa}$ for $\nu < -\frac{1}{2}$ are simply related to those for $\nu > \frac{1}{2}$ by the formula ([5], [6, p. 67]

$$j_{-\nu,\nu+\kappa} = j_{\nu\kappa},\ \nu \ge 0,\ \kappa > 0. \tag{4.2}$$

In Figure 1, we do not go beyond $\kappa = 1$. For larger values of κ, the curves become increasingly indistinguishable from straight lines. It may be more insightful to place the graphs on the surface of a circular cylinder as in Figure 2. The zeros of a particular solution are determined as the intersections of a generator of the cylinder with the graph. An alternative two-dimensional presentation is given in Figure 3. Graphing the zeros against $\cos\alpha = \cos(k - \kappa)\pi$ rather than against κ or α is suggested by [22, Figure 1]. This gives the picture a three-dimensional appearance as does the alternation in thickness of the graph segments on the front and back of the cylinder; cf. the opening paragraphs in [3].

The graphs in Figure 1 complements those in [24, p. 510] where $\pm j_{\nu k}$ is graphed against ν for $k = 1, 2, 3, 4$, and in [7] where $j_{\nu\kappa}$ is graphed against ν for various values of κ between 0 and 1. It is also worth mentioning that Olver [20] considered what amounts to $j_{\nu\kappa}$ for continuous κ while in [21], he considered it for continuous ν, both situations arising from a need to evaluate the zeros numerically.

5. Derivatives with respect to κ

Differentiating (2.9) with respect to κ gives

$$x'(\kappa) = \pi p[x(\kappa)],\quad 0 < \kappa < K. \tag{5.1}$$

When applied to solutions of equation (4.1) this gives a result mentioned by Elbert [6, (1.4)]

$$\frac{dj_{\nu\kappa}}{d\kappa} = dj/d\kappa = (\pi^2/2) j_{\nu\kappa}[J_\nu^2(j_{\nu\kappa}) + Y_\nu^2(j_{\nu\kappa})],\quad \kappa > 0. \tag{5.2}$$

From this it is clear that $j_{\nu\kappa}$ is an infinitely differentiable function of κ, $\kappa > 0$.

In the general case, we may differentiate the equation (5.1) repeatedly with respect to κ to get expressions for the higher derivatives of $x(\kappa)$ with respect to κ; see [23]. Thus, in case $p(x)$ has higher monotonicity properties (sign regularity of its higher derivatives) we get corresponding properties of $x'(\kappa)$:

Theorem 5.1. *Let y_1, y_2 be the solutions of (2.1) introduced in §2 and let p, as given by (2.4), satisfy*

$$(-1)^n p^{(n)}(x) \ge 0,\ a < x < b,\ n = 0, \ldots, N. \tag{5.3}$$

Then, with $x(\kappa)$ defined by (2.9) or by (2.14), we have

$$(-1)^n x^{(n+1)}(\kappa) \ge 0,\ a < \kappa < b,\ n = 0, \ldots, N. \tag{5.4}$$

In case (5.3) is replaced by

$$p(x) > 0,\ p'(x) < 0,\ (-1)^n p^{(n)}(x) \ge 0,\ n = 2, \ldots, N,\ a < x < b, \tag{5.5}$$

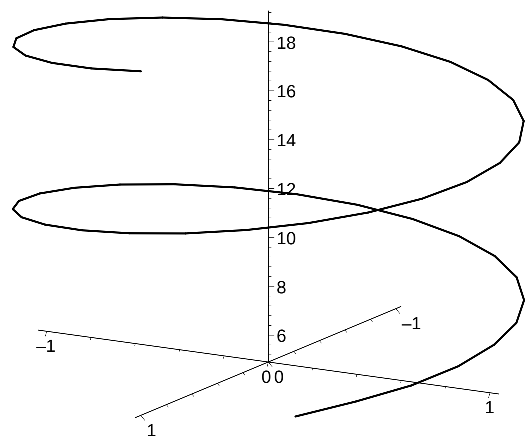

FIGURE 2. Approximate graph of the curve
$x = \cos\kappa,\quad y = \sin\kappa,\quad z = j_{5\kappa},\ 0.2 \le \kappa \le 4.$

then we get strict inequality in (5.4). The results remains true when the factors $(-1)^n$, $(-1)^{n+1}$ *are simultaneously removed from (5.3), (5.4) and (5.5).*

This result is contained essentially in [23, §3] and the proof is essentially the same as that of [17, Lemma 2.1]. It is based on the successive formulas $x'(\kappa) = p(x)$, $x''(\kappa) = p'(x)x'(\kappa)$, $x'''(\kappa) = p''(x)[x'(\kappa)]^2 + p'(x)x''(\kappa), \ldots$. By induction, one can show that, for each n, $(-1)^n x^{(n+1)}(\kappa)$ is a sum of non-negative terms. In the case of the stronger hypotheses (5.5), one can show, as in [18] that one of these terms is positive.

Theorem 5.1 generalizes the case of Bessel functions ($\nu > 1/2$) considered in [12, Corollary 3.3]. It extends to derivatives with respect to κ the special case $\lambda = 0$ of [17, Theorem 2.1], which dealt with finite differences $\Delta^n x_k$.

6. Hermite functions

The equation

$$y'' + (2\lambda + 1 - x^2)y = 0, \tag{6.1}$$

has linearly independent solutions

$$y_1(x) = \left[\frac{\pi^{1/2}}{2^{\lambda+1}\Gamma(\lambda+1)}\right]^{1/2} e^{-x^2/2} H_\lambda(x) = \cos\frac{\lambda\pi}{2}\, u(x,\lambda) + \sin\frac{\lambda\pi}{2}\, v(x,\lambda), \tag{6.2}$$

(6.3)

$$y_2(x) = \left[\frac{\pi^{1/2}}{2^{\lambda+1}\Gamma(\lambda+1)}\right]^{1/2} e^{-x^2/2} G_\lambda(x) = -\sin\frac{\lambda\pi}{2}\, u(x,\lambda) + \cos\frac{\lambda\pi}{2}\, v(x,\lambda),$$

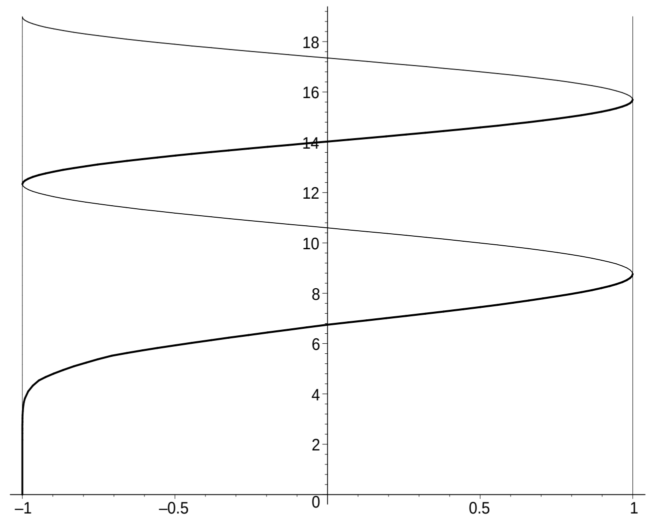

FIGURE 3. Approximate graph of the curve
$x = \cos\pi(1-\kappa), \quad y = j_{5\kappa}, \ 0 \le \kappa \le 4.$

where

$$\begin{aligned} u(x,\lambda) &= \phi(\lambda)e^{-x^2/2}{}_1F_1(-\tfrac{1}{2}\lambda, \tfrac{1}{2}, x^2); \\ v(x,\lambda) &= \tfrac{1}{\phi(\lambda)}e^{-x^2/2}x\ {}_1F_1(-\tfrac{1}{2}\lambda+\tfrac{1}{2}, \tfrac{3}{2}, x^2); \end{aligned} \tag{6.4}$$

and

$$\phi(\lambda) = \left[\frac{\frac{1}{2}\Gamma(\frac{1}{2}(\lambda+1))}{\Gamma(\frac{1}{2}(\lambda+2))}\right]^{1/2}. \tag{6.5}$$

This choice of $\phi(\lambda)$ ensures that $W(y_1, y_2) = 1$, in view of the formula

$$W(H_\lambda, G_\lambda) = \frac{2^{\lambda+1}}{\sqrt{\pi}}\Gamma(\lambda+1)e^{x^2}, \tag{6.6}$$

a slight correction of [4, (50)]. $H_\lambda(x)$, considered in [4], [14] and [10], is a generalization to real λ of the Hermite polynomial to which it reduces in case $\lambda = 0, 1, 2, \ldots$. The solution $y_1(x)$ is a principal solution of (6.1) at $+\infty$; this follows from the asymptotic expansion for ${}_1F_1(a, c; z)$ [11, p. 278] leading to the following asymptotic formulas for $H_\lambda(t)$ and $G_\lambda(t)$:

$$H_\lambda(t) \sim (2t)^\lambda, \quad t \to +\infty, \tag{6.7}$$

$$G_\lambda(t) \sim \frac{1}{\sqrt{\pi}}\Gamma(\lambda+1)t^{-\lambda-1}e^{t^2}, \quad t \to +\infty. \tag{6.8}$$

The nonoscillatory end-point a is now at $+\infty$ and we would need to replace x by $-x$ to satisfy the requirements of §2. Equation (6.1) may be transformed into the trigonometric equation $d^2u/dt^2 + u = 0$ by the transformation

$$t = \int_x^\infty \frac{du}{p_\lambda(u)}, \quad u(t) = \sqrt{p_\lambda(x)}y(x),$$

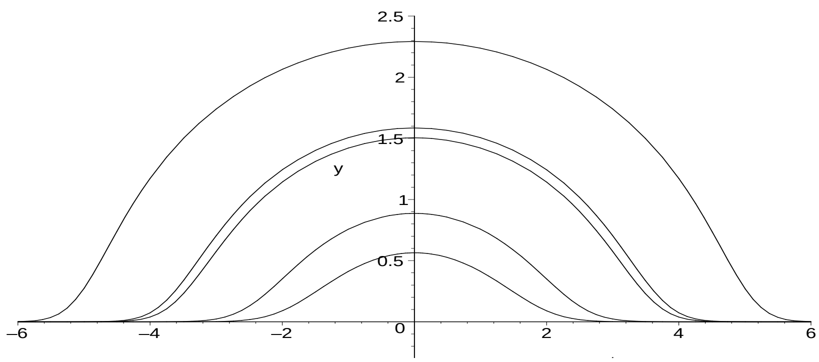

FIGURE 4. Approximate graphs of $1/p_\lambda(x)$, as a function of x, for (from below) $\lambda = 0, 1, 4, 4.5, 10$.

where

$$p_\lambda(x) = \frac{2^{-\lambda-1}\sqrt{\pi}}{\Gamma(\lambda+1)} e^{-x^2}[H_\lambda^2(x) + G_\lambda^2(x)]. \tag{6.9}$$

In particular, we have

$$H_\lambda(x) = \text{const.}\sqrt{p_\lambda(x)}\sin\int_x^\infty \frac{du}{p_\lambda(u)} \tag{6.10}$$

so the zeros of $H_\lambda(x)$ in *decreasing* order are $h_k(\lambda)$, where

$$\int_{h_k(\lambda)}^\infty \frac{du}{p_\lambda(u)} = k\pi, \quad k = 1, 2, \ldots, \lceil\lambda\rceil. \tag{6.11}$$

(Recall [10, Theorem 3.1] that $H_\lambda(t)$ has $\lceil\lambda\rceil$ real zeros, for $\lambda > -1$.)

It was shown by Durand [4] that

$$p_\lambda(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-(2\lambda+1)\tau + x^2\tanh\tau}\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}}, \ \lambda > -1. \tag{6.12}$$

(The condition $\lambda > -1$ is not mentioned in [4], but it is necessary for the convergence of the integral.) This makes it clear that $1/p_\lambda(x)$ is an increasing function of λ for each fixed x and an even positive function of x for each fixed λ. The representation (6.12) also shows that

$$\frac{d^k}{dx^k}p_\lambda(x) > 0, \ 0 < x < \infty, \quad k = 1, 2, \ldots \tag{6.13}$$

and

$$(-1)^k\frac{d^k}{dx^k}p_\lambda(x) > 0, \ -\infty < x < 0, \quad k = 1, 2, \ldots. \tag{6.14}$$

The asymptotic formulas for H_λ and G_λ show that $1/p_\lambda(x)$ approaches 0 very quickly (like $x^{2\lambda+2}e^{-x^2}$) as $x \to \pm\infty$. Figure 4 gives the graphs of $1/p_\lambda(x)$ for $\lambda = 0, 1, 4, 4.5, 10$.

As in the case of Bessel functions, the zeros $h(\lambda, k, \alpha)$ of a solution of

$$\cos\alpha H_\lambda(x) - \sin\alpha G_\lambda(x), \quad 0 \le \alpha < \pi,$$

appear to depend on three variables, λ, α and the *rank* k (first, second, etc.) of a zero. But in fact α and k can be subsumed in a single variable κ if we define $h(\lambda,\kappa)$ by

$$\int_{h(\lambda,\kappa)}^{\infty} \frac{du}{p_\lambda(u)} = \kappa\pi. \tag{6.15}$$

When $\kappa = k$, a positive integer, we get the zeros, in decreasing order of H_λ.

Since

$$\frac{d}{dx}\arctan\frac{y_2(x)}{y_1(x)} = \frac{1}{p_\lambda(x)},$$

we have

$$\int_0^{\infty} \frac{du}{p_\lambda(u)} = \arctan\frac{G_\lambda(x)}{H_\lambda(x)}\Big|_0^{\infty} = \frac{(\lambda+1)\pi}{2}, \tag{6.16}$$

leading to

$$\int_0^{\infty} \frac{e^{u^2}du}{H_\lambda^2(u)+G_\lambda^2(u)} = \frac{(\lambda+1)\pi^{3/2}}{2^{\lambda+2}\Gamma(\lambda+1)},\ \ \lambda > -1. \tag{6.17}$$

The general form of the graph of $h(\lambda,\kappa)$ as a function of λ, for fixed κ can be seen from [10, Figure 1], which covers the case when κ is a small positive integer. The graphs are similar for real $\kappa > 0$. Comparing equation (6.16) with the definition (6.15) we see that $h(\lambda,\kappa) = 0$ when $\kappa = (\lambda+1)/2$, or

$$h(2\kappa-1,\kappa) = 0,\ \kappa > 0. \tag{6.18}$$

From (6.15), we have

$$\int_{h(\lambda,2\kappa)}^{\infty} \frac{du}{p_\lambda(u)} = 2\int_{h(\lambda,\kappa)}^{\infty} \frac{du}{p_\lambda(u)}, \tag{6.19}$$

Also as $\lambda \to 2\kappa-1$, we have $h(\lambda,\kappa) \to 0$ and since the integrand in (6.19) is an even function of u, we find that $h(\lambda,2\kappa) \to -\infty$. This can be expressed as

$$\lim_{\lambda\to\kappa-1} h(\lambda,\kappa) = -\infty,\ \kappa > 0. \tag{6.20}$$

Thus for fixed $\kappa > 0$, $h(\lambda,\kappa)$ increases from $-\infty$ to ∞ as λ increases from $\kappa-1$ to ∞ taking the value 0 when $\lambda = 2\kappa-1$.

It is also of interest to record the form of the graphs of $h(\lambda,\kappa)$ as a function of κ for fixed λ. From (6.18) and (6.20), we get $h(\lambda,(\lambda+1)/2)) = 0$ and $h(\lambda,\kappa) \to -\infty$ as $\kappa \to (\lambda+1)^-$. For each fixed $\lambda > -1$, $h(\lambda,\kappa)$ decreases from $+\infty$ to $-\infty$ as κ increases from 0 to $\lambda+1$.

The statement of [10, Corollary 7.2] (where it was framed in terms of zeros of fixed rank of linear combinations of H_λ and G_λ), can be stated more succinctly as follows:

Theorem 6.1. *For each fixed positive* κ*,*

$$(-1)^{r+1}\frac{d^r}{d\lambda^r}h(\lambda,\kappa) > 0,\ r = 1,2,\ldots,\ \lambda > \kappa-1. \tag{6.21}$$

Similar remarks may be made concerning $h(\lambda,\kappa)$ as a function of κ. From (6.18) and (6.20), we see that, for each fixed $\lambda > -1$, $h(\lambda,\kappa)$ decreases from ∞ to $-\infty$ as κ increases from 0 to $\lambda+1$ taking the value 0 when $\kappa = (\lambda+1)/2$. We also have higher monotonicity properties with respect to κ.

Theorem 6.2. *If we define $h(\lambda, \kappa)$ by (6.15), then*

$$(-1)^r \frac{d^r}{d\kappa^r} h(\lambda, \kappa) > 0, \quad 0 < \kappa < \frac{\lambda+1}{2}, \ r = 0, 1, 2, \ldots, \tag{6.22}$$

and

$$\frac{d^r}{d\kappa^r} h(\lambda, \kappa) < 0, \quad \frac{\lambda+1}{2} < \kappa < \lambda + 1, \ r = 0, 1, 2, \ldots, \tag{6.23}$$

The Theorem is proved in the same way as Theorem (5.1), on taking account of (6.13) and (6.14).

The first part of Theorem 6.1 is a continuous analogue of Durand's result [4, pp. 371–372] that for the positive zeros of $H_n(x)$, with fixed n,

$$(-1)^r \Delta^r_{(k)} x_{nk} > 0, \ r = 1, 2, \ldots \tag{6.24}$$

(It should be noted that Durand lists the zeros in increasing order so the $(-1)^r$ does not appear in his result.)

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Department of Mathematics & Statistics, York University, Toronto, Ontario M3J 1P3, Canada

E-mail address: muldoon@yorku.ca