Nicholson-type Formulas for Special Functions

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International Conference on Asymptotics and Special Functions
City University of Hong Kong

May 30 - June 3, 2011

Nicholson's Formula

Bessel functions $J_\nu(z)$, $Y_\nu(z)$. J. W. Nicholson [20] (1910):

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t\,dt, \quad \mathrm{Re}\,\mathrm{z} > 0, \quad (1)$$

where

$$K_0(x) = \int_0^\infty e^{-\mathrm{x}\cosh t}\,dt.$$

G. N. Watson [23] (1922):

$$J_\nu(z)\frac{\partial Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial\nu} = -\frac{4}{\pi}\int_0^\infty K_0(2z\sinh t)e^{-2\nu t}\,dt \qquad (2)$$

Both formulas are important in the study of zeros of Bessel functions.

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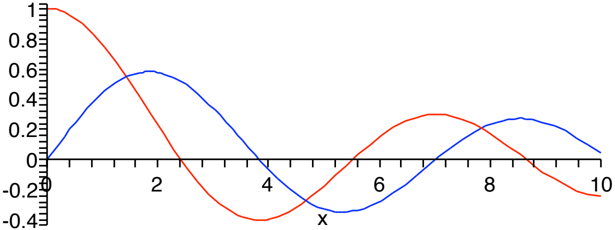
Both formulas are important in the study of zeros of Bessel functions.

Zeros of Bessel functions

For $\nu > -1$, $J_\nu(z)$ has infinitely many positive zeros $0 < j_{\nu 1} < j_{\nu 2} < j_{\nu 3} < \cdots$.



Graphs of $J_0(x), J_1(x)$.

How do the zeros of $J_\nu(x)$ vary with ν?

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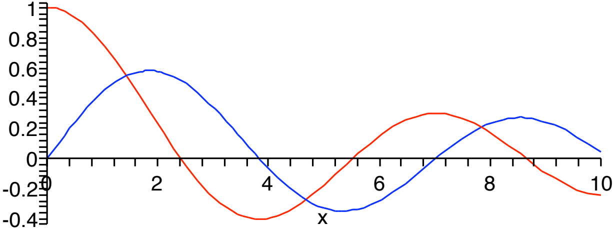

Graphs of $J_0(x), J_1(x)$.
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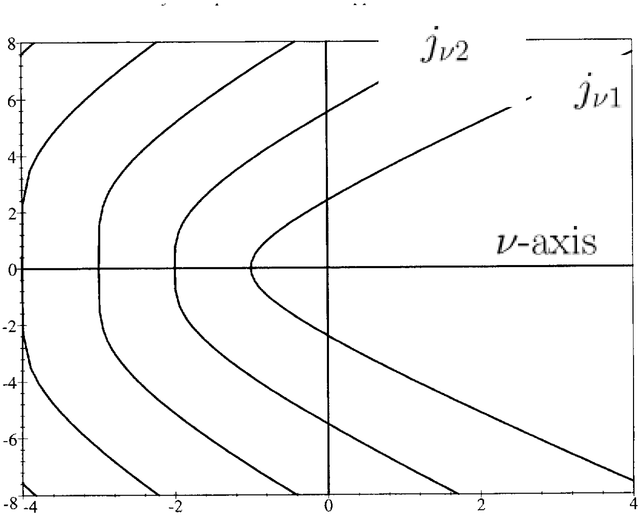
Zeros as functions of ν

Watson (1922)



Zeros of cylinder functions

Watson (1922). If $c = c(\nu, \alpha)$ is a zero of $\cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x)$, then

$$\frac{dc}{d\nu} = 2c\int_0^\infty K_0(2c\sinh t)e^{-2\nu t}dt. \tag{3}$$

Á. Elbert [8](1977) used this formula to show that $j_{\nu k}$ is a concave increasing function of ν on $-k < \nu < \infty$.
Á. Elbert and A. Laforgia use this formula very effectively during the 1980s and 1990s to get inequalities and other properties for the zeros of Bessel functions. See [9] for references. Compare with Nicholson's formula:

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t dt, \quad \mathrm{Re}\, z > 0.$$

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Deriving (3) from (2)

Let $c = c(\nu, \alpha)$ be a zero of $\cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x)$. Then c is a function of ν such that $\arctan[Y_\nu(c)/J_\nu(c)]$ is constant. Differentiation with respect to ν, gives

$$\frac{2}{\pi c}\frac{dc}{d\nu} + \left[J_\nu(z)\frac{\partial Y_\nu(z)}{\partial \nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial \nu}\right]_{z=c} = 0$$

so, using (2),

$$\frac{dc}{d\nu} = 2c\int_0^\infty K_0(2c\,\mathrm{sinh}\,t)e^{-2\nu\, t}\,dt. \tag{3}$$

Nicholson-type formulas

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t dt, \quad \mathrm{Re}\, z > 0. \quad (1)$$

$$J_\nu(z)\frac{\partial Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial\nu} =$$

$$; \qquad -\frac{4}{\pi}\int_0^\infty K_0(2z\sinh t)e^{-2\nu t}dt. \qquad (2)$$

$$\frac{dc}{d\nu} = 2c\int_0^\infty K_0(2c\,\mathrm{sinh}t)e^{-2\nu\, t}\, dt. \qquad (3)$$

Methods of proof

Watson: "[formula (1)] is difficult to establish rigorously".
J. E. Wilkins, Jr. (1948) [20] gave a differential equations proof of (1). With $z = e^{\theta}$ both sides satisfy

$$w''' + 4[e^{2\theta} - \nu^2]w' + 4e^{2\theta}w = 0$$

and the initial values can be checked. A differential equations proof of (2) (and hence (3)) was given by MEM [17](1981). Both sides of (2) satisfy the nonhomogeneous d.e.

$$w''' + 4[e^{2\theta} - \nu^2]w' + 4e^{2\theta}w = -8\nu/\pi.$$

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Dixon and Ferrar [5, 1930]

$$J_\mu(z)J_\nu(z) + Y_\mu(z)Y_\nu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2z\sinh t)[e^{(\mu+\nu)t} + e^{-(\mu-\nu)t}\cos(\mu-\nu)\pi]dt$$

- reduces to Nicholson's formula (1) when $\mu = \nu$.

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Dixon and Ferrar [5, 1930] (cont.)

$$J_\mu(z)Y_\nu(z) - J_\nu(z)Y_\mu(z) = \frac{4}{\pi^2}\sin(\mu-\nu)\pi\int_0^\infty K_{\nu-\mu}(2z\sinh t)\, e^{-(\mu+\nu)t}\,dt$$

This reduces to

$$J_\nu(z)\frac{\partial Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial\nu} = -\frac{4}{\pi}\int_0^\infty K_0(2z\sinh t)e^{-2\nu t}dt \quad (2)$$

on dividing by $\mu - \nu$ and letting $\mu \to \nu$.
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Other Nicholson-type formulas for Bessel functions

P. Hartman [14, 1973]

$$\frac{\pi}{2}[J_\nu^2(x) + Y_\nu^2(x)] = \int_0^\infty e^{-sx} P_{\nu-1/2}(1 + s^2/2)ds, \quad x > 0.$$

Müller and Richberg [19] (1980).

Beyond Bessel

L. Durand [6, 1975]; [7, 1978]: ultraspherical, Laguerre, Jacobi.
Legendre functions:

$$P_n^2(x)+\frac{4}{\pi^2}Q_n^2(x)=\frac{4}{\pi^2}\int_0^\infty Q_n(x^2+(1-x^2)z)\frac{dz}{(z^2-1)^{1/2}}$$

Hermite functions: For $\lambda>-1$,

$$e^{-x^2}\left[H_\lambda^2(x)+G_\lambda^2(x)\right]=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-(2\lambda+1)\tau+x^2\tanh\tau}\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}}.$$

$$=\frac{1}{\sqrt{\pi}}\int_0^1 e^{x^2u}\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du,\qquad \lambda>-1, \tag{1'}$$

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Hermite functions: For $\lambda > -1$,

$$e^{-x^2} \left[H_\lambda^2(x) + G_\lambda^2(x) \right] = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(2\lambda+1)\tau + x^2 \tanh \tau} \frac{d\tau}{\sqrt{\sinh \tau \cosh \tau}}.$$

$$= \frac{1}{\sqrt{\pi}} \int_0^1 e^{x^2 u} \frac{(1 - u)^\lambda}{u^{1/2}(1 + u)^{\lambda+1}} du, \qquad \lambda > -1, \qquad (1')$$

Elbert, MEM [11, 1999]: (2') and (3')

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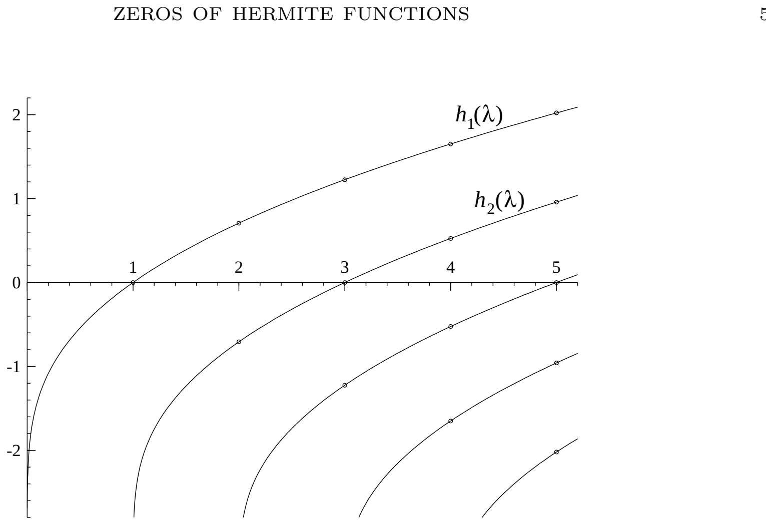
$$e^{-x^2}\left[H_\lambda(x)\frac{\partial G_\lambda(x)}{\partial\lambda} - G_\lambda(x)\frac{\partial H_\lambda(x)}{\partial\lambda}\right] =$$

$$\frac{\sqrt{\pi}}{2}\int_0^1 e^{x^2 u}\mathrm{erfc}(xu^{1/2})\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du, \qquad (2')$$

If $h(\lambda)$ a zero of a linear combination of H_λ and G_λ, then,

$$\frac{dh}{d\lambda} = \frac{\sqrt{\pi}}{2}\int_0^1 e^{x^2 u}\mathrm{erfc}(xu^{1/2})\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du. \qquad (3')$$

Hermite zeros as functions of λ



ZEROS OF HERMITE FUNCTIONS 5

Hermite functions (cont.)

The formula (3') for $dh/d\lambda$ was found useful [12] in finding an asymptotic expansion for the zeros (as $\lambda \to +\infty$). he first five terms are given by [12]

$$h(k,\alpha) = \Lambda - a\Lambda^{-1/3} - \frac{1}{10}a^2\Lambda^{-5/3} + \left[\frac{9}{280} - \frac{11}{350}a^3\right]\Lambda^{-3} + \left[\frac{277}{12600}a - \frac{823}{63000}a^4\right]\Lambda^{-13/3} + \cdots,$$

where $\Lambda = \sqrt{2\lambda+1}$ and a is a zero of an Airy function.

Bateman's method: [2, 1909], [15, Ch. 8]

We seek a solution of the form

$$w(x) = \int_{\alpha}^{\beta} k(x,t) v(t) dt \tag{5}$$

of the linear ode $L_x w = 0$. The method is to find a linear differential operator $M_t = \sum_{k=0}^{m} m_k(t) D_t^k$ and a function $\kappa(x,t)$ such that

$$L_x k(x,t) = M_t \kappa(x,t).$$

Then we determine $v(t)$ as a solution of $\overline{M}_t v = 0$. α and β are chosen so that (with $' = d/dt$)

$$\left[\sum_{k=1}^{m} \sum_{\ell=0}^{m-1} (-1)^{\ell} (m_k v)^{(\ell)} \kappa^{(k-\ell-1)} \right]_{\alpha}^{\beta} = 0.$$

Application to products of Hermite functions

The differential equation

$$y'' - 2ty' + 2\lambda y = 0.$$

has a solution

$$H_\lambda(t) = -\frac{\sin \pi\lambda \ \ \Gamma(1+\lambda)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left((n-\lambda)/2\right)}{\Gamma(n+1)} (-2t)^n$$

which reduces to the Hermite polynomials for $\lambda = 0, 1, 2, \ldots$. In terms of confluent hypergeometric functions,

$$H_\lambda(t) = \frac{2^\lambda}{\sqrt{\pi}} \left[\cos \frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2} + \frac{1}{2}) {}_1F_1(-\frac{\lambda}{2}, \frac{1}{2}; t^2) \right.$$

$$\left. + 2t \sin \frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2} + 1) {}_1F_1(-\frac{\lambda}{2} + \frac{1}{2}, \frac{3}{2}; t^2) \right].$$

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Finding integral representations

A product w of solutions of

$$y'' + (2\lambda + 1 - t^2)y = 0,$$

and

$$y'' + (2\mu + 1 - t^2)y = 0,$$

satisfies

$$\begin{aligned} L_x w := \frac{d^4 w}{dx^4} + 4(\lambda + \mu + 1 - x^2)\frac{d^2 w}{dx^2} \\ -12x\frac{dw}{dx} + 4[(\lambda - \mu)^2 - 1]w = 0. \end{aligned}$$

Finding integral representations (cont.)

In the present case, we find that

$$L_x f(x^2 t) = M_t f(x^2 t)$$

where

$$M_t y = 16(t^4 - t^2)\frac{d^2 y}{dt^2} + 8(6t^3 + 2(\lambda + \mu + 1)t^2 - 4t)\frac{dy}{dt}$$

$$+(12t^2 + 8(\lambda + \mu + 1)t + 4(\lambda - \mu)^2 - 4)y,$$

where $f(x)$ is any solution of

$$f'' - 2xf' - 2f = 0,$$

i.e., $f(x)$ is a linear combination of e^{x^2} and $e^{x^2}\mathrm{erf}(x)$.

Elbert, MEM [11, 1999]: (2') and (3')

For $\lambda > -1$,

$$e^{-x^2}\left[H_\lambda^2(x) + G_\lambda^2(x)\right] = \frac{1}{\sqrt{\pi}}\int_0^1 e^{x^2 u}\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du, \qquad (1')$$

$$e^{-x^2}\left[H_\lambda(x)\frac{\partial G_\lambda(x)}{\partial\lambda} - G_\lambda(x)\frac{\partial H_\lambda(x)}{\partial\lambda}\right] =$$

$$\frac{\sqrt{\pi}}{2}\int_0^1 e^{x^2 u}\mathrm{erfc}(xu^{1/2})\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du, \qquad (2')$$

If $h(\lambda)$ a zero of a linear combination of H_λ and G_λ, then,

$$\frac{dh}{d\lambda} = \frac{\sqrt{\pi}}{2}\int_0^1 e^{x^2 u}\mathrm{erfc}(xu^{1/2})\frac{(1-u)^\lambda}{u^{1/2}(1+u)^{\lambda+1}}du. \qquad (3')$$

Finding integral representations (cont.)

The expression for M_t leads to

$$\overline{M_t}w = 16(t^4 - t^2)\frac{d^2w}{dt^2} + 8(10t^3 - 2(\lambda+\mu+1)t^2 - 4t)\frac{dw}{dt} +$$

$$[60t^2 - 24(\lambda+\mu+1)t + 4(\lambda-\mu)^2 - 4]w.$$

This equation has linearly independent solutions

$$w_1(t) = F(t,\lambda,\mu) = t^{-(\lambda-\mu+1)/2}(1-t)^{(\lambda-\mu-2)/2}$$

$$\times {}_2F_1\left(\begin{matrix}\mu+1,\ 1-\lambda/2+\mu/2\\ 1-\lambda+\mu\end{matrix}\ \Big|\ \frac{2t}{t-1}\right),$$

Finding integral representations (cont.)

and

$$w_2(t) = F(t,\mu,\lambda) = t^{-(\mu-\lambda+1)/2}(1-t)^{(\mu-\lambda-2)/2}$$

$${}_2F_1\left(\begin{matrix}\lambda+1,\ 1-\mu/2+\lambda/2 \\ 1-\mu+\lambda\end{matrix}\ \middle|\ \frac{2t}{t-1}\right).$$

w_2 is obtained from w_1 by simply interchanging λ and μ. MAPLE was found useful here. Note that

$$F(t,\lambda,\lambda) = t^{-1/2}(1-t)^{-1}{}_2F_1\left[\lambda+1,1;1;\frac{2t}{t-1}\right] = \frac{(1-t)^{\lambda}}{t^{1/2}(1+t)^{\lambda+1}}$$

Finding integral representations (cont.)

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Finding integral representations (cont.)

Thus

$$e^{-x^2}[H_\lambda(x)H_\mu(x)+G_\lambda(x)G_\mu(x)]$$

and

$$e^{-x^2}[H_\lambda(x)G_\mu(x)-G_\lambda(x)H_\mu(x)]$$

are both of the form

$$\int_0^1 e^{x^2u}[c_1+c_2\mathrm{erfc}(xu^{1/2})][c_3F(u,\lambda,\mu)+c_4F(u,\mu,\lambda)]du,$$

where the constants remain to be determined.

Using symmetry

Consider the linear combinations H_λ^*, G_λ^* given by

$$H_\lambda^*(t) = {}_1F_1(-\frac{\lambda}{2}, \frac{1}{2}; t^2), \quad G_\lambda^*(t) = t\ {}_1F_1(-\frac{\lambda}{2} + \frac{1}{2}, \frac{3}{2}; t^2)$$

H_λ^* is even, G_λ^* odd in x. Since the left-hand sides of the following equation is even in x and symmetric in λ and μ, we are led to

$$\begin{aligned} &e^{-x^2}[H_\lambda^*(x)H_\mu^*(x) + G_\lambda^*(x)G_\mu^*(x)] \\ &\qquad = \int_0^1 e^{x^2u}[c(\lambda,\mu)F(u,\lambda,\mu) + c(\mu,\lambda)F(u,\mu,\lambda)]du, \end{aligned}$$

where $c(\lambda, \mu)$ is to be determined

Using symmetry (cont.)

Similarly

$$
\begin{aligned}
&e^{-x^2}[H_\lambda^*(x)G_\mu^*(x) - G_\lambda^*(x)H_\mu^*(x)] \\
&\qquad = \int_0^1 e^{x^2 u}\mathrm{erfc}(xu^{1/2})[k(\lambda,\mu)F(u,\lambda,\mu) - k(\mu,\lambda)F(u,\mu,\lambda)]du,
\end{aligned}
$$

where $k(\lambda, \mu)$ is to be determined.

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