On the zeros of a transcendental function

Mark V. DeFazio and Martin E. Muldoon

York University, Toronto, Canada
`muldoon@yorku.ca`

Summary. We examine the zeros of the function $g(z) = z + \beta e^{-z}$ where $\beta \in \mathbf{C}$. In particular, we discuss the movement of the zeros as β moves over straight lines through the origin and find that all the zeros of $g(z)$ are in the left half–plane if and only if $|\beta| + |\arg\beta| < \pi/2$.

1 Introduction: A transcendental function.

In this paper we study the location in the complex plane of the zeros of the function

$$g(z) = z + \beta e^{-z}. \tag{1}$$

According to Liouville's theorem [3], the zeros are not elementary transcendental functions of β. When $\beta = 0$, there is a single zero at the origin, while for $\beta = \infty$, $g(z)$ has no zeros in the finite part of the plane. For every other complex β, $g(z)$ has a countably infinite number of zeros. So it is of interest to examine how the zeros change with changes in the parameter β. Here we study their behaviour as the parameter β varies over straight lines through the origin in the complex plane. It is convenient to distinguish four cases according as to whether the line of variation of β is the real axis, or the imaginary axis or lies in the odd or even quadrants. The behaviour of the zeros when β is real is qualitatively different from the other three cases. We use our results to show that the zeros of $g(z)$ have negative real part if and only if $|\beta| + |\arg\beta| < \pi/2$.

The interest in the kind of question discussed here arises from the stability theory of delay differential equations where the location of the complex zeros of a transcendental function becomes important; see, e.g., [1], [2], [4]. In particular, stability requires that the zeros be located in the left half plane, $\mathrm{Re}(z) < 0$.

This paper goes beyond the scope of the talk given by the first-named author at ISAAC-4. That talk dealt mainly with the case where β is real. It was a question from Ilpo Laine at the end of the talk that led us to consider the zeros of $g(z)$ for β complex as well.

2 The zeros of $z + \beta e^{-z}$.

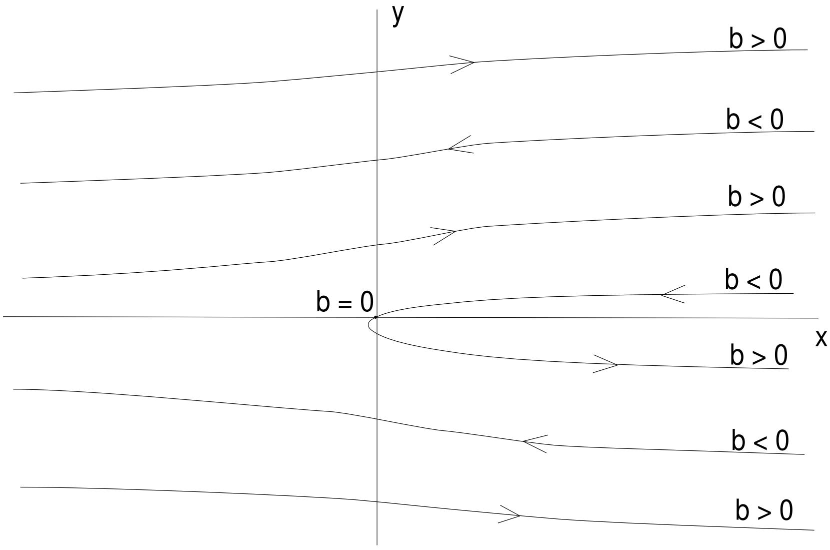

Fig. 1. Part of the graph of $x = -y \cot(y-1)$ (not drawn to scale), with horizontal asymptotes $y = n\pi + 1$; in case $\alpha = 1$, the zeros of $g(z)$ lie on this graph. The alternate transverse branches describe the paths of zeros as b increases from $-\infty$ to 0, and from 0 to infinity. The convex branch is special; one of the zeros moves along it from $1 + i\infty$ to $1 - \pi + i\infty$ as b increases from $-\infty$ to ∞.

Let us put polar coordinates on the parameter space by writing $\beta = be^{i\alpha}$ with $-\infty < b < +\infty$ and $0 \leq \alpha < \pi$. Thus for a fixed α, allowing b to vary from $-\infty$ to $+\infty$ causes β to trace out the line at angle $\alpha > 0$ to the positive real axis, starting from the lower half β–plane, and passing through the origin when $b = 0$. When $\alpha = 0$, β is real and it traverses the real axis in the usual sense. The imaginary axis is traced out when $\alpha = \pi/2$ and β traces out a line in the odd and even quadrants when $0 < \alpha < \pi/2$ and $\pi/2 < \alpha < \pi$, respectively.

We first note that all the zeros of $g(z)$ are simple except for a zero of order two at $z = -1$ in the special case $\beta = 1/e$. We also note that $g(z)$ has real zeros only if β is real. Now let $z = x + iy$ and note that

$$g(z) = x + be^{-x}\cos(y-\alpha) + i\left(y - be^{-x}\sin(y-\alpha)\right). \tag{2}$$

Let us consider two special cases: $\alpha = 0,\ \pi/2$. In case $\alpha = 0$, we have $g(\overline{z}) = \overline{g(z)}$ and the zeros of $g(z)$ occur in complex conjugate pairs. In case $\alpha = \pi/2$, we have $\beta = ib$ for $b \in \mathbf{R}$ and $g(z) = x + be^{-x}\sin y + i\,(y + be^{-x}\cos y)$. Thus if z is a zero of $g(z)$ with $\beta = -ib$ then $\overline{z}$ is a zero of $g(z)$ with $\beta = ib$. In both cases, the locus of the zeros of $g(z)$ is symmetric with respect to the real axis.

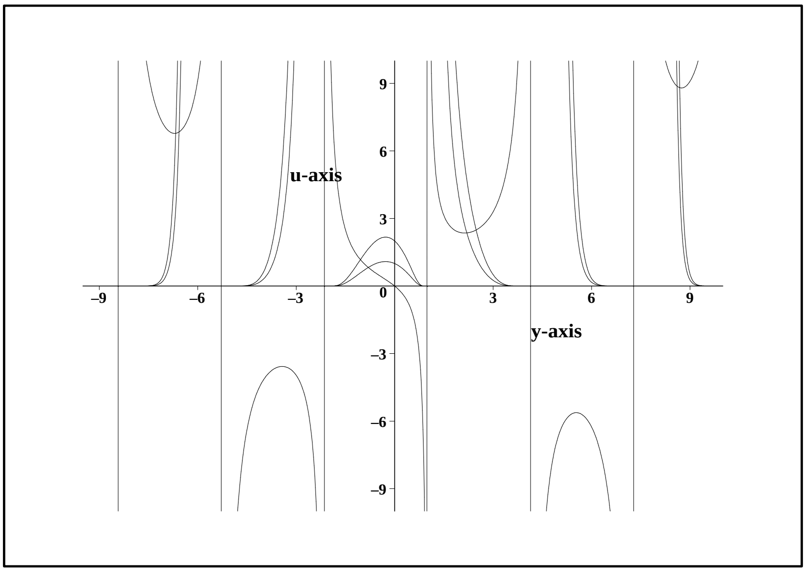

Fig. 2. The graphs of $u = y\csc(y-1)$ and $u = be^{y\cot(y-1)}$ for $b = 1, 2$.

To investigate the zeros of $g(z)$ further, we equate real and imaginary parts in $g(z) = 0$ to get

$$x = -be^{-x}\cos(y - \alpha), \tag{3}$$

and

$$y = be^{-x}\sin(y - \alpha). \tag{4}$$

To solve this system of equations we need to consider two separate cases, $\alpha = 0$ (i.e., β real) and $\alpha \neq 0$ (i.e., β non-real).

3 Non-real β

Let us first consider β non-real. For $0 < \alpha < \pi$, $y \neq 0$ and we can divide (3) by (4) to get

$$x = -y\cot(y - \alpha). \tag{5}$$

If we now substitute (5) in (3) we get

$$y \csc(y-\alpha) = b e^{y \cot(y-\alpha)}. \tag{6}$$

Equation (5) represents a curve Γ with multiple branches in $\mathbf{C}$. All the zeros of $g(z)$ must lie on Γ which is depicted in Figure 1 in the special case $\alpha = 1$. For $0 < \alpha < \pi$, Γ has horizontal asymptotes $L_n = \{z \in \mathbf{C} \mid \mathrm{Im}(z) = \alpha + n\pi\}$, n an integer. Each strip of the complex plane between two consecutive asymptotes L_{n-1} and L_n contains a single branch Γ_n of Γ. For $n \neq 0$, these branches are transverse, crossing the imaginary axis at $(\alpha + [n-1/2]\pi)i$. The transverse branches have y an increasing function of x for $n = 1, 2, \ldots$ and a decreasing function of x for $n = -1, -2, \ldots$. Γ_0 is concave to the right, crossing the imaginary axis at the origin and at $(\alpha - \pi/2)i$. Note that for $\alpha = \pi/2$, Γ_0 does not actually cross the imaginary axis, but is tangent to it at the origin and lies entirely in the right half plane $\{z \in \mathbf{C} \mid \mathrm{Re}(z) \geq 0\}$. The branch Γ_0 has its vertex at $\eta = -\cos^2(\zeta - \alpha) + i\zeta$ where ζ is the unique solution of $2y = \sin 2(y - \alpha)$. It is of interest to note that as α increases between 0 and π, the point η traverses the circle of radius $1/2$ and center $z = -1/2$ from $z = -1 - 0i$ to $z = -1 + 0i$, in the positive direction, passing through the origin when $\alpha = \pi/2$.

We now turn to equation (6) and consider the intersections of the two curves $u = y \csc(y - \alpha)$ and $u = b e^{y \cot(y-\alpha)}$ and how they depend on b. (Figure 2 illustrates the cases $\alpha = 1$ and $b = 1, 2$. The U-shaped and inverted U-shaped branches represent the curve $u = y \csc(y - 1)$. The general shape of the graphs of $u = b e^{y \cot(y-1)}$ follows from the discussion above of equation (5); see Figure 1.) We see that, for a fixed α, $0 < \alpha < \pi$, Γ is the locus of the zeros of $g(z)$ as b varies over $\mathbf{R}$.

Figure 2 suggests that, in case $b < 0$, (6) has a unique zero in $(\alpha + n\pi, \alpha + (n+1)\pi)$ for $n = 1, 3, \ldots$ and $n = -2, -4, \ldots$, while in case $b > 0$, (6) has a unique zero in $(\alpha + n\pi, \alpha + (n+1)\pi)$ for $n = 0, 2, \ldots$ and $n = -3, -5, \ldots$.

To verify this, we have to show, for example that, with $b > 0$, the equation (6)

$$y \csc(y-\alpha) = b e^{y \cot(y-\alpha)}$$

has a unique root on the interval $(\alpha, \alpha + \pi)$. Now the rhs $\to 0$ and the lhs $\to +\infty$ as $y \to (\alpha + \pi)^-$. Also both the rhs and the lhs $\to +\infty$ as $y \to \alpha^+$. However the rhs $\to +\infty$ much faster (like $b e^{1/(y-\alpha)}$ as opposed to just $1/(y - \alpha)$). Thus it is clear that equation (6) has a root on the interval $(\alpha, \alpha + \pi)$. It remains to show that this root is unique. To this end, consider

$$f(y) = y \csc(y-\alpha) - b e^{y \cot(y-\alpha)}.$$

We have $f(\alpha^+) = -\infty$ and $f((\alpha + \pi)^-) = +\infty$. The uniqueness of the root will be clear if we can show that $f'(x) > 0$ at all points where $f(x) = 0$. A direct computation shows that at such a point

$$f'(x) = \csc(x-\alpha)[x^2 + (1 - x\cot(x-\alpha))^2] > 0.$$

The direction of the movement of the zeros on each branch can be further analyzed by looking at dz/db for z a zero. Then $z = -\beta e^{-z} = -b e^{i\alpha - z}$ and

$$\frac{dz}{db} = -e^{i\alpha - z} + be^{i\alpha - z}\frac{dz}{db} = \frac{z}{b} - z\frac{dz}{db}.$$

Thus

$$\frac{dz}{db} = \frac{z}{b(1+z)} = \frac{1}{b}\left\{1 - \frac{1}{1+z}\right\} = \frac{1}{b}\left\{1 - \frac{1+x-iy}{|1+z|^2}\right\},$$

where we have written $z = x + iy$. Equating real and imaginary parts thus gives

$$\frac{dy}{db} = \frac{1}{b}\frac{y}{(1+x)^2 + y^2} \quad \text{and}$$

$$\frac{dx}{db} = \frac{1}{b}\left\{1 - \frac{1+x}{(1+x)^2+y^2}\right\} = \frac{1}{b}\frac{x^2+x+y^2}{(1+x)^2+y^2} = \frac{1}{b}\frac{(x+\frac{1}{2})^2 + y^2 - \frac{1}{4}}{(1+x)^2+y^2}.$$

For a zero on a transverse branch the formula for dy/db tells us that in the upper half plane, the imaginary part of the zero decreases when $b < 0$ and increases when $b > 0$ and the situation is reversed in the lower half plane. Let us now consider dx/db. All of the transverse branches are exterior to the circle with center $-1/2$ and radius $1/2$ and thus the real part of the zeros is strictly increasing(decreasing) for $b > 0 (< 0)$.

Now the concave branch Γ_0 crosses the circle with center $-1/2$ and radius $1/2$ at the origin when $b = 0$ and at the vertex η when $b = b_0 = \pm e^{-\cos^2(\zeta - \alpha)}$ where we take the plus sign for $0 < \alpha < \frac{\pi}{2}$ and the minus sign for $\frac{\pi}{2} < \alpha < \pi$. A careful analysis of dx/db shows that the real part of the zero is strictly decreasing for $b < b_0$ and strictly increasing for $b > b_0$.

The conclusion is that for $b < 0$, we have a single zero, z_n, on each branch Γ_n for $n = 2, 4, \dots$. that starts from $(n\pi + \alpha)i + \infty$ when $b = -\infty$ and goes to $((n-1)\pi + \alpha)i - \infty$ when $b = 0$ crossing the imaginary axis at the point $(\alpha + (n - 1/2)\pi)\, i$ when $b = -(\alpha + (n - 1/2)\pi)$. Similarly, still with $b < 0$, we have a single zero, z_n, on each branch Γ_n for $n = -1, -3, \dots$. that starts from $((n-1)\pi + \alpha)i + \infty$ when $b = -\infty$ and goes to $(n\pi + \alpha)i - \infty$ when $b = 0$ crossing the imaginary axis at the point $-(\alpha - (n + 1/2)\pi)\, i$ when $b = -(\alpha - (n + 1/2)\pi)$.

Thus, in the case $0 < \alpha < \pi/2$, all of the zeros on the transverse branches are in the left half plane $\mathrm{Re}(z) < 0$ for $\alpha - 3\pi/2 < b < 0$. Also we have $\mathrm{Re}(z_n)$ is close to $-\infty$ when b is close to 0^-, $n \neq 0$.

We can do a similar analysis for $\pi/2 < \alpha < \pi$ to show that all of the zeros on the transverse branches are in the left half plane $\mathrm{Re}(z) < 0$ for $\alpha + \pi/2 > b > 0$. Also we have $\mathrm{Re}(z_n)$ is close to $-\infty$ when b is close to 0^-.

Let us now turn to the motion of the zero on Γ_0. We see from (6) that there is a single zero z_0 on Γ_0 and that it starts on the upper part at $\alpha i + \infty$ when $b = -\infty$ and ends up at $(\alpha - \pi)i + \infty$ when $b = +\infty$. The zero z_0 is in the left half plane $\mathrm{Re}(z) < 0$ for b between 0 and $(\pi/2 - \alpha)$, $0 < \alpha < \pi$, $\alpha \neq \pi/2$. When $\alpha = \pi/2$, the branch Γ_0 is tangent to the imaginary axis and z_0 is always in the right half plane. It is clear then that, as b increases from $-\infty$ to ∞, z_0 is the last zero to leave the right half plane and the first to re-enter.

Thus all of the zeros of $g(z)$ are in the left half plane for b between 0 and $(\pi/2-\alpha)$, $0<\alpha<\pi,\ \alpha\neq\pi/2$.

4 Real β

Let us now consider the case where β is real. For $\alpha=0$, equations (3) and (4) become

$$x=-be^{-x}\cos y \tag{7}$$

and

$$y=be^{-x}\sin y. \tag{8}$$

Now equation (8) has the solution $y=0$, in which case (7) becomes $x=-be^{-x}$ and describes the real zeros of $g(z)$, and the solution $y=be^{-x}\sin y$ which together with (7) gives the system

$$x=-y\cot y \tag{9}$$

and

$$y\csc y=be^{y\cot y} \tag{10}$$

which describes the non-real zeros of $g(z)$.

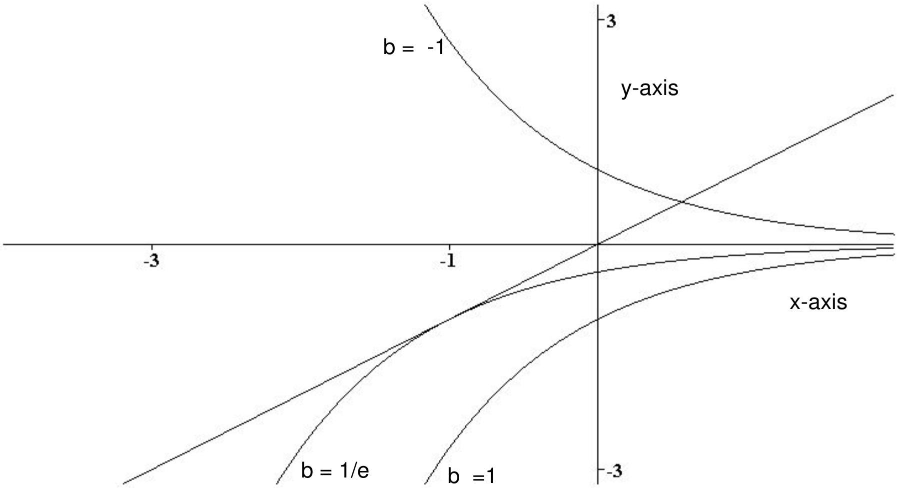

Fig. 3. The graphs of $y=x$ and $y=-be^{-x}$ for $b=-1,1/e,1$.

The situation for real zeros is easy. The real zeros of $g(z)$ are the $x-$coordinate of the points of intersection of the graph of $y = x$ with $y = -be^{-x}$. For $b \leq 0$, $g(z)$ has only one simple, positive real zero x_b^+. As b increases from $-\infty$ to 0^-, x_b^+ decreases from ∞ to 0^+ and x_b^+ approaches 0 and like $-b$. As b increases past 0, x_b^+ continues to decrease and a second negative real zero, x_b^-, starts out from $-\infty$ and these two zeros approach each other, meeting when $b = 1/e$ and merge into the zero $x_{1/e}^+ = x_{1/e}^- = -1$ of order two. As b increases past $1/e$, this double real zero splits into 2 simple complex zeros. These zeros depart at right angles to the real axis .

Now we turn to the complex zeros. The analysis is similar to that in Section 3. The curve $x = -y \cot y$ has an infinite number of transverse branches but unlike the cases considered above, the concave branch occupies a strip that is 2π units wide. (See Figure 4.) Let us call the transverse branch in the strip $k\pi < \mathrm{Im}(z) < (k+1)\pi$, Γ_k, and the concave branch in the strip $-\pi < \mathrm{Im}(z) < \pi$, Γ_0. If we compare the graphs of $u = y \csc y$ and $u = be^{y \cot y}$ we see that there is one zero on each Γ_k for k odd when $b < 0$ and as b varies from $-\infty$ to 0 this zero moves from $(2k+2)\pi i + \infty$ to $(2k+1)\pi i - \infty$, crossing the imaginary axis from right to left at $(4k+3)\pi/2i$ when $b = -(4k+3)\pi/2$. Furthermore, the last pair of non-real zeros leave the right half plane $\mathrm{Re}(z) > 0$ when $b = -3\pi/2$ but the real zero is in the right half plane $\mathrm{Re}(z) > 0$ for all $b < 0$. For $b > 0$ there is one zero on each Γ_k for k even, $k \neq 0$ and this zero moves from $2k\pi i - \infty$ to $(2k+1)\pi i + \infty$ crossing the imaginary axis from left to right at $(4k+1)\pi/2i$ when $b = (4k+1)\pi/2$. We also see that there are no zeros on Γ_0 for $b < 1/e$ and two zeros on Γ_0 for $b > 1/e$. These two zeros split off from the double zero at $z = -1$ and move off to $\pm\pi i + \infty$ as b goes to $+\infty$, crossing the imaginary axis at $\pm i\pi/2$ when $b = \pi/2$. Thus all of the zeros of $g(z)$ are in the left half plane for $0 < b < \pi/2$. Note that for b close to but not equal to zero, $g(z)$ has one zero near the origin and a countable number of zeros in a neighbourhood of $-\infty$.

5 A general stability result

It is clear form the preceding discussion that all of the zeros of $g(z) = z + \beta e^{-z}$ are in the left half plane $\mathrm{Im}(z) < 0$ for b between 0 and $\pi/2 - \alpha$. We formally state this result as follows.

Theorem 1. *All the zeros of the function $g(z) = z + \beta e^{-z}$, $\beta = be^{i\alpha}$, $0 \leq \alpha < \pi$, have negative real part if and only if $|\beta| + |\arg \beta| < \pi/2$.*

The region $|\beta| + |\arg \beta| < \pi/2$ is illustrated in Figure 5. It is bounded by the piecewise polar curve given by $r = \pi/2 - |\theta|$, $|\theta| \leq \pi/2$ consisting of segments of the spirals $r = \pi/2 \pm \theta$.

Remark. The doubling of the width of the convex branch in the case $\alpha = 0$ (Figure 4) as opposed to the case $\alpha \neq 0$ (Figure 1) may seem counter-intuitive. It can be understood if one considers that as $\alpha \to 0^+$, the two

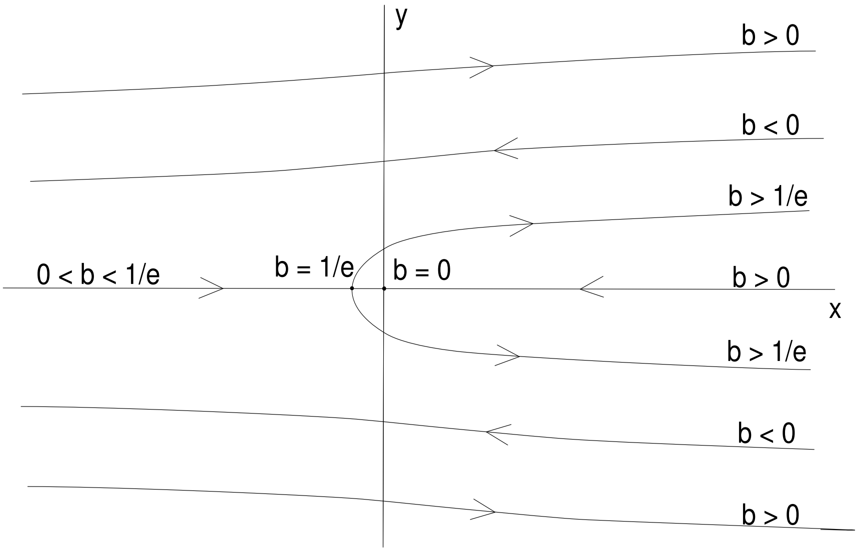

Fig. 4. Part of the graph of $x = -y \cot y$ (not drawn to scale); in case $\alpha = 0$, most zeros of $g(z)$ lie on this graph. The alternate transverse branches describe the paths of zeros as b increases from $-\infty$ to 0, and from 0 to infinity. The convex branch is special; two initially real zeros move along it as b increases from $1/e$ to ∞.

branches Γ_0 and Γ_1 in Figure 1 coalesce into a single convex branch Γ_0 in Figure 4. The separate simple zeros on the branches Γ_0 and Γ_1 in Figure 1 are replaced by a pair of zeros which are initially real ($b < 1/e$) then collide to form a double zero ($b = 1/e$) and the two complex conjugate zeros ($b > 1/e$). The limit $\alpha \to 0^+$ brings no qualitative change in the other branches and the zeros lying on them.

6 A result of Hale and Verduyn Lunel

In [2], the function

$$h(t) = t + a + be^{-t} \tag{11}$$

is considered. We consider a to be real and fixed and let b vary over $\mathbf{R}$. In [2, Theorem A.5, p. 416], there are given necessary and sufficient conditions for all the zeros of $h(z)$ to have negative real parts. The conditions are

$$-1 < a \text{ and } -a < b < \zeta \sin \zeta - a \cos \zeta, \tag{12}$$

where ζ is the unique root of $-\zeta \cot \zeta = a$. We will obtain this result using the results of Section 4 and obtain the simpler form $-a \sec \zeta$ for the upper bound for b in (12).

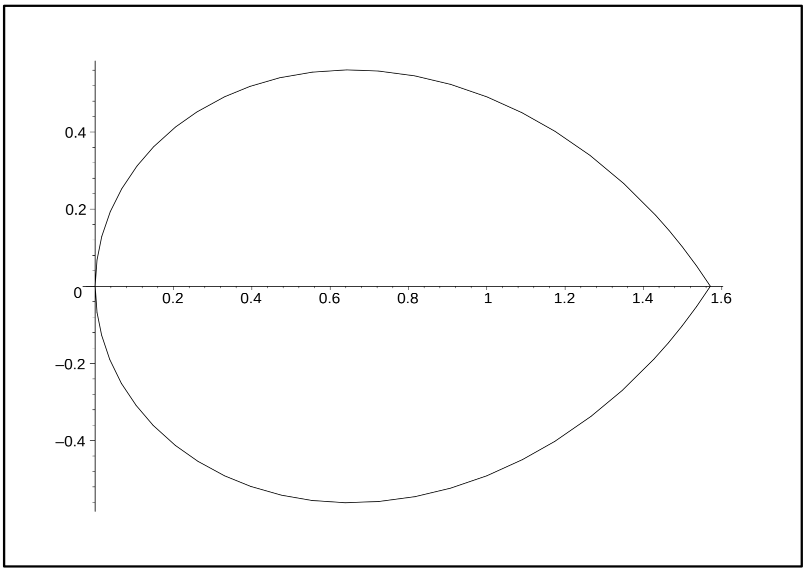

Fig. 5. The region $|\beta| + |\arg\beta| < \pi/2$ for which $g(z) = z + \beta e^{-z}$ is stable.

Let us note that $h(t) = 0$ if and only if $g_{be^a}(t + a) = 0$ where we have written $g_b(z) = g(z) = z + be^{-z}$ to emphasize the dependence on $b \in \mathbf{R}$. Thus $h(t)$ is stable if and only if all the zeros of $g_{be^a}(z)$ lie in the left half plane $\mathrm{Re}(z) < a$. A first reduction occurs when we note that if $a \le -1$, there will always be a zero in $\mathrm{Re}(z) \ge a$ by following the trajectory of the real zero that starts from $+\infty$ as b increases over $\mathbf{R}$. Thus a must be greater than -1. Now if $a > -1$, it is clear from our discussion of the movement of the zeros of $g(z)$ that the positive real zero of $g_{be^a}(z)$ that starts from $+\infty$ crosses the line $\mathrm{Re}(z) = a$ from right to left when $b = -a$, as b increases. Furthermore, at that point all of the zeros of $g_{be^a}(z)$ lie in the left half plane $\mathrm{Re}(z) < a$ and stay there until the pair of zeros on $\mathit{\Gamma}_0$ cross the line $\mathrm{Re}(z) = a$ for some positive value of b. The only question left is how big b is allowed to get.

Consider the zero on the upper half of $\mathit{\Gamma}_0$. It will cross the line $\mathrm{Re}(z) = a$ when $\mathrm{Im}(z) = \zeta$ where ζ is the unique solution of $a = -y \cot y$ and $0 < \zeta < \pi$. Note that

$$0 < \zeta < \pi/2 \text{ for } \; -1 < a < 0 \;\text{ and }\; \pi/2 < \pi \;\text{ for }\; a > 0. \qquad (13)$$

To find the value of b when it crosses the line $\mathrm{Re}(z) = a$ we note that

$$b = (be^a)e^{-a} = (be^a)e^{\zeta \cot \zeta} = \zeta \csc \zeta = -a \sec \zeta$$

by (10). It is now simple trigonometry to show that
$-a \sec \zeta = \zeta \sin \zeta - a \cos \zeta$ and (13) gives us $-a \sec \zeta > 0$ as expected by our discussion above.

Acknowledgment. We are indebted to a referee for pointing out a number of errors and omissions in an earlier version of this article.

References

1. O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H.-O.Walther, *Delay Equations: Functional-. Complex-, and Nonlinear Analysis*, Applied Mathematical Sciences, vol. 110, Springer-Verlag, 1995.
2. J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, 1993.
3. M. Matsuda, Liouville's theorem on a transcendental equation log $y = y/x$, *J. Math. Kyoto Univ.* **16** (1976), no. 3, 545–554. MR 55 #2864
4. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences, vol. 119, Springer-Verlag, 1996.