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Inequalities and monotonicity properties for zeros of Hermite functions[1]

by

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Abstract

We study the variation of the zeros of the Hermite function $H_\lambda(t)$ with respect to the positive real variable λ. We show that, for each nonnegative integer n, $H_\lambda(t)$ has exactly $n+1$ real zeros when $n < \lambda \le n+1$ and that each zero increases from $-\infty$ to ∞ as λ increases. We establish a formula for the derivative of a zero with respect to the parameter λ; this derivative is a completely monotonic function of λ. By-products include some results on the regular sign behaviour of differences of zeros of Hermite polynomials as well as a proof of some inequalities, related to work of W. K. Hayman and E. L. Ortiz for the largest zero of $H_\lambda(t)$. Similar results on zeros of certain confluent hypergeometric functions are given too. These specialize to results on the first, second, etc., positive zeros of Hermite polynomials.

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1 Introduction

The Hermite function $H_\lambda(t)$ can be defined (see, e.g., [16]) by

$$H_\lambda(t) = -\frac{\sin \pi\lambda\ \Gamma(1+\lambda)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left((n-\lambda)/2\right)}{\Gamma(n+1)} (-2t)^n \tag{1.1}$$

or, in terms of the confluent hypergeometric functions ([4]), by

$$\begin{aligned} H_\lambda(t) \;=\; & \frac{2^\lambda}{\sqrt{\pi}} \left[\cos\frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2}+\frac{1}{2})\, {}_1F_1(-\frac{\lambda}{2}, \frac{1}{2}; t^2) \right. \\ & \left. + 2t \sin\frac{\lambda\pi}{2} \Gamma(\frac{\lambda}{2}+1)\, {}_1F_1(-\frac{\lambda}{2}+\frac{1}{2}, \frac{3}{2}; t^2) \right]. \end{aligned} \tag{1.2}$$

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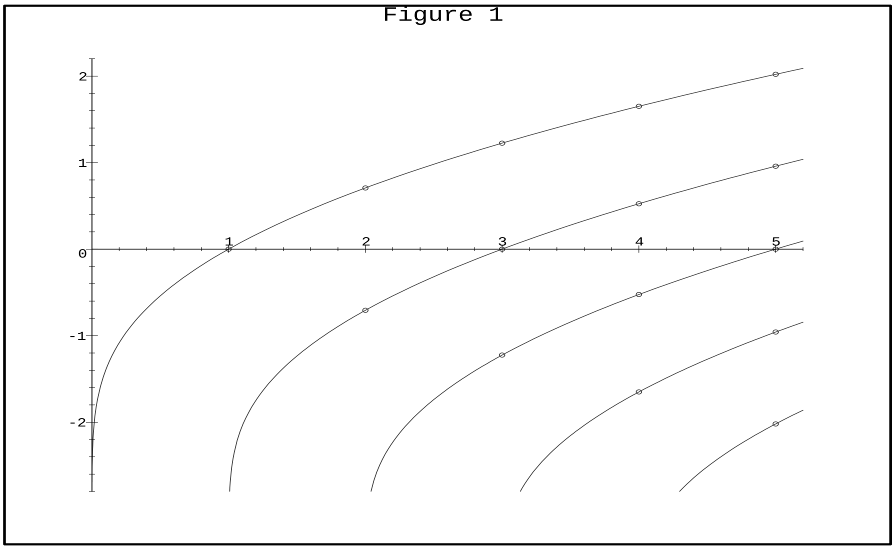


Figure 1

Formula (1.1) is to be understood in a limiting sense when λ is an integer and the constant multiplying the sum is chosen so that $H_\lambda(t)$ reduces to the Hermite polynomials (with the notation of, e.g., [26]) in case λ is a nonnegative integer. Thus $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12t$, etc. We note also that, from the definition (1.1), we have

$$H'_\lambda(t) = 2\lambda H_{\lambda-1}(t). \tag{1.3}$$

In the polynomial case ($\lambda = n$), the zeros of $H_\lambda(t)$ are real and located symmetrically with respect to the origin. Our main object here is the study of the real zeros of $H_\lambda(t)$ in the case where λ is a positive real number. We use the notation $h(\lambda)$ for the largest real zero of $H_\lambda(t)$; it is of importance in the study of subharmonic functions [13], [16].

It will turn out that, when $n < \lambda \leq n+1$, with n a nonnegative integer, $H_\lambda(t)$ has $n+1$ real zeros which increase with λ. As λ passes through each nonnegative integer n a new leftmost zero appears at $-\infty$ while the right-most zero passes through the largest zero of $H_n(t)$. See Figure 1 which provides graphs of the real zeros of $H_\lambda(t)$ (solid lines) as functions of λ. The small circles mark the zeros of Hermite polynomials. (This and the other figures, were produced using MAPLE V, Release 4.)

For each fixed λ, $H_\lambda(t)$ is that solution of the Hermite differential equation

$$y'' - 2ty' + 2\lambda y = 0, \tag{1.4}$$

which grows relatively slowly as $t \to +\infty$. It will be useful also to record the self-adjoint form of (1.4):

$$(e^{-t^2}y')' + 2\lambda e^{-t^2}y = 0. \tag{1.5}$$

We consider also a solution of (1.4) which is linearly independent of $H_\lambda(z)$:

$$G_\lambda(t) = \frac{2^\lambda}{\sqrt{\pi}}\left[-\sin\frac{\lambda\pi}{2}\Gamma(\frac{\lambda+1}{2})_1F_1(\frac{-\lambda}{2},\frac{1}{2};t^2) + 2t\cos\frac{\lambda\pi}{2}\Gamma(\frac{\lambda+2}{2})_1F_1(\frac{-\lambda+1}{2},\frac{3}{2};t^2)\right]. \tag{1.6}$$

The functions $e^{-t^2/2}H_\lambda(t)$ and $e^{-t^2/2}G_\lambda(t)$, which have the same zeros as $H_\lambda(t)$ and $G_\lambda(t)$ are linearly independent solutions of the modified Hermite equation

$$y'' + (2\lambda + 1 - t^2)y = 0. \tag{1.7}$$

It was shown by Durand [4] that

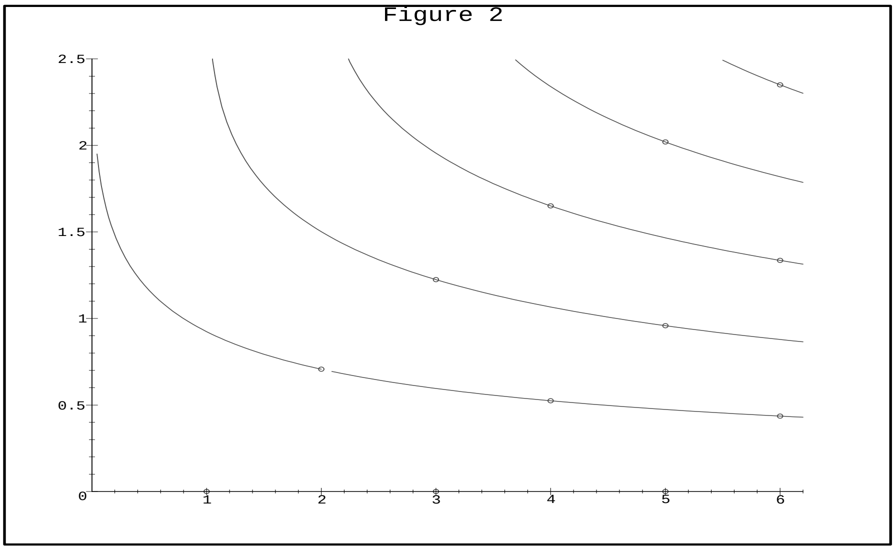
$$\frac{2^{-\lambda}\sqrt{\pi}}{\Gamma(\lambda+1)}e^{-t^2}[H_\lambda^2(t)+G_\lambda^2(t)] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-(2\lambda+1)\tau+t^2\tanh\tau}\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}}. \tag{1.8}$$

The main result of the present paper is a formula for the derivative of a zero of a solution of (1.7) with respect to λ. It involves an integral closely related to that in (1.8).

We are interested in the zeros of two kinds of solutions of (1.4) or (1.7)

(i) On the one hand, we consider the zeros of $H_\lambda(t)$, $\lambda > 0$, or, more generally, the zeros of linear combinations of H_λ and G_λ. The situation is illustrated in Figure 1, where the curves indicate the zeros of H_λ and the dots are the zeros of the Hermite polynomials.

(ii) On the other hand, we consider the zeros of ${}_1F_1(-\lambda/2, 1 \pm 1/2; t^2)$, $\lambda > 0$, as in Figure 2, where the dots are again the zeros of the Hermite polynomials. It is clear from (1.2) that the first, second, etc., positive zeros of the even or odd order Hermite polynomials are interpolated by curves of the family of zeros of ${}_1F_1(-\lambda/2, 1/2; t^2)$ or ${}_1F_1(-\lambda/2, 3/2; t^2)$ according as the degree is even or odd.



The second situation is somewhat simpler and, after some preliminary results, is dealt with first (§6). But the first situation is more interesting and for it we have more complete results (§§7–8).

2 Principal solutions and zeros

The notion of *principal solution* was introduced by W. Leighton and M. Morse [19] in connection with a problem of variational calculus. Further investigations have been carried out by P. Hartman and A. Wintner; see [15]. A differential equation of the form

$$(r(t)x')' + q(t)x = 0 \tag{2.1}$$

defined on (α, ∞) with $r(t) > 0$ and $r(t)$, $q(t)$ piece-wise continuous functions on (α, ∞), is nonoscillatory at $t = \infty$ if there is a solution $x(t)$ of (2.1) such that $x(t)$ has finitely many zeros (say, k, $k \geq 0$) on the interval $[T, \infty)$ where $\alpha < T < \infty$. Then the well-known

Sturmian theory ([15], [26]) assures that any other solution of (2.1) has also finitely many zeros (say, k') on $[T, \infty)$ and $|k - k'| \leq 1$.

A solution $x_0(t)$ of (2.1) is called a *principal solution* (at ∞) if

$$\frac{x_0(t)}{x(t)} \to 0, \quad \text{as} \quad t \to \infty, \tag{2.2}$$

where $x(t)$ is an arbitrary solution of (2.1) which is linearly independent of $x_0(t)$. Such a solution always exists for a nonoscillatory equation (2.1) and is uniquely determined up to a constant factor by the condition (2.2) [15, p. 355].

Lemma 2.1. *If a principal solution $x_0(t)$ of (2.1) has a zero t_1 in (α, ∞), then any other (linearly independent) solution $x(t)$ has a zero $\tilde{t}_1$ such that $t_1 < \tilde{t}_1 < \infty$.*

In this respect, a principal solution behaves as if it had a zero at $+\infty$. This result goes back to work of Leighton and Morse [19]. A more complete exposition is found in work of Lorch and Newman [23].

In order to apply these ideas to the Hermite differential equation, we need to take account of the asymptotic expansion for ${}_1F_1(a, c; z)$ [11, p. 278] leading to the following asymptotic formulas for $H_\lambda(t)$ and $G_\lambda(t)$:

$$H_\lambda(t) \sim (2t)^\lambda, \quad t \to +\infty, \tag{2.3}$$

$$G_\lambda(t) \sim \frac{1}{\sqrt{\pi}} \Gamma(\lambda + 1) t^{-\lambda - 1} e^{t^2}, \quad t \to +\infty. \tag{2.4}$$

These show that $e^{-t^2/2} H_\lambda(t)$ is a principal solution at $+\infty$ of equation (1.7).

3 The number of real zeros of a Hermite function

Here we prove:

Theorem 3.1. *For $n < \lambda \leq n + 1$, $n = 0, 1, \ldots,$ $H_\lambda(t)$ has $n + 1$ real zeros, and it has no real zeros when $\lambda \leq 0$. Each zero is an increasing function of λ on its interval of definition.*

Remark. As Figure 1 suggests, a new real zero appears as λ increases through each integer value.

Proof of Theorem 3.1. (a) The result is obvious in case $\lambda = 0$. Next we consider the case where $\lambda < 0$. In this case, the function $q(t)$ in (1.5) is negative so we can use [15, Corollary 6.4, p. 357] to see that the principal solution $H_\lambda(t)$ has no real zeros.

(b) We consider next the case $n = 0$; that is, we show that $H_\lambda(z)$ has one real zero, for $0 < \lambda \leq 1$. Recalling that

$${}_1F_1(a, c; z) = 1 + \frac{a}{c} \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \cdots, \tag{3.1}$$

we see that the series arising from the first ${}_1F_1$ in (1.2) has one constant positive term. The other terms are negative multiples of even powers of t, hence they are increasing for $-\infty < t < 0$. In the second series, once the factor t is incorporated, the terms are positive multiples of odd powers of t, hence they too are increasing for $-\infty < t < 0$. This shows that $H_\lambda(t)$ increases for $-\infty < t < 0$ and so it has at most one zero on that interval. Moreover $H_\lambda(t) \to -\infty$ as $t \to -\infty$ so it has exactly one zero on this interval.

(c) In order to deal with the cases $n \geq 1$, we use the Sturm comparison and separation theorems in a slightly extended form which covers zeros "at infinity" of principal solutions.

Lemma 3.2. (a) *Let $f(t)$ and $F(t)$ be piece-wise continuous functions in $t_0 < t < \infty$ with $f(t) \leq F(t)$, but $f(t) \not\equiv F(t)$. Let the functions $y(t)$ and $Y(t)$, both not identically zero, satisfy the differential equations*

$$y'' + f(t)y = 0, \tag{3.2}$$

and

$$Y'' + F(t)Y = 0, \tag{3.3}$$

respectively, on $t_0 < t < \infty$. Then $Y(t)$ has at least one zero between each two zeros of $y(t)$. (b) *Suppose, in addition, that both equations are nonoscillatory at $+\infty$ and that y and Y are principal solutions at $+\infty$ of the respective differential equations. Let t_1 be the largest zero of $y(t)$. Then Y has a zero which is larger than t_1 provided $f(t) \not\equiv F(t)$ on (t_1, ∞).*

Remarks on proof. The result (a) is the well-known Sturm comparison theorem (see, e.g., [26]). To prove part (b), consider the expression $w(t) = y'(t)Y(t) - y(t)Y'(t)$. By our assumption $f(t) \not\equiv F(t)$ on (t_1, ∞), we have $w(t) \not\equiv 0$. Let $t_0 \in (t_1, \infty)$, with $w(t_0) \neq 0$. If $Y(t_0) = 0$, we are finished. Suppose $Y(t_0) > 0$. (If $Y(t_0) < 0$, we consider $-Y(t)$ instead.) Let $z(t)$ be another solution of (3.2) which satisfies $z(t_0) = Y(t_0) > 0$, $z'(t_0) = Y'(t_0)$. Clearly $y(t)$ and $z(t)$ are linearly independent solutions of (3.2). By Lemma 2.1, $z(t)$ vanishes at $\tilde{t}_1$ in (t_1, ∞). By the Sturm comparison theorem, $z(t) \geq Y(t)$ as long as $Y(t) > 0$. Hence, $Y(t)$ has a zero between t_0 and $\tilde{t}_1$.

We use $h(\lambda)$ and $\tilde{h}(\lambda)$ for the largest and smallest real zeros of $H_\lambda(t)$. Clearly $\tilde{h}(n) = -h(n)$. We apply Lemma 3.2 to pairs of differential equation of the form (1.7) with respective solutions of the form $e^{-t^2/2}H_n(t)$. The conclusion of Lemma 3.2 shows that the number of zeros is nondecreasing as λ increases and that each zero which is continuous in λ is an increasing function of λ. Thus there are at least n zeros of $H_\lambda(t)$ in $(-h(n), \infty)$ and

$$h(n) < h(\lambda) < h(n+1).$$

To show that $\tilde{h}(\lambda)$ is in $(-\infty, -h(n))$, we consider the solution $e^{-t^2/2}H_\lambda(-t)$ of (1.7). For $n < \lambda < n+1$, it is linearly independent of $e^{-t^2/2}H_\lambda(t)$, hence not a principal solution, so from Lemma 2.1, it has a zero on $(h(\lambda), \infty)$. Thus $h(\lambda) < -\tilde{h}(\lambda) < \infty$ or

$$-\infty < \tilde{h}(\lambda) < -h(\lambda) < -h(n).$$

All of this shows that $H_\lambda(t)$ has *at least* $n+1$ real zeros. To see that it has exactly $n+1$ of them, we observe that the zeros are continuous and strictly increasing functions of λ and that there are $n+1$ of them when $\lambda = n+1$.

This completes the proof of Theorem 3.1.

In fact, we can compare all the zeros of $H_\lambda(t)$ with those of $H_\lambda(-t)$ for $n < \lambda < n+1$. Using the notation $h_1(\lambda), \ldots, h_{n+1}(\lambda)$ for the zeros of $H_\lambda(t)$ in decreasing order, we have

$$h_{n+1}(\lambda) < -h_1(\lambda) < h_n(\lambda) < -h_2(\lambda) < \ldots < h_1(\lambda) < -h_{n+1}(\lambda).$$

For later purposes, we need the following information about $h(\lambda)$.

Lemma 3.3. *Let $h(\lambda)$ denote the largest real zero of $H_\lambda(t)$. Then*

$$h(0.17) < -1.13. \tag{3.4}$$

Proof. First, we establish the inequalities, for $0 < \lambda < 1, \ \ t > 0$:

$$ {}_1F_1(-\frac{\lambda}{2} + \frac{1}{2}, \frac{3}{2}; t) \leq 1 + \frac{1-\lambda}{3} t + \frac{1}{5}(e^t - t - 1), \tag{3.5}$$

$$ {}_1F_1(-\frac{\lambda}{2}, \frac{1}{2}; t) \geq 1 - \lambda t - \frac{\lambda(2-\lambda)}{3}(e^t - t - 1). \tag{3.6}$$

To prove (3.5), we note that the remainder after the first two terms in the power series for ${}_1F_1$ there is less, term–by–term, than the series for $\frac{1}{5}(e^t - t - 1)$. To prove (3.6), write the series as $1 - \lambda t S_2$ where S_2 is a series of positive terms. The remainder after the first term of S_2 is majorized by the power series for

$$\frac{2-\lambda}{3} t^{-1}(e^t - t - 1).$$

Thus (3.6) follows. Using (3.5) and (3.6), we see that, for $t < 0$, $H_\lambda(t)$ exceeds a positive multiple of the more elementary function

$$\cos\frac{\lambda\pi}{2}\,\Gamma(\frac{\lambda}{2} + \frac{1}{2})\left[1 - \lambda t^2 - \frac{\lambda(2-\lambda)}{3}(e^{t^2} - t^2 - 1)\right] +$$

$$+2t\ \sin\frac{\lambda\pi}{2}\,\Gamma(\frac{\lambda}{2} + 1)\left[1 + \frac{1-\lambda}{3} t^2 + \frac{1}{5}(e^{t^2} - t^2 - 1)\right]$$

so that any zero of this function on $(-\infty, 0)$ will be an upper bound for a zero of $H_\lambda(t)$. In case $\lambda = 0.17$, this function is found (numerically) to have a zero for $t = -1.1388$, approximately. This proves the lemma.

4 A Schläfli–type formula

We begin by considering the differential equations

$$y_\lambda'' + (2\lambda + 1 - t^2) y_\lambda = 0 \tag{4.1}$$

$$y_\mu'' + (2\mu + 1 - t^2) y_\mu = 0 \tag{4.2}$$

satisfied by $y_\lambda(t) = e^{-t^2/2} H_\lambda(t)$ and $y_\mu(t) = e^{-t^2/2} H_\mu(t)$. Multiplying (4.1) by y_μ and (4.2) by y_λ and subtracting, we get

$$D_t[y_\lambda' y_\mu - y_\lambda y_\mu'] + 2(\lambda - \mu) y_\lambda y_\mu = 0, \tag{4.3}$$

and hence

$$y_\lambda'(t) y_\mu(t) - y_\lambda(t) y_\mu'(t) = 2(\lambda - \mu)\int_t^\infty y_\lambda(s) y_\mu(s)\, ds, \tag{4.4}$$

since it is clear from (2.3) and (1.3) that $y_\lambda'(t) y_\mu(t) - y_\lambda(t) y_\mu'(t) \to 0$ as $t \to +\infty$. Also, from (1.3), we have

$$y_\lambda'(t) = 2\lambda y_{\lambda-1}(t) - t y_\lambda(t), \tag{4.5}$$

so

$$\lambda y_{\lambda-1} y_\mu - \mu y_\lambda y_{\mu-1} = (\lambda - \mu)\int_t^\infty y_\lambda(s) y_\mu(s)\, ds. \tag{4.6}$$

Dividing (4.6) by $\lambda - \mu$ and letting $\mu \to \lambda$, we get

$$y_\lambda y_{\lambda-1} + \lambda y_\lambda \frac{\partial y_{\lambda-1}}{\partial \lambda} - \lambda \frac{\partial y_\lambda}{\partial \lambda} y_{\lambda-1} = \int_t^\infty [y_\lambda(s)]^2 \, ds. \tag{4.7}$$

When we choose $t = c$, a zero of y_λ, this becomes

$$-\lambda \left. \frac{\partial y_\lambda(t)}{\partial \lambda} \right|_{t=c} y_{\lambda-1}(c) = \int_c^\infty [y_\lambda(s)]^2 \, ds. \tag{4.8}$$

Since c is a zero of $H_\lambda(t)$, we have $y_\lambda(c) = 0$, so differentiating with respect to λ, we have

$$\left. \frac{\partial y_\lambda(t)}{\partial \lambda} \right|_{t=c} + y'_\lambda(c) \frac{dc}{d\lambda} = 0. \tag{4.9}$$

Combining this with (4.5) and (4.8), we get

$$[y'_\lambda(c)]^2 \frac{dc}{d\lambda} = 2 \int_c^\infty y_\lambda^2(t) dt, \tag{4.10}$$

or

$$\frac{dc}{d\lambda} = 2[H'_\lambda(c)]^{-2} \int_c^\infty e^{c^2-t^2} H_\lambda^2(t) dt. \tag{4.11}$$

This formula shows again that the zeros increase with λ. It is analogous to that of Schläfli [25], [27, p.508]

$$\frac{dj}{d\nu} = \frac{2j}{\nu J_{\nu+1}^2(j)} \int_0^j s^{-1} J_\nu^2(s) ds, \quad \nu > 0, \tag{4.12}$$

for zeros of the Bessel function $J_\nu(z)$.

The form of equation (4.11) suggests strongly that it can be derived from the Hellmann–Feynman theorem in the way described in [17], [18]. We have not succeeded in deriving (4.11) in this way.

5 A cross-product of Hermite functions

We seek an analogue for Hermite functions of a formula due to Watson [27, Ch. 13],

$$\frac{dc}{d\nu} = 2c \int_0^\infty K_0(2c \sinh t) e^{-2\nu t} dt, \tag{5.1}$$

for a zero c of a cylinder function (linear combination of $J_\nu(z)$ and $Y_\nu(z)$). As well as applying to a broader class of functions, formula (5.1) is more useful than (4.12). The simple nature (positive, decreasing, etc.) of the modified Bessel function $K_0(t)$, has led to the use of (5.1) in several discussions of monotonicity, convexity, etc. of the zeros; see [6], [7], [8], [9] and references therein.

In [10] we considered a pair of linearly independent solutions $x(t, \lambda)$, $y(t, \lambda)$ of the differential equation

$$z'' + q(t, \lambda) z = 0, \quad t \in I, \tag{5.2}$$

satisfying the initial conditions

$$x(a, \lambda) = \phi(\lambda), \; y(a, \lambda) = 0, \tag{5.3}$$

$$x_t(a, \lambda) = 0, \; y_t(a, \lambda) = 1/\phi(\lambda). \tag{5.4}$$

for each λ in some interval J. The function $\phi(\lambda)$ was supposed to be positive and differentiable on J. We suppose, as in [10], that $q(t,\lambda)$ is of class C^1 in a domain of (t,λ)–space which includes $I \times J$. Clearly

$$\begin{vmatrix} x & y \\ x_t & y_t \end{vmatrix} = 1, \quad \text{for each } \nu \in J. \tag{5.5}$$

Consider the linear combination

$$z(t,\lambda) = \quad \cos\alpha\, x(t,\lambda) - \sin\alpha\, y(t,\lambda). \tag{5.6}$$

Let $c = c(\lambda,\alpha)$ be a zero of $z(t,\lambda)$, for some fixed α. Then, as in [10],

$$\frac{dc}{d\lambda} = -Q(c,\lambda) \tag{5.7}$$

where

$$Q(t,\lambda) = x(t,\lambda)\frac{\partial y(t,\lambda)}{\partial\lambda} - y(t,\lambda)\frac{\partial x(t,\lambda)}{\partial\lambda} \tag{5.8}$$

Now we consider equation (1.7), the special case $q(t,\lambda) = 2\lambda + 1 - t^2$ of (5.2). We introduce two solutions $x(t,\lambda)$, $y(t,\lambda)$ of (1.7) as follows:

$$x(t,\lambda) = \phi(\lambda)\, e^{-t^2/2}\, {}_1F_1(-\frac{\lambda}{2},\frac{1}{2};t^2);\ y(t,\lambda) = \frac{1}{\phi(\lambda)}e^{-t^2/2}t\, {}_1F_1(-\frac{\lambda}{2}+\frac{1}{2},\frac{3}{2};t^2), \tag{5.9}$$

where

$$\phi(\lambda) = \left[\frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right)/\Gamma\left(\frac{\lambda+2}{2}\right)\right]^{1/2}. \tag{5.10}$$

These solutions satisfy the conditions (5.3), (5.4) and (5.5). Using standard identities for the gamma function, we get, by (1.2), (1.6),

$$e^{-t^2/2}H_\lambda(t) = (\pi^{-1/2}2^{\lambda+1}\Gamma(\lambda+1))^{1/2}\left[\cos\frac{\lambda\pi}{2}\ x(t,\lambda) + \sin\frac{\lambda\pi}{2}\ y(t,\lambda)\right], \tag{5.11}$$

$$e^{-t^2/2}G_\lambda(t) = (\pi^{-1/2}2^{\lambda+1}\Gamma(\lambda+1))^{1/2}\left[-\sin\frac{\lambda\pi}{2}\ x(t,\lambda) + \cos\frac{\lambda\pi}{2}\ y(t,\lambda)\right]. \tag{5.12}$$

Theorem 5.1. *With $x(t,\lambda)$ and $y(t,\lambda)$ as in* (5.9)*, and $Q(t,\lambda)$ as in* (5.8)*, we have*

$$Q(t,\lambda) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-(2\lambda+1)\tau + t^2\tanh\tau}\operatorname{erf}(t(\tanh\tau)^{1/2})\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}}. \tag{5.13}$$

Proof. We note that results of [10] show that Q satisfies the inhomogeneous differential equation

$$w''' + 4(2\lambda+1-t^2)w' - 4tw = 4, \tag{5.14}$$

and the initial conditions

$$Q(0,\lambda) = Q_{tt}(0,\lambda) = 0,\ Q_t(0,\lambda) = \frac{1}{2}\left[\Psi\left(\frac{\lambda+2}{2}\right) - \Psi\left(\frac{\lambda+1}{2}\right)\right], \tag{5.15}$$

where Ψ is the logarithmic derivative of the gamma function. So it remains to show that $w(t,\lambda)$, the function on the right-hand side of (5.13), satisfies these same conditions.

We have

$$w(t,\lambda)=\int_0^\infty e^{-(2\lambda+1)\tau}f(t(\tanh\tau)^{1/2})\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}},\tag{5.16}$$

where

$$f(u)=\frac{\sqrt{\pi}}{2}e^{u^2}\mathrm{erf}u=e^{u^2}\int_0^u e^{-t^2}dt.\tag{5.17}$$

Clearly,

$$[D_u^2-2uD_u-2]f(u)=0,\ f(0)=0,\ f^{'}(0)=1,\tag{5.18}$$

Writing

$$u=t(\tanh\tau)^{1/2},\quad \mu(\tau)=(\sinh\tau\cosh\tau)^{-1/2}$$

we have, using w for $w(t,\lambda)$,

$$w=\int_0^\infty e^{-(2\lambda+1)\tau}f(u)\mu(\tau)d\tau,\tag{5.19}$$

$$w_t=\int_0^\infty e^{-(2\lambda+1)\tau}f^{'}(u)(\tanh\tau)^{1/2}\mu(\tau)d\tau,\tag{5.20}$$

$$w_{tt}=\int_0^\infty e^{-(2\lambda+1)\tau}f^{''}(u)(\tanh\tau)\mu(\tau)d\tau,\tag{5.21}$$

and

$$w_{ttt}=\int_0^\infty e^{-(2\lambda+1)\tau}f^{'''}(u)(\tanh\tau)^{3/2}\mu(\tau)d\tau.\tag{5.22}$$

Using integration by parts, we see from (5.20) that

$$\begin{aligned}2(2\lambda+1)w_t &= 2f^{'}(0)+\int_0^\infty e^{-(2\lambda+1)\tau}f^{''}(u)\,t\,\mathrm{sech}^2\tau\mu(\tau)d\tau \\ &\quad -2\int_0^\infty e^{-(2\lambda+1)\tau}f^{'}(u)(\tanh\tau)^{3/2}\mu(\tau)d\tau.\end{aligned}\tag{5.23}$$

Now, using (5.22), (5.23), (5.20) and (5.19), we find that the differential equation (5.14) is satisfied, provided that

$$2f^{'}(0)+\frac{1}{2}\int_0^\infty e^{-(2\lambda+1)\tau}(\tanh\tau)^{-1/2}F(\tau,u)\mu(\tau)d\tau=2,\tag{5.24}$$

where

$$F(\tau,u)=(\tanh\tau)^2[f^{'''}(u)-2uf^{''}(u)-4f^{'}(u)]+2u[f^{''}(u)-2uf^{'}(u)-2f(u)].$$

But

$$F(\tau,u)=[(\tanh\tau)^2D_u+2u][D_u^2-2uD_u-2]f(u)=0,$$

where the last equality follows from (5.18). Thus the differential equation (5.14) is satisfied by $w(t,\lambda)$. We also have $w(0,\lambda)=0$, and

$$\begin{aligned}w_t(0,\lambda) &= f^{'}(0)\int_0^\infty e^{-(2\lambda+1)\tau}\mathrm{sech}\tau d\tau \\ &= \frac{1}{2}\left[\Psi\left(\frac{\lambda+2}{2}\right)-\Psi\left(\frac{\lambda+1}{2}\right)\right],\end{aligned}\tag{5.25}$$

[11, p. 163, (7)], where Ψ is the logarithmic derivative of the gamma function. Thus we have $w_t(0,\lambda)=-2\phi'(\lambda)/\phi(\lambda)$, where $\phi(\lambda)$ is given by (5.10). Also, from (5.21) and the differential equation (5.18) we get $f''(0)=0$ and $w_{tt}(0,\lambda)=0$. Hence $Q(t,\lambda)\equiv w(t,\lambda)$.

This completes the proof of Theorem 5.1.

6 Applications to ${}_1F_1$ solutions

Using (5.7) and Theorem 5.1, we are led to the following result:

Theorem 6.1. *Let $c(\lambda)$ be a zero of a linear combination* (5.6) *of $x(t,\lambda)$ and $y(t,\lambda)$, as given by* (5.9). *Then*

$$\frac{dc}{d\lambda} = -\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-(2\lambda+1)\tau} f(c\sqrt{\tanh\tau})\frac{d\tau}{\sqrt{\sinh\tau\,\cosh\tau}}, \tag{6.1}$$

where

$$f(u) = \frac{\sqrt{\pi}}{2}e^{u^2}\mathrm{erf}u = e^{u^2}\int_0^u e^{-t^2}dt.$$

This may be used to discover monotonicity properties of the zeros:

Theorem 6.2. *Let $c(\lambda)$ be a positive zero of a linear combination* (5.6) *as in* Theorem 6.1. *Then*

(i) $c(\lambda)$ *decreases as λ increases.*

(ii) $c(\lambda)$ *is a convex function of λ.*

(iii) $\sqrt{2\lambda+1}\;c(\lambda)$ *decreases as λ increases for all $c(\lambda)$ satisfying $\sqrt{2\lambda+1}\;c(\lambda) \geq 3/2$.*

In particular, these results apply to the functions which interpolate the zeros of either the even or the odd Hermite polynomials as in Figure 2.

Proof of Theorem 6.2. Part (i) is obvious from (6.1). Also f is a strictly increasing function so $dc/d\lambda$ is an increasing function of λ, and hence we get (ii). To show (iii), we need the following three facts:

(a) $f(u)/u$ is strictly increasing for $u > 0$;

(b)$f(u) > u + (2/3)u^3$;

(c)$\int_0^\infty e^{-(2\lambda+1)t}(2\lambda+1+\tanh t)\mathrm{sech}t\,dt = 1$.

Now (a) is an easy consequence of the differential equation in (5.18). The same equation, when differentiated, gives, for $u > 0$, $f'''(u) = 2uf''(u) + 4f'(u) > 4f'(0) = 4$. By repeated integration, we get $f''(u) > 4u$, $f'(u) > 1 + 2u^2$, and finally (b). To prove (c), we only need to notice that the integrand on its left–hand side is the derivative with respect to t of $-e^{-(2\lambda+1)t}\mathrm{sech}t$.

To prove (iii), we need to show that

$$-\frac{2\lambda+1}{c}\frac{dc}{d\lambda} > 1. \tag{6.2}$$

We have

$$\begin{aligned}
-\frac{2\lambda+1}{c}\frac{dc}{d\lambda} &= \int_0^\infty (2\lambda+1)e^{-(2\lambda+1)\tau}\frac{f(u)}{u}\mathrm{sech}\tau\,d\tau \\
&\geq \int_0^\infty (2\lambda+1)e^{-(2\lambda+1)\tau}\frac{f(\sqrt{(3/2)\tanh\tau/(2\lambda+1)})}{\sqrt{(3/2)\tanh\tau/(2\lambda+1)}}\mathrm{sech}\tau\,d\tau,
\end{aligned}$$

on using (a) and the consequence

$$u = c\sqrt{\tanh\tau} \geq \sqrt{(3/2)\tanh\tau/(2\lambda+1)} \tag{6.3}$$

of the assumption $\sqrt{2\lambda+1}c(\lambda) \geq 3/2$. Thus we get, on using (b),

$$-\frac{2\lambda+1}{c}\frac{dc}{d\lambda} > \int_0^\infty e^{-(2\lambda+1)\tau}(2\lambda+1+\tanh\tau)\mathrm{sech}\tau d\tau = 1, \tag{6.4}$$

the last inequality following from (c). This completes the proof of (iii) and of Theorem 6.2.

From [26, p. 130], we have for the first positive zeros $t_{1,n}$ of the Hermite polynomials,

$$\lim_{n\to\infty} t_{1,n}\sqrt{2n+1} = \begin{cases} \pi/2, & n \text{ even,} \\ \pi, & n \text{ odd.} \end{cases}$$

i.e., Theorem 7.2 (iii) can be applied, and we get

$$\sqrt{2n+1}t_{1,n} > \begin{cases} \frac{\pi}{2}, & n \text{ even,} \\ \pi, & n \text{ odd,} \end{cases}$$

We have

$$\left.\begin{matrix} \sqrt{5/2} \\ \sqrt{21/2} \end{matrix}\right\} \geq t_{1,n}\sqrt{2n+1} > t_{1,n+2}\sqrt{2n+5} > \ldots > \begin{cases} \frac{\pi}{2}, & n \text{ even,} \\ \pi, & n \text{ odd}(\geq 3). \end{cases}$$

Also it follows that $\{t_{1,2}, t_{1,4}, t_{1,6}, \ldots\}$ and $\{t_{1,3}, t_{1,5}, t_{1,7}, \ldots\}$ are convex decreasing sequences. Similar statements hold also for the second, third, ..., positive zeros as in [26, p. 130].

7 Application to Hermite functions

We now want to discuss the zeros $\tilde{c}(\lambda)$ of the linear combination

$$\cos\alpha \;\; H_\lambda(t) - \sin\alpha \;\; G_\lambda(t) = 0 \tag{7.1}$$

or, using (5.11) and (5.12),

$$\cos(\alpha - \frac{\lambda\pi}{2}) \;\; x(t,\lambda) - \sin(\alpha - \frac{\lambda\pi}{2}) \;\; y(t,\lambda) = 0. \tag{7.2}$$

Proceeding as in §5, we differentiate (7.2) with respect to λ to find that at a zero $\tilde{c}$ of the linear combination (7.1), there holds:

$$\begin{aligned} \cos(\alpha - \frac{\lambda\pi}{2}) \, [\, x'(\tilde{c},\lambda)\frac{d\tilde{c}}{d\lambda} + \frac{\partial x(\tilde{c},\lambda)}{\partial\lambda} \quad &+ \quad \frac{\pi}{2} y(\tilde{c},\lambda) \,] \\ - \sin(\alpha - \frac{\lambda\pi}{2}) \, [\, y'(\tilde{c},\lambda)\frac{d\tilde{c}}{d\lambda} \quad &+ \quad \frac{\partial y(\tilde{c},\lambda)}{\partial\lambda} - \frac{\pi}{2} x(\tilde{c},\lambda) \,] = 0. \end{aligned} \tag{7.3}$$

Since

$$\cos^2(\alpha - \frac{\lambda\pi}{2}) + \sin^2(\alpha - \frac{\lambda\pi}{2}) \neq 0,$$

we see that, in order for (7.2) and (7.3) to hold simultaneously, we must have

$$\frac{d\tilde{c}}{d\lambda} + Q(\tilde{c},\lambda) - \frac{\pi}{2}(x^2+y^2) = 0. \tag{7.4}$$

Making use of (5.11), (5.12) and (1.8), we find

$$\frac{\pi}{2}(x^2+y^2) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-(2\lambda+1)\tau + t^2\tanh\tau} \frac{d\tau}{\sqrt{\sinh\tau\,\cosh\tau}}.$$

Then by (5.13) of Theorem 5.1 we are led to the following theorem.

Theorem 7.1. *Let* $\tilde{c}$ *be a zero of the linear combination* (7.1)*. Then*

$$\frac{d\tilde{c}}{d\lambda} = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-(2\lambda+1)\tau}\phi(\tilde{c}\sqrt{\tanh\tau})\frac{d\tau}{\sqrt{\sinh\tau\,\cosh\tau}} \tag{7.5}$$

where

$$\phi(x) = e^{x^2}\operatorname{erfc}(x).$$

Here erfc is the complementary error function:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.$$

Now, from [1, 7.4.2]

$$\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2-2xt}dt$$

so ϕ is completely monotonic on $(-\infty,\infty)$, i.e. $(-1)^n\phi^{(n)}(x) \geq 0,\ n = 0,1,\ldots$. This observation enables us to establish the following Corollary.

Corollary 7.2. *Let* $\tilde{c}(\lambda)$ *be as in* Theorem 7.1*. Then* $\tilde{c}'(\lambda)$ *is completely monotonic on its interval of definition.*

Proof. We proceed by induction. Obviously $\tilde{c}'(\lambda) > 0$. Now suppose that

$$(-1)^k\tilde{c}^{(k+1)}(\lambda) \geq 0,\ k = 1,2,\ldots,n.$$

Then, using a standard lemma on composition of n–times monotonic functions [3], [22], [21, Lemma 2.1], we see that $\phi(\tilde{c}\sqrt{\tanh\tau})$ is an $(n+1)$–times monotonic function of λ for each fixed τ. Also $e^{-(2\lambda+1)\tau}$ is a completely monotonic function of λ for each τ. Thus the product $e^{-(2\lambda+1)\tau}\phi(\tilde{c}\sqrt{\tanh\tau})$ is $(n+1)$–times monotonic for each τ and from the representation (7.5) we see that

$$(-1)^k\tilde{c}^{(k+1)}(\lambda) \geq 0,\ k = 1,2,\ldots,n+1.$$

This completes the inductive proof.

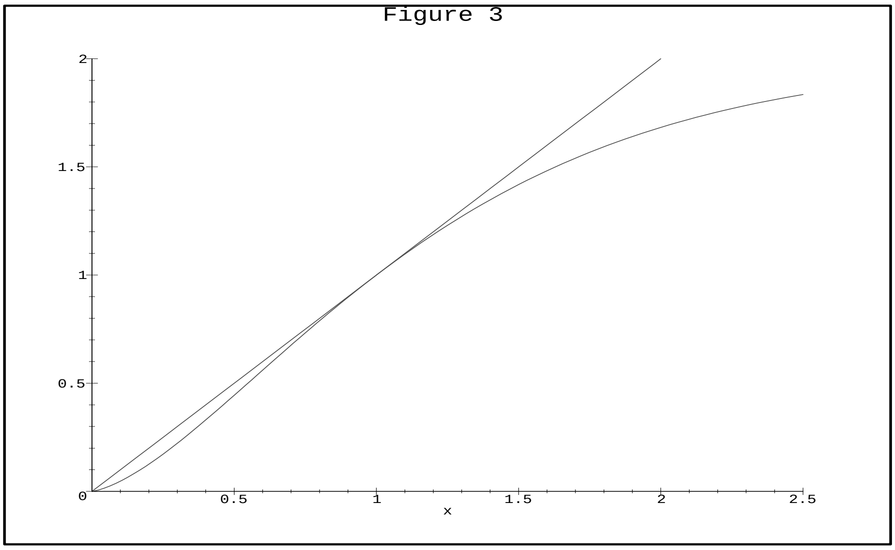
A further corollary is that if $x_{nk},\ k = 1,\ldots,n$ are the zeros in decreasing order of $H_n(t)$, then for fixed k, with $\Delta_{(n)}\mu_{nk} = \mu_{n+1,k} - \mu_{nk}$, we have

$$(-1)^{r+1}\Delta^r_{(n)}x_{nk} > 0, r = 1,2,\ldots.$$

We may compare this with the result of Durand [4, pp. 371–372] that for the positive zeros, with fixed n,

$$(-1)^r\Delta^r_{(k)}x_{nk} > 0, r = 1,2,\ldots.$$

(It should be noted that Durand lists the positive zeros in increasing order, so the $(-1)^r$ does not occur in his formulas.)



8 The Hayman–Ortiz inequality

W. K. Hayman and E. L. Ortiz [16] give a proof of the following result (and of a stronger one for λ close to 0) which is partly analytic and partly numerical and mention the desirability of giving an analytic proof of the whole result. The inequality is important in the study of the growth of subharmonic functions ([13], [16]).

Theorem 8.1. *Let* $h = h(\lambda)$ *be the largest zero of the Hermite function* $H_\lambda(t)$*. Then* $\lambda \geq 1 + \operatorname{erf}[h(\lambda)], \;\; 0 < \lambda < \infty$ *with equality only for* $\lambda = 1$*, and for* $\lambda \to 0^+$*.*

Remark. This result has been proved by Hayman and Ortiz by dividing the interval $-\infty < h < \infty$, or equivalently, $0 < \lambda < \infty$ up into a number of subintervals for which separate proofs are given. The intervals for which they provided only numerical proofs are $-1.1 \leq h \leq -0.1$ and $0.1 \leq h \leq 1/\sqrt{2}$. Using our results, we can give quite a short proof for $h \geq 0$ and a more complicated one for $-1.1 \leq h < 0$. Thus the theorem is proved analytically for all h. Figure 3 gives graphs of $\lambda - 1$ and $\operatorname{erf}[h(\lambda)]$, for $0 < \lambda < 2$.

Proof of Theorem 8.1. Since $H_1(t) = t$ we have

$$h(1) = 0. \tag{8.1}$$

With $\alpha = 0$ in (7.1), we get $h(\lambda) = \tilde{c}(\lambda)$ and we can apply Theorem 7.1 and Corollary 7.2 to $h(\lambda)$. We have

$$h'(1) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-3\tau}\frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}} = \frac{\sqrt{\pi}}{2}. \tag{8.2}$$

Case (i): $h > 0$. Writing $G(\lambda) = \lambda - 1 - \operatorname{erf}(h(\lambda))$, we have

$$\exp[h^2(\lambda)]G'(\lambda) = \exp[h^2(\lambda)] - \frac{2}{\sqrt{\pi}}h'(\lambda). \tag{8.3}$$

Thus, for $h > 0$ or $\lambda > 1$, both terms on the right of (8.3) and hence also $\exp[h^2(\lambda)]G'(\lambda)$ are increasing functions of λ. Thus the latter function is convex in λ, $\lambda \geq 1$. Also, using (8.1) and (8.2), $G(1) = G'(1) = 0$ and so $G(\lambda) > 0$ for $\lambda > 1$.

Case (ii): $h < 0$. The proof is more difficult in this case. Observing that $h'(\lambda)$ is a completely monotonic function of λ on $(0, \infty)$, it is analytic as long as h is finite so we have the series expansion

$$h(\lambda) = \sum_{n=1}^{\infty} h^{(n)}(1) \frac{(\lambda - 1)^n}{n!}.$$

For $0 < \lambda < 1$, the terms here are all negative so we get the upper bound

$$h(\lambda) < P(\lambda) = h^{'}(1)(\lambda - 1) + \frac{1}{2} h^{''}(1)(\lambda - 1)^2 + \frac{1}{6} h^{'''}(1)(\lambda - 1)^3. \tag{8.4}$$

By differentiation with respect to λ in (7.5), we obtain

$$h^{''}(1) = -h^{'}(1)(A + 2B)$$

$$h^{'''}(1) = 4h^{'}(1)(C + D) + 2\left(h^{'}(1)\right)^3 E - h^{''}(1)A$$

where

$$A = \int_0^{\infty} e^{-3\tau} \frac{d\tau}{\cosh\tau},$$

$$B = \int_0^{\infty} \tau e^{-3\tau} \frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}},$$

$$C = \int_0^{\infty} \tau^2 e^{-3\tau} \frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}},$$

$$D = \int_0^{\infty} \tau e^{-3\tau} \frac{d\tau}{\cosh\tau},$$

and

$$E = \int_0^{\infty} e^{-3\tau} \frac{\tanh\tau}{\sqrt{\sinh\tau\cosh\tau}}\, d\tau.$$

Clearly, $A = 1 - \log 2$, and using [14, 3.452.4] we get $B = (1 - \log 2)/2$. Also the transformation

$$e^{-4\tau} = 1 - t^2$$

shows that

$$C = \int_0^{\infty} \tau^2 e^{-3\tau} \frac{d\tau}{\sqrt{\sinh\tau\cosh\tau}} = \frac{1}{16} \int_0^1 [\log(1 - t^2)]^2\, dt$$

which, using integration by parts, is equal to

$$-\frac{1}{4} \int_0^1 \frac{t}{1+t} \log(1 - t^2)\, dt$$

that is

$$\frac{1}{4} \left[\int_0^1 \frac{1}{1+t} \log(1 - t) dt + \int_0^1 \frac{1}{1+t} \log(1 + t) dt - \int_0^2 \log s\, ds \right].$$

The first integral here can be evaluated using a series expansion and [14, 0.241]:

$$\int_0^1 \log s\, \frac{1}{2 - s}\, ds = -\sum_{m=1}^{\infty} \frac{1}{2^m m^2} = -\frac{\pi^2}{12} + \frac{1}{2} (\log 2)^2.$$

The two other integrals are elementary so we get

$$C = \frac{1}{4} [-\frac{\pi^2}{12} + (\log 2)^2 - 2\log 2 + 2].$$

Also

$$D = -\frac{1}{2}\int_0^1 \log s \frac{s}{1+s}\, ds = \frac{1}{2}[\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \ldots] = \frac{1}{2}[1 - \frac{\pi^2}{12}],$$

and the substitution $u = \tanh^{1/2}\tau$ gives

$$E = \int_0^1 \frac{2u^2(1-u^2)}{(1+u^2)^2}\, du = \pi - 3.$$

Hence

$$P(\lambda) = \frac{\sqrt{\pi}}{2}(\lambda - 1)[1 - a(\lambda - 1) + b(\lambda - 1)^2]$$

where

$$a = 1 - \log 2 = 0.3068528\ldots,$$

and

$$b = \frac{\pi^2}{24} - \frac{\pi}{4} + \frac{1}{2} + \frac{1}{2}(1 - \log 2)^2 = 0.1729146....$$

Now we show that on the interval $0.17 \leq \lambda < 1$,

$$\psi(\lambda) = \lambda - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{P(\lambda)} e^{-s^2} ds > 0,$$

which, since $h(\lambda) < P(\lambda)$, implies the Hayman–Ortiz inequality on this interval. Clearly we have

$$\psi' = 1 - \frac{2}{\sqrt{\pi}} P' e^{-P^2}$$

$$\psi'' = \frac{2}{\sqrt{\pi}}(2P(P')^2 - P'')e^{-P^2}.$$

On the other hand $\psi(1) = \psi'(1) = 0$, $\psi''(1) = -(2/\sqrt{\pi})P''(1) > 0$; hence $\psi(\lambda) > 0$ certainly holds in some left neighbourhood of $\lambda = 1$. Moreover,

$$\begin{aligned} \frac{2}{\sqrt{\pi}}(2P(P')^2 - P'') &= \frac{\pi}{2}\left[(L - aL^2 + bL^3)(1 - 2aL + 3bL^2)^2 - \frac{2}{\pi}(-2a + 6bL)\right] \\ &= \frac{\pi}{2}\left[\frac{4a}{\pi} + (1 - \frac{12b}{\pi})L - 5aL^2 + (8a^2 + 7b)L^3 - (4a^3 + 22ab)L^4 + \right. \\ &\quad \left. + (16a^2b + 15b^2)L^5 - 21ab^2L^6 + 9b^3L^7\right] \end{aligned}$$

where $L = \lambda - 1$. The derivative of this function has alternating coefficients and hence is positive for $L < 0$. Thus $2P(P')^2 - P''$ is strictly increasing, consequently $\psi(\lambda)$ is concave for small values of λ and then convex for larger values in the interval $0 < \lambda < 1$. Thus if $\psi(\lambda)$ is positive for a single value of λ it will be positive for all larger values on this interval. Since $P(0.17) = -1.0105\ldots$, we have

$$\begin{aligned} \psi(0.17) &> 0.17 - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{-1.01} e^{-s^2} ds = 0.17 - (1 - \mathrm{erf}(1.01)) \\ &= 0.8468\ldots - 0.83 > 0, \end{aligned}$$

and we conclude that $\psi(\lambda) > 0$ for $\lambda \in [0.17, 1)$ which was to be proved.

It remains to show that the interval $0.17 < \lambda < 1$ covers all of the interval $-1.1 \leq h < 0$. But this follows since h is a monotonic function of λ and $h(0.17) < -1.1$ (Lemma 2.4).

Remark. The interest in the inequality just proved is that it shows that $\lambda \geq 2(1-S)$, where

$$S = S(h) = \frac{1}{\sqrt{\pi}} \int_h^\infty e^{-t^2} dt,$$

and the lower bound $2(1-S)$ is a convex decreasing function of S for $0 < S < 1$. See [13], [16] for the relevance of this to the theory of subharmonic functions. Hayman and Ortiz [16], by a combination of analytic and numerical techniques, supplied the more complicated lower bound

$$\lambda \geq \begin{cases} 2(1-S), & \frac{1}{4} < S < 1, \\ \frac{1}{2}\log(1/S), & 0 < S < \frac{1}{4}. \end{cases}$$

The question naturally arises whether λ is itself a convex decreasing function of S. It is certainly decreasing as can be seen from

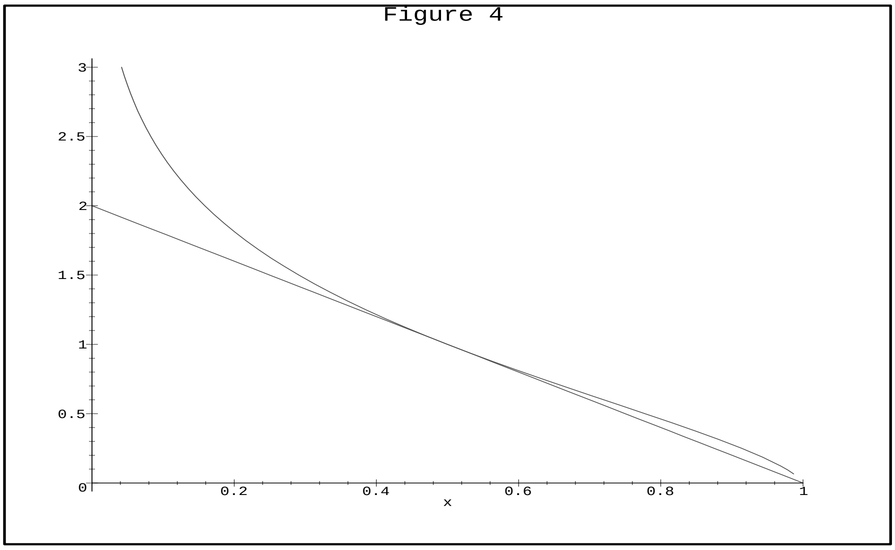
$$\frac{d\lambda}{dS} = \frac{d\lambda/dh}{dS/dh} = -\sqrt{\pi} e^{h^2} \frac{d\lambda}{dh}.$$

The function $\lambda(S)$ is not convex on the whole interval $0 < S < 1$ since $\lambda(S) \geq 2(1-S)$, with equality at 1/2 and 1, so that the graph of $\lambda = \lambda(S)$ actually lies above the chord joining the points $S = 1/2$ and $S = 1$. However, it is convex for $0 < S < \frac{1}{2}$. To see this, it is enough to show that S is a convex decreasing function of λ for $\lambda > 1$. This can be seen from the representation

$$\frac{d^2S}{d\lambda^2} = -\frac{1}{\sqrt{\pi}} e^{-h^2} h''(\lambda) + \frac{2}{\sqrt{\pi}} e^{-h^2} [h'(\lambda)]^2 h(\lambda),$$

Theorem 9.2, and the fact that $h(\lambda) > 0$ for $\lambda > 1$.

Figure 4 gives a graph of λ versus S.



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