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PRINCIPAL PAIRS FOR OSCILLATORY SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS[1]

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Abstract

Nonoscillatory second order differential equations always admit "special", principal solutions. For a certain type of oscillatory equation principal pairs of solutions were introduced by Á. Elbert, F. Neuman and J. Vosmanský, *Diff. Int. Equations* **5** (1992), 945–960. In this paper, the notion of principal pair is extended to a wider class of oscillatory equations. Also an interesting property of some of the principal pairs is presented that makes the notion of these "special" pairs more understandable.

1 Introduction

For nonoscillatory equations, the notion of principal solution was introduced by W. Leighton, M. Morse and P. Hartman [5]. Principal pairs were defined for certain types of oscillatory equations in [4].

In this paper, we extend the notion of principal pairs to a wider class of oscillatory equations. Under certain conditions, general enough to cover many important special equations (Bessel, Airy, etc.) we also study the question of identifying the second member of such a pair when the first member is given. We find that among all the solutions, the second member is the one whose zeros approximate most closely the zeros of the derivative of the first member of the pair. In these cases, this property may also serve as an alternative definition, connecting in a certain sense the Kummer and the Prüfer transformations and bringing more light to this particular choice of "good" pairs of solutions of linear second order oscillatory differential equations.

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2 Basic facts and definitions

Consider the equation

$$y'' + q(x)y = 0 \tag{1}$$

on $[x_0, \infty)$, where $q \in C^0[x_0, \infty)$.

If equation (1) is nonoscillatory, then a solution y_1 of (1) is called *principal*, in accordance with the terminology of Leighton, Morse and Hartman (see, e.g., [5]), if

$$\int^{\infty} y_1^{-2}(x)dx = \infty, \tag{2}$$

or, equivalently, if $\lim_{x\to\infty} y_1/y_2 = 0$ for every solution y_2 of (1) linearly independent of y_1. In the nonoscillatory case such a principal solutions always exists and is unique up to multiplication by a constant factor.

Now, consider an *oscillatory* equation (1) with two linearly independent solutions y_1, y_2 normalized by having their Wronskian

$$w = w(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{3}$$

equal to ±1, i.e.,

$$|w| = 1. \tag{4}$$

It was proved in [4] that if

$$\lim_{x\to\infty} [y_1^2(x) + y_2^2(x)] = L \neq 0, \tag{5}$$

for a pair of solutions y_1, y_2 of an oscillatory equation (1), satisfying (4), then the pair y_1, y_2 is unique up to orthogonal transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \text{ with } A^T A = I, \tag{6}$$

i.e., if (y_1, y_2) is one such pair then every such pair is of the form $(ay_1 + by_2,\ cy_1 + dy_2)$ with $a^2 + c^2 = 1 = b^2 + d^2,\ ab + cd = 0$.

In [4], each of these pairs (y_1, y_2) was called a *principal* pair of solutions of (1). Some sufficient conditions for the existence of principal pairs (y_1, y_2) of solutions of (1) with $q \in C^1[x_0, \infty)$, were also given there, based on the fact that $v(x) = y_1^2(x) + y_2^2(x)$ is a solution of the corresponding Appell equation (see [2], [5])

$$v''' + 4q(x)v' + 2q'(x)v = 0 \tag{7}$$

on $[x_0, \infty)$, admitting three linearly independent solutions $y_1^2(x), y_2^2(x)$ and $y_1(x)y_2(x)$. Here we extend the investigation of pairs of solutions to the cases where the L in (5) is 0 or $+\infty$.

We shall need the following important facts concerning the Kummer transformation of the second order equations of the form (1), derived by O. Borůvka, and summarized in his monograph [3]; see also [8, Chapter 2]. Consider an equation (1) (either oscillatory or nonoscillatory) and a pair of linearly independent solutions, y_1 and y_2. Denote by $w = w(y_1, y_2)$ their Wronskian

$$w(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = w \neq 0. \tag{8}$$

Following O. Borůvka, define the (first) phase α of (1) with respect to y_1, y_2 as a continuous function $\alpha \in C^0[0,\infty)$ satisfying the relation

$$\tan\alpha = y_1(x)/y_2(x), \tag{9}$$

whenever this quantity is defined, i.e., for $y_2(x) \neq 0$. Then $\alpha \in C^3[x_0,\infty)$, $\alpha'(x) \neq 0$ on $[x_0,\infty)$, and

$$\begin{aligned} y_1(x) &= \varepsilon|\alpha'(x)|^{-1/2}\sin\alpha(x), \\ y_2(x) &= \varepsilon|\alpha'(x)|^{-1/2}\cos\alpha(x), \\ \varepsilon &= \pm\sqrt{-w\cdot\ \mathrm{sign}\ \alpha'}, \end{aligned}$$

$y(x) = k_1|\alpha'(x)|^{-1/2}\sin(\alpha(x)+k_2)$ being a general solution of (1) for arbitrary constants k_1 and k_2. Moreover, equation (1) is oscillatory on $[x_0,\infty)$ (as $x \to \infty$) if and only if $\lim_{x\to\infty}|\alpha(x)| = \infty$; see [3].

Each solution of (1) is bounded if and only if $|\alpha'(x)|^{-1}$ is bounded on $[x_0,\infty)$; see [8].

Each solution of (1) tends to 0 as $x \to \infty$ if and only if $\lim_{x\to\infty}|\alpha'(x)|^{-1} = 0$; see [8].

In particular, we have

$$y_1^2(x) + y_2^2(x) = -\frac{w}{\alpha'(x)}$$

on $[x_0,\infty)$ and for y_1 and y_2 with $w(y_1,y_2) = -1$, we get

$$\begin{aligned} y_1(x) &= \pm|\alpha'(x)|^{-1/2}\sin\alpha(x), \\ y_2(x) &= \pm|\alpha'(x)|^{-1/2}\cos\alpha(x) \end{aligned}$$

and

$$y_1^2(x) + y_2^2(x) = (\alpha'(x))^{-1}.$$

3 Principal pairs of solutions — extension

In the previous section, the number L in the limit relation (5) was supposed to satisfy $0 < L < \infty$. This raises the question of extension to the cases $L = 0,\ +\infty$. In both of these cases, the limit relation (5) must be supplemented by an additional condition in order to characterize sums of squares up to orthogonal transformation.

Consider a pair $\overline{y}_1, \overline{y}_2$ of linearly independent solutions of (1) with Wronskian $w(\overline{y}_1,\overline{y}_2)$ satisfying

$$|w(\overline{y}_1,\overline{y}_2)| = 1. \tag{10}$$

We can write

$$\begin{aligned} \overline{y}_1 &= ay_1 + by_2, \\ \overline{y}_2 &= cy_1 + dy_2, \end{aligned}$$

where

$$\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} w(y_1,y_2)\right| = |ad-bc| = 1.$$

Hence

$$\overline{y}_1(x) = \varepsilon|\alpha'(x)|^{-1/2}[a\sin\alpha(x) + b\cos\alpha(x)],$$
$$\overline{y}_2(x) = \varepsilon|\alpha'(x)|^{-1/2}[c\sin\alpha(x) + d\cos\alpha(x)],$$

and

$$\begin{aligned}\overline{v}(x) &= \overline{y}_1^2(x) + \overline{y}_2^2(x)\\ &= v(x)\left[(a^2+c^2)\sin^2\alpha(x) + (ab+cd)\sin 2\alpha(x) + (b^2+d^2)\cos^2\alpha(x)\right].\end{aligned} \tag{11}$$

In case $v(x) \to L$ with $0 < L < +\infty$, it is clear that $\overline{v}(x)$ approaches a finite limit if and only if $a^2 + c^2 = b^2 + d^2 = K > 0$, and $ab + cd = 0$. However, in case $L = 0$ the condition $v(x) \to 0$ implies that $\overline{v}(x) \to 0$, regardless of the values of a, b, c, d. In case $L = \infty$, $v(x) \to \infty$ implies that $\overline{v}(x) \to \infty$, for any values of a, b, c, d satisfying $a^2 + c^2 = b^2 + d^2 > |ab + cd|$. Thus the condition (5), with $L = 0$ or $L = \infty$, does not characterize the pair y_1, y_2 up to orthogonal transformation. In as much as this excludes some important examples, such as the Airy equation ($q(x) = x$) and the Cauchy–Euler equation ($q(x) = \gamma^2/x^2$), we now present a theorem which characterizes the pair y_1, y_2 by a condition on v':

Theorem 1. *Let (1) be an oscillatory equation and let y_1, y_2 be a pair of linearly independent solutions normalized by the unit Wronskian $|w(y_1, y_2)| = 1$. Let $v(x) = y_1^2(x) + y_2^2(x)$ and suppose that*

$$\lim_{x\to\infty} v'(x) = K, \quad 0 \le K < \infty. \tag{12}$$

Then the pair y_1, y_2 is unique up to orthogonal transformation.

Proof. Let α denote a phase of equation (1) corresponding to y_1 and y_2. In accordance with §2, we have $\alpha \in C^3[x_0, \infty)$, $\alpha'(x) \neq 0$ on $[x_0, \infty)$, and

$$\lim_{x\to\infty} |\alpha(x)| = \infty, \tag{13}$$

because (1) is an oscillatory equation. Moreover,

$$y_1(x) = \varepsilon|\alpha'(x)|^{-1/2}\sin\alpha(x),$$

$$y_2(x) = \varepsilon|\alpha'(x)|^{-1/2}\cos\alpha(x),$$

$\varepsilon = \pm 1$. Hence $v(x) = y_1^2(x) + y_2^2(x) = |\alpha'(x)|^{-1}$. Now, for a pair $\overline{y}_1, \overline{y}_2$, we get, from (11)

$$\begin{aligned}\overline{v}'(x) &= v'(x)\left[(a^2+c^2)\sin^2\alpha(x) + 2(ab+cd)\sin\alpha(x)\cos\alpha(x) + (b^2+d^2)\cos^2\alpha(x)\right]\\ &\quad + \frac{1}{|\alpha'(x)|}\left[(a^2+c^2-b^2-d^2)\sin 2\alpha(x) + 2(ab+cd)\cos 2\alpha(x)\right]\cdot\alpha'(x)\end{aligned}$$

i.e.,

$$\overline{v}'(x) = v'(x)F(x) + G(x) \tag{14}$$

where

$$F(x) = (a^2+c^2)\sin^2\alpha(x) + 2(ab+cd)\sin\alpha(x)\cos\alpha(x) + (b^2+d^2)\cos^2\alpha(x)$$

is a bounded function on $[x_0, \infty)$, and

$$G(x) = K_1\sin 2\alpha(x) + K_2\cos 2\alpha(x),$$

where $K_1 = a^2 + c^2 - b^2 - d^2$, $K_2 = 2(ab + cd)$. Now we show that if

$$\lim_{x\to\infty} \overline{v}'(x) = \overline{K}, \quad |\overline{K}| < \infty. \tag{15}$$

i.e., if (12) holds for $\overline{y}_1, \overline{y}_2$, then $K_1 = K_2 = 0$.

We have

$$2F(x) = (a^2 + b^2 + c^2 + d^2) - K_1 \cos 2\alpha(x) + K_2 \sin 2\alpha(x).$$

Then, using (14), we get

$$2\overline{v}'(x) = v'(x)[a^2 + b^2 + c^2 + d^2] + \cos 2\alpha(x)[K_2 - v'(x)K_1] + \sin 2\alpha(x)[K_1 + v'(x)K_2]. \tag{16}$$

Letting $x \to \infty$, we get $K_1 = K_2 = 0$. Hence $a^2 + c^2 = b^2 + d^2 = K_3 \neq 0$, and $ab + cd = 0$, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} K_3 & 0 \\ 0 & K_3 \end{pmatrix}.$$

From the normalization of the Wronskian, we get $|ad - bc| = K_3 = 1$; hence (y_1, y_2) is unique up to orthogonal transformation. This completes the proof of Theorem 1.

3.1 An extended definition

Definition. *The pair y_1, y_2 will be called principal for an oscillatory equation (1) if $v(x) := y_1^2(x) + y_2^2(x)$ satisfies either*

$$\lim_{x\to\infty} v(x) = L, \ (0 < L < \infty), \tag{17}$$

or

$$\lim_{x\to\infty} v'(x) = K, \ (0 \leq K < \infty). \tag{18}$$

Due to [4], and Theorem 1, this pair is unique up to orthogonal transformation.

Since all principal pairs for an oscillatory equation (if they exist) differ only by an orthogonal transformation, the expression

$$v(x) = y_1^2(x) + y_2^2(x)$$

remains the same for all principal pairs.

The following sufficient condition for the existence of principal pairs is simply a restatement of a result of P. Hartman [6, Theorem 20.1$_0$]:

Corollary 1. *Let $q \in C^1[x_0, \infty)$, $q'(x) \geq 0$, $q''(x) \leq 0$ and $\lim_{x\to\infty} q(x) = \infty$. Then (1) has a principal pair of solutions y_1, y_2 such that*

$$v(x) \geq 0, \quad \lim_{x\to\infty} v(x) = 0,$$

$$v'(x) \leq 0, \quad \lim_{x\to\infty} v'(x) = 0,$$

$$\lim_{x\to\infty} v''(x) = 0$$

and

$$\lim_{x\to\infty} v'''(x) = 0.$$

Another sufficient condition for the existence of principal pairs follows from another result of Hartman [6, Theorem 22.1_0]:

Corollary 2. *Let* $q \in C^2[x_0, \infty)$, $q'(x) \leq 0$, $q(x)q''(x) - 3q'(x)^2 \geq 0$. *Then (1) has a principal pair of solutions* y_1, y_2 *such that*

$$v(x) \geq 0,$$

$$\lim_{x \to \infty} \left[q^{-1}(x) \left(\frac{d}{dx} \right)^n v(x) \right] = 0, \; n = 1, 2, 3,$$

and v *approaches a finite limit or* ∞ *according as* q *approaches a positive limit or* 0.

3.2 Examples

Example 1. The generalized Airy equation

This is an example with $L = 0$. The equation in question is

$$y'' + (2\nu)^{-2} x^{1/\nu - 2} y = 0, \tag{19}$$

where we suppose that $0 < \nu \leq 1/2$. The usual Airy equation corresponds to $\nu = 1/3$. With the usual notation for the Bessel functions [9], the pair $x^{1/2} J_\nu(2\nu x^{1/(2\nu)})$, $x^{1/2} Y_\nu(2\nu x^{1/(2\nu)})$ satisfy (19). In view of the known result [9, p. 446] that, for $0 \leq \mu < 1/2$, the function $t[J_\mu^2(t) + Y_\mu^2(t)]$ increases to $2/\pi$ on $(0, \infty)$, we see that

$$v(x) = x[J_\nu^2(2\nu x^{1/(2\nu)}) + Y_\nu^2(2\nu x^{1/(2\nu)})]$$

approaches 0 as $x \to \infty$. In fact, for $\frac{1}{3} \leq \nu < \frac{1}{2}$, $v(x)$ is completely monotonic (i.e., its successive derivatives alternate in sign) [7, Theorem 5.1 (i)]. The asymptotic formula

$$x[J_\nu^2(2\nu x^{1/(2\nu)}) + Y_\nu^2(2\nu x^{1/(2\nu)})] = \frac{1}{\nu\pi} x^{1 - 1/(2\nu)} \left[1 + O(x^{-1/\nu}) \right], \tag{20}$$

[9, p. 224 (5)] shows that $v(x)$ and $v'(x)$ approach 0 as $x \to \infty$. Thus, by Theorem 1, the pair $x^{1/2} J_\nu(2\nu x^{1/(2\nu)}), x^{1/2} Y_\nu(2\nu x^{1/(2\nu)})$ constitute a principal pair for (19).

Example 2.

The equation

$$y'' + x^{-1} y = 0, \tag{21}$$

has a principal pair $x^{1/2} J_1(2x^{1/2}), x^{1/2} Y_1(2x^{1/2})$. Here $v(x) \to \infty$, but $v'(x) \to 0$ as $x \to \infty$, so the hypotheses of Theorem 1 are satisfied again.

Example 3. The Cauchy–Euler equation

This is an example with $L = \infty$, to which Theorem 1 is applicable. The Cauchy–Euler equation is

$$y'' + \frac{\gamma^2}{x^2} y = 0. \tag{22}$$

In case $\gamma^2 > 1/4$, the equation is oscillatory and a principal pair of solutions is given by $y_1(x) = x^{1/2}\sin(s\log x)$, $y_2(x) = x^{1/2}\cos(s\log x)$, where $s = \sqrt{\gamma^2 - 1/4}$, so $v(x) = x$ and $v'(x) = 1$, so the hypotheses of Theorem 1 are satisfied.

It is clear that every pair

$$\overline{y}_1 = ay_1 + by_2,$$

$$\overline{y}_2 = cy_1 + dy_2,$$

with $a^2 + c^2 = b^2 + d^2 > |ab + cd|$ has the property that $\overline{v}(x) \to \infty$ as $x \to \infty$, but it is only in the case $a^2 + c^2 = c^2 + d^2,\ ab + cd = 0$ that $\overline{v}'(x)$ approaches a finite limit.

4 Properties of principal pairs

Suppose that an oscillatory equation (1) admits a pair y_1, y_2, for which

$$\lim_{x\to\infty} v'(x) = \lim_{x\to\infty} (y_1^2(x) + y_2^2(x))' = 0. \tag{23}$$

Due to the extended definition in §3.1, the pair y_1, y_2 is a principal pair. Let α denote the phase of (1) corresponding to y_1, y_2. Then

$$y_1(x) = \varepsilon|\alpha'(x)|^{-1/2}\sin\alpha(x),$$

$$y_2(x) = \varepsilon|\alpha'(x)|^{-1/2}\cos\alpha(x),$$

$$\varepsilon = \pm 1,$$

$$\alpha \in C^3[x_0,\infty),\ \alpha'(x) \neq 0 \text{ on } [x_0,\infty),$$

$$v(x) = y_1^2(x) + y_2^2(x) = |\alpha'(x)|^{-1}.$$

Condition (23) gives

$$\lim_{x\to\infty} \frac{\alpha''(x)}{(\alpha'(x))^2} = 0. \tag{24}$$

Consider the first derivative of y_1:

$$y_1'(x) = \varepsilon\left[-\frac{1}{2}|\alpha'(x)|^{-1/2}\frac{\alpha''(x)}{\alpha'(x)}\sin\alpha(x) + |\alpha'(x)|^{1/2}\cos\alpha(x)\cdot\ \text{sign}\ \alpha'\right].$$

All of its zeros are at points x_j where

$$\frac{\cos\alpha(x_j)}{\sin\alpha(x_j)} = \frac{1}{2}\frac{\alpha''(x_j)}{(\alpha'(x_j))^2}. \tag{25}$$

From (24) we have

$$|\alpha(x_j) + \pi/2 - j\pi\ \text{sign}\ \alpha'| \to 0, \quad \text{as} \ \ j \to \infty.$$

A general solution of (1) can be written in the form

$$y(x; c_1, c_2) = c_1|\alpha'(x)|^{-1/2}\sin(\alpha(x) + c_2).$$

Its zeros are at points x_j, where

$$\alpha(x_j) + c_2 = j\pi\ \text{sign}\ \alpha',\ j = j_0, j_0 + 1, \ldots.$$

Hence

$$y(x;c_1,\pi/2)=c_1|\alpha'(x)|^{-1/2}\cos\alpha(x)$$

is that solution (up to constant multiple) whose zeros are approached by the zeros of the derivative of y_1. However, after the normalization $c_1=\pm1$, y_1 together with $y(x,\pm1,\pi/2)=\pm y_2(x)$ represent a principal pair of (1).

We may summarize our considerations in the following Theorem:

Theorem 2. *If an oscillatory equation (1) has a pair of solutions* y_1, y_2 *for which*

$$\lim_{x\to\infty} v'(x)=0, \tag{26}$$

then they have the property that the difference of the zeros of the derivative of one of them and the zeros of the other tend to 0 *as* $x\to\infty$. *In other words, among all sequences of zeros of solutions of (1), the best approximation to the zeros of* y_1' *is given by zeros of* y_2.

Remark. Evidently, the conditions in Corollary 1 are sufficient to guarantee the conclusion of Theorem 2. Further sufficient conditions can be obtained by using Hartman [6, Theorem 18.1_0]. In fact, this Theorem, together with his Theorem 20.1_0, show that the requirements

$$q'\geq 0,\ q''\leq 0,\ \text{and}\ \lim_{x\to\infty} q(x)=q(\infty),\ \text{where}\ 0<q(\infty)<\infty$$

are sufficient to imply the conclusion of our Theorem 2.

The property described in Theorem 2 links two transformations of second order equations.

The first is the Kummer (sometimes called Stäckel) transformation consisting in the change of independent and dependent variables of the solution $z(t)$ into the solution $y(x)$ expressible in the form

$$y(x)=f(x)z(h(x))$$

or, better,

$$y_1(x)=f(x)z_1(h(x))$$

and

$$y_2(x)=f(x)z_2(h(x)),$$

for two linearly independent solutions y_1, y_2 and z_1, z_2 of the transformed equations. In particular,

$$y_1(x)=f(x)\sin h(x)$$

and

$$y_2(x)=f(x)\cos h(x),$$

when the equation $z''+z=0$ is taken as canonical (as can always be done). The second is the Prüfer transformation, considering again a two dimensional underlying space for two functions, now

$$y_1(x)=y(x)=\rho(x)\sin\varphi(x)$$

and

$$y_2(x)=y'(x)=\rho(x)\cos\varphi(x).$$

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