Abstract. We study the asymptotic behaviour of random walks in i.i.d. random environments on $\mathbb{Z}^d$. The environments need not be elliptic, so some steps may not be available to the random walker. Our results include generalisations (to the non-elliptic setting) of 0-1 laws for directional transience, and in 2-dimensions the existence of a deterministic limiting velocity. We prove a monotonicity result for the velocity (when it exists) for any 2-valued environment. We give a proof of directional transience and the existence of positive speeds under strong, but non-trivial conditions on the distribution of the environment. Particular emphasis is placed on the 2-dimensional setting in 2-valued environments where the random walk evolves by choosing uniformly among a random subset of nearest neighbours of their current location. We give explicit velocity calculations in some such cases where renewals occur in a straightforward way.

1. Introduction

We will study simple random walks in random environments that are degenerate, in the sense that no ellipticity condition is assumed. We start with some elementary examples, to illustrate the kind of questions we will consider.

Example 1.1. $(\uparrow, \downarrow, \rightarrow, \leftarrow)$: At each vertex of $\mathbb{Z}^2$ independently choose an arrow from $\{\uparrow, \downarrow, \rightarrow, \leftarrow\}$ with non-zero probabilities $p_1, \ldots, p_4$. Then perform a random walk $X_n$ that follows the arrow for the current vertex.

Clearly the walk in Example 1.1 is deterministic once the arrows and initial state are given. Equally clearly, the walk eventually gets stuck in some loop. For example, if the arrow at $(0, 0)$ is $\rightarrow$ and the arrow at $(1, 0)$ is $\leftarrow$ then the walk from $X_0 = (0, 0)$ just oscillates between these two sites. This is not particularly interesting, so our first task will be to impose conditions that rule out such environments.

Example 1.2. $(\uparrow \rightarrow, \uparrow \leftarrow)$: Perform site percolation with parameter $p$ on the lattice $\mathbb{Z}^2$. From each occupied vertex $x = (x^{[1]}, x^{[2]})$, insert two directed edges, one pointing up $\uparrow$ and one pointing right $\rightarrow$. If $x$ is not occupied, insert directed edges pointing up $\uparrow$ and left $\leftarrow$ (see Figure 1).

In the random directed graph arising in Example 1.2, made up of configurations $\uparrow \rightarrow$ and $\uparrow \leftarrow$, there is now an arrow pointing up from every vertex, so the walker cannot get stuck in a loop. In particular from any vertex $x$ the set of vertices $C_x$ that can be reached from $x$ is infinite. Likewise for any $y$, the set of vertices $B_y = \{x : y \in C_x\}$ from which $y$ can be reached is also infinite. However, for each $x$, $\mathcal{M}_x := C_x \cap B_x$ is finite.

The random walk $X_n$ that chooses uniformly among available steps in such a random environment is clearly transient, and in this particular case much more can be said. Since at each step

\begin{flushright}
2000 Mathematics Subject Classification. 60K37.
Key words and phrases. Random walk, non-elliptic random environment, zero-one law, transience.
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the random walk has probability $1/2$ of moving up and probability $0$ of moving down, the random walk in this random environment trivially has a limiting velocity in the vertical direction given by $v^{[2]} = 1/2$. In addition, each upward step constitutes a renewal for the walk since the environment seen thereafter has no intersection with the past. This renewal structure and the fact that the expectations can be calculated explicitly (see Section 6 and also Figure 1) yields

$$v^{[1]} = \frac{(2p - 1)(p^2 - p + 6)}{6(2 - p)(1 + p)}.$$

Comparing this with the speed $\tilde{v}^{[1]} = p - \frac{1}{2}$ of a true random walk that goes up with probability $1/2$, right with probability $p/2$, and left with probability $(1 - p)/2$, we see that the speeds agree for $p = 0, 1/2$, and $1$, but the RWRE is slower in between. But the basic message is that in some examples one obtains transience and an asymptotic speed by elementary calculation.

Example 1.3. $((\uparrow \rightarrow \downarrow), (\uparrow \leftarrow \downarrow))$: Again start with site percolation on $\mathbb{Z}^2$ with each site occupied with probability $p$. From each occupied site insert the edges pointing left $\leftarrow$, down $\downarrow$, and right $\rightarrow$. From each vacant site insert the edges pointing left $\leftarrow$, up $\uparrow$ and right $\rightarrow$. 

**Figure 1.** A finite region of a degenerate environment in two dimensions such that $\mu(\{\uparrow, \rightarrow\}) = p = .75, \mu(\{\leftarrow, \uparrow\}) = 1 - p = .25$, and the first coordinate of the velocity as a function of $p$. 


A random walker, choosing uniformly from the available edges, will never get stuck walking on a random graph as in Example 1.3. Unlike the preceding example, there seems to be no elementary argument that shows the existence of a limiting velocity (though it is easy to see that the horizontal speed is 0). Nevertheless we will prove general results that imply the existence of such a velocity, and that its vertical component is monotone in $p$. We conjecture, but cannot prove this, that the velocity is strictly monotone, and that the random walk in this particular environment is recurrent when $p = \frac{1}{2}$ and otherwise is transient. In some other environments, where again there is no elementary argument that limiting speeds exist, our results imply a bit more. For example, in the model $\left(\uparrow, \downarrow\right)$ our general results still imply that a velocity $v$ exists, that $v^{[1]} = 0$, and that $v^{[2]}$ is monotone in $p$. We conjecture that the random walk is transient for all $p > 0$, with $v^{[2]} \neq 0$. In this model we can then go on to prove a portion of this statement via coupling, namely transience for $p > \frac{3}{4}$ and that $v^{[2]} \neq 0$ for $p > \frac{6}{7}$.

In all of the above examples, the random environment is degenerate in the sense that some edges are missing. It is natural then to first consider the connectivity structure of these directed random graphs, and this was the main focus of [4, 5]. The goal of the present paper was initially to study simple random walks on these graphs, and since that is our main interest, the examples we give will all be of that type. But virtually all the arguments apply more generally to random walks in random environments, without any ellipticity assumption on the environment. So that is the context in which we will formulate our results.

This paper is organised as follows. In Section 2 we introduce both the random environments in which our random walks evolve, and the definition of random walk in these random environments. We also state our main results here. In Section 3 we recall from [4, 5] notions and results about connectivity in certain random directed graphs, and examine RWRE results that can be inferred directly from the connectivity properties of the environments. In Section 4 we adapt established techniques from the elliptic setting to prove 0-1 laws for directional transience and recurrence in our setting. In Section 5 we use coupling methods to prove transience, ballisticity and monotonicity of speeds for certain models. Finally in Section 6 we give explicit computations for speeds in 2-dimensional settings where there are clear renewal events (such as the first example in the introduction).

2. The model

For fixed $d \geq 2$ let $\mathcal{E} = \{\pm e_i : i = 1, \ldots, d\}$ be the set of unit vectors in $\mathbb{Z}^d$. Let $\mathcal{P} = M_1(\mathcal{E})$ denote the set of probability measures on $\mathcal{E}$, and let $\mu$ be a probability measure on $\mathcal{P}$. Let $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ be equipped with the product measure $\nu = \mu^{\otimes \mathbb{Z}^d}$ (and the corresponding product $\sigma$-algebra). A random environment $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$ is an element of $\Omega$. We write $\omega_x(e)$ for $\omega_x(\{e\})$. Note that $(\omega_x)_{x \in \mathbb{Z}^d}$ are i.i.d. with law $\mu$ under $\nu$.

The random walk in environment $\omega$ is a time-homogeneous Markov chain with transition probabilities from $x$ to $x + e$ defined by

\begin{equation}
(2.1) \quad p_\omega(x, x + e) = \omega_x(e).
\end{equation}

Given an environment $\omega$, we let $\mathbb{P}_\omega$ denote the law of this random walk $X_n$, starting at the origin. Let $P$ denote the law of the annealed random walk, i.e. $P(\cdot; \star) := \int_\mathcal{P} \mathbb{P}_\omega(\cdot) d\nu$. Since $P(A) = E_\nu[\mathbb{P}_\omega(A)]$ and $0 \leq f(\omega) = \mathbb{P}_\omega(A) \leq 1$, $P(A) = 1$ if and only if $\mathbb{P}_\omega(A) = 1$ for $\nu$-almost
every $\omega$. Similarly $P(A) = 0$ if and only if $P_\omega(A) = 0$ for $\nu$-almost every $\omega$. If we start the RWRE at $x \in \mathbb{Z}^d$ instead, we write $P_x$ for the corresponding probability, so $P = P_o$. We associate to each environment $\omega$ a directed graph $G(\omega)$ (with vertex set $\mathbb{Z}^d$) as follows. For each $x \in \mathbb{Z}^d$, the directed edge $(x, x + u)$ is in $G_x$ if and only if $\omega_x(u) > 0$, and the edge set of $G(\omega)$ is $\bigcup_{x \in \mathbb{Z}^d}G_x(\omega)$. For convenience we will also write $G = (G_x)_{x \in \mathbb{Z}^d}$. Note that under $\nu$, $(G_x)_{x \in \mathbb{Z}^d}$ are i.i.d. subsets of $\mathcal{E}$. The graph $G(\omega)$ is equivalent to the entire graph $\mathbb{Z}^d$, precisely when the environment is elliptic, i.e. $\nu(\omega_x(u) > 0) = 1$ for each $u \in \mathcal{E}, x \in \mathbb{Z}^d$. Much of the current literature assumes either the latter condition, or the stronger property of uniform ellipticity, i.e. that $\exists \epsilon > 0$ such that $\nu(\omega_x(u) > \epsilon) = 1$ for each $u \in \mathcal{E}, x \in \mathbb{Z}^d$.

On the other hand, given a directed graph $G = (G_x)_{x \in \mathbb{Z}^d}$ (with vertex set $\mathbb{Z}^d$, and such that $G_x \neq \emptyset$ for each $x$), we can define a uniform random environment $\omega = (\omega_x(G_x))_{x \in \mathbb{Z}^d}$. Let $|A|$ denote the cardinality of $A$, and set

$$
\omega_x(e) = \begin{cases} |G_x|^{-1}, & \text{if } e \in G_x \\ 0, & \text{otherwise} \end{cases}
$$

The corresponding RWRE then moves by choosing uniformly from available steps at its current location. This gives us a way of constructing rather nice and natural examples of random walks in non-elliptic random environments: first generate a random directed graph $G = (G_x)_{x \in \mathbb{Z}^d}$ where $G_x$ are i.i.d., then run a random walk on the resulting random graph (choosing uniformly from available steps). This natural class of RWRE receives special attention in this paper, and will henceforth be referred to as uniform RWDRE. Note that we have chosen above to forbid $G_x = \emptyset$. In the setting of uniform RWDRE it would be reasonable to instead allow $G_x = \emptyset$ and define $\omega_x(o) = 1$ in this case, with the walker getting absorbed at $x$. However (see Example 1.1 and Lemma 3.1 below) if this happens with positive probability then the random walker gets stuck on a finite set of vertices almost surely.

**Definition 2.1.** We say that the environment is 2-valued when $\mu$ charges exactly two points, i.e. there exist $\gamma_1, \gamma_2 \in \mathcal{P}$ and $p \in (0, 1)$ such that $\mu(\{\gamma_1\}) = p$, $\mu(\{\gamma_2\}) = 1 - p$. We say that the graph is 2-valued when there exist $E_1, E_2 \subset \mathcal{E}$ and $p \in (0, 1)$ such that $\mu(G_o = E_1) = p$ and $\mu(G_o = E_2) = 1 - p$.

Note that when an environment is 2-valued, the corresponding graph is at most 2-valued. When a graph is 2-valued, the uniform environment corresponding to that graph is also 2-valued. We will obtain monotonicity of speeds when the environment is 2-valued.

**Definition 2.2.** Given a directed graph $G$:

- We say that $x$ is connected to $y$, and write $x \rightarrow y$ if: there exists an $n \geq 0$ and a sequence $x = x_0 x_1, \ldots, x_n = y$ such that $x_{i+1} - x_i \in G_x$, for $i = 0, \ldots, n - 1$. Let $C_x = \{y \in \mathbb{Z}^d : x \rightarrow y\}$, and $B_y = \{x \in \mathbb{Z}^d : x \rightarrow y\}$.
- We say that $x$ and $y$ are mutually connected, or that they communicate, and write $x \leftrightarrow y$ if $x \rightarrow y$ and $y \rightarrow x$. Let $M_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\} = B_x \cap C_x$.
- A cluster $\mathcal{M}$ is said to be giant in $\mathbb{Z}^d$ if every connected component of $\mathbb{Z}^d \setminus \mathcal{M}$ is finite.

The following are some examples of 2-valued graphs.

**Example 2.3.** ($\uparrow\downarrow, \uparrow\downarrow$): When $\mu(G_o = \uparrow\downarrow) \equiv \mu(G_o = \{e_1, e_2\}) = p$ and $\mu(G_o = \downarrow\uparrow) = 1 - p$, clearly each $C_x$ is infinite and each $M_x$ is finite. The uniform RWDRE in this case is the second example discussed in the introduction (see also Figure 1).
Example 2.4. $\uparrow \downarrow$: When $\mu(G_o = \uparrow) = p$ and $\mu(G_o = \downarrow) = 1 - p$, a giant mutually connected cluster $M$ exists if $p \approx \frac{1}{2}$, and not if $p \approx 0$ or $p \approx 1$, and there are sharp phase transitions in between [4]. We’ll show that any corresponding RWRE is a.s. directionally transient when $p$ is not close to $\frac{1}{2}$. We conjecture that this is in fact true for $p \neq \frac{1}{2}$ and that in the uniform case there is an infinite recurrent set when $p = \frac{1}{2}$.

Example 2.5. $\leftrightarrow 
\uparrow \downarrow$: When $\mu(G_o = \leftrightarrow) = p$ and $\mu(G_o = \uparrow) = 1 - p$, there is a giant mutually connected cluster $M$ for every $p \in (0, 1)$ [4]. The corresponding uniform RWDRE is a degenerate version of the “good-node bad-node” model of Lawler [9]. We believe that it follows from recent results of Berger and Deuschel [2] that there is an infinite recurrent set for this model for every such $p$.

Example 2.6. $\uparrow \downarrow 
\uparrow$: When $\mu(G_o = \uparrow \downarrow) = p$ and $\mu(G_o = \uparrow) = 1 - p$, there is a giant mutually connected cluster $M$ for every $p$ larger than some $p_c$ [4]. For $p \approx 0$ we’ll show that the random walk is transient in direction $\uparrow$.

The present paper addresses the question of ballistic behaviour. We conjecture that the following is true for general IID environments:

Conjecture 2.7. There exists $v \in \mathbb{R}^d$ such that $P(v = \lim_{n \to \infty} n^{-1} X_n) = 1$.

This is trivial for walks that have renewals, or which only visit a finite collection of sites. We will be able to show it in non-trivial cases using coupling and other techniques. This will be sufficient to prove the conjecture when $d = 2$, but we will not resolve it for general environments in general dimensions.

The following theorem is one of the main results of this paper.
Lemma 3.1. Fix any 2-valued environment with $\mu(\{\gamma^1\}) = p$, $\mu(\{\gamma^2\}) = 1 - p$. If for every $p$ there exists $v[p]$ such that $P(v[p] = \lim_{n \to \infty} n^{-1} X_n) = 1$, then each coordinate of $v[p]$ is monotone in $p$.

In fact we’ll prove a more comprehensive version of this in Section 5.1, via a simple coupling argument. See Corollary 5.2.

Fix $\ell \in \mathbb{R}^d \setminus o$. Let $A_+$ and $A_-$ denote the events that $X_n \cdot \ell \to \infty$ and $X_n \cdot \ell \to -\infty$ respectively. We prove the following generalisations (to the non-elliptic setting) of results of Sznitman and Zerner [10], Zerner [12] and Zerner and Merkl [15].

Theorem 4.8. There exist deterministic $v_+, v_-$ such that

$$\lim_{n \to \infty} \frac{X_n \cdot \ell}{n} = v_+ I_{A_+} + v_- I_{A_-}, \quad P - a.s.$$  

Theorem 4.10. When $d = 2$, $P(A_\ell) \in \{0, 1\}$.

Note that these two theorems imply (as claimed above) that a deterministic velocity $v$ always exists in the 2-dimensional setting (see Corollary 4.11). Moreover by Theorem 2.8 the velocity of any two-valued environment in 2-dimensions is monotone in $p$ (in each coordinate direction).

Of course, existence of speeds does not imply transience, unless one has a way of checking that $v \neq 0$. When there is sometimes a drift in direction $u$ but never a drift in direction $-u$ the walk should be almost surely transient in direction $u$, with positive speed. We will attempt to prove this in a subsequent paper. Instead, in this paper we give a relatively simple proof under stronger conditions (see Proposition 5.5), relying on results from [6, 13, 1]. The stronger condition is that with sufficiently large probability we have a sufficiently large drift at the origin.

3. Random walk properties obtained from $C_x$ and $M_x$

In this section we present a number of results for RWRE, that depend only on the clusters $(C_x)_{x \in \mathbb{Z}^d}$ and $(M_x)_{x \in \mathbb{Z}^d}$ of the graph $G(\omega)$ induced by the environment $\omega$.

Whether the RWRE $X$ gets stuck on a finite set of sites can be characterized completely in terms of the law of the connected cluster $C_\omega$. If $C_\omega$ is almost surely infinite, then so is $C_x$ for each $x$, so the random walk will eventually escape from any finite set of sites. On the other hand if $C_\omega$ is finite with positive probability, then we will see that there is some $\delta > 0$ such that each time the walk reaches a new $\|X_n\|_{\infty}$ maximum it has probability at least $\delta$ of being at a site with finite $C$, whence the walk will eventually get stuck. These arguments are formalised in the following Lemma.

Lemma 3.1. Fix $d \geq 2$, and let $X_n$ be the random walk in i.i.d. environment.

(i) If $\nu(|C_\omega| < \infty) = 0$, then $P(\sup_{n \geq 1} |X_n| < \infty) = 0$, (i.e. the RWRE visits infinitely many sites) $\nu$-almost surely.

(ii) If $\nu(|C_\omega| < \infty) > 0$, then $P(\sup_{n \geq 1} |X_n| < \infty) = 1$ (i.e. the RWRE gets stuck on a finite set of points) $\nu$-almost surely.

Proof. To prove the first claim, suppose $X_n$ visits only finitely many sites. Then it must visit some site $x$ i.o. Let $z \in C_x$. There is an admissible path connecting $x$ to $z$ which has a fixed positive probability of being followed on any excursion of $X_n$ from $x$. Therefore eventually it will be followed, so $X_n$ will visit $z$. This is true for every $z$ in the infinite set $C_z$, contradicting the assumption.
To prove (ii), suppose that \( \nu(|C_o| < \infty) > 0 \). Define

\[
    n_0 = \inf\{n \geq 1 : \exists F \subset \mathbb{Z}^d \text{ with } |F| = n \text{ and } \nu(C_o = F) > 0\}
\]

and choose \( F \) satisfying \( |F| = n_0 \) and \( \delta = \nu(C_o = F) > 0 \). Note that if \( \mathcal{M}_y = F \) for some \( y \in F \), then \( \mathcal{M}_{y'} = F \) for each \( y' \in F \). So by translation invariance of \( \nu \), for each \( y \in F \),

\[
    \nu(\mathcal{M}_y = F) = \nu(\mathcal{M}_y = F) = \nu(\mathcal{M}_o = F - y).
\]

Furthermore, for \( y \in F \), if \( F = C_y \supseteq \mathcal{M}_y \), then there exists \( y' \in C_y \setminus \mathcal{M}_y \). Since \( y \notin C_y \subset C_o \) the set \( G = C_{y'} \subset F \) satisfies \( |G| < n_0 \), and \( \nu(C_o = G - y') = \nu(C_{y'} = G) > 0 \). This would contradict the definition of \( n_0 \). So in fact \( \nu(\mathcal{M}_y = F) = \nu(C_y = F) \) for each \( y \in F \). Therefore by (3.1),

\[
    \nu(C_o = F) = \nu(C_o = F - y),
\]

for every \( y \in F \).

For each \( x \in \mathbb{Z}^d \) we can find a \( y \in F \) such that \( x \) minimizes \( \|z\|_\infty \) over \( z \in F + x - y \) (just find a unit vector \( e \in \mathbb{Z}^d \) such that \( \|x + e\|_\infty > \|x\|_\infty \), and then choose \( y \) so that the projection of \( F - y \) in the direction of \( e \) is \( \geq 0 \). Then

\[
    \nu(C_x = F + x - y) = \nu(C_o = F - y) = \nu(C_o = F) = \delta.
\]

For \( k \geq 1 \) define

\[
    T_k := \inf\{m \geq 1 : \|X_m\|_\infty = k(n_0 + 1)\}.
\]

Let \( \mathcal{F}_k \) reveal \( X_m \) for \( m < T_k \), and the environment \( \omega_z \) for \( \|z\|_\infty < k(n_0 + 1) \). Then

\[
    P(T_k = \infty \mid \mathcal{F}_{k-1}) \geq P(|C_{X_{T_{k-1}}} \setminus n_0 | \mathcal{F}_{k-1}) \geq \sum_{y \in F} P(C_{X_{T_{k-1}}} = X_{T_{k-1}} + F - y \mid \mathcal{F}_{k-1}).
\]

We have shown above that \( y \) can be chosen so that \( X_{T_{k-1}} + F - y \) is disjoint from \( \{z : \|z\|_\infty < (k - 1)(n_0 + 1)\} \) (the region whose environment is revealed by \( \mathcal{F}_{k-1} \)). Applying (3.2) to that \( y \) we conclude that

\[
    P(T_k < \infty \mid \mathcal{F}_{k-1}) \leq (1 - \delta)1_{\{T_{k-1} < \infty\}}.
\]

Iterating \( k \) times, \( P(T_k < \infty) \leq (1 - \delta)^k \), and sending \( k \to \infty \) we obtain

\[
    0 = P(\cap_{k=1}^\infty \{T_k < \infty\}) = 1 - \nu(\cup_{k=1}^\infty \{T_k = \infty\})
\]

which establishes the result.

On the event that the walk gets stuck on a finite set of sites, the asymptotic velocity is trivially zero and the walk is not directionally transient in any direction, almost surely. Hence, by Lemma 3.1, Theorems 4.8 and 4.10 hold trivially when \( \nu(|C_o| < \infty) > 0 \). Our principal interest will therefore be in situations where the following condition holds:

\[
    \nu(|C_o| < \infty) = 0.
\]

Note that our general hypotheses rule out the possibility that \( C_x = \{x\} \). As remarked above, at the cost of more cumbersome notation, we could have included the possibility of \( \mathcal{G}_x = \emptyset \) in our models. In this case \( X_a \) would be stuck as soon as it reaches \( x \), and the argument just given would show that \( \nu(|C_o| < \infty) = 1 \). Thus the condition (3.3) rules out (environments giving rise to) percolation-type graphs where \( \mu(\mathcal{G}_o = \{\emptyset\}) > 0 \). So there is in fact no loss of generality in formulating our general hypotheses the way we have, since we will typically also impose (3.3).
The following simple criterion from [4] is equivalent to (3.3), and hence to the statement that the random walk visits infinitely many sites almost surely.

**Lemma 3.2.** Fix $d \geq 2$. If there exists an orthogonal set $V$ of unit vectors such that $\mu(\mathcal{G}_o \cap V \neq \emptyset) = 1$ then $|\mathcal{C}_o| = \infty$, $\nu$-almost surely. Conversely, if $|\mathcal{C}_o| = \infty$, $\nu$-almost surely then such an orthogonal set $V$ exists.

It follows immediately that the uniform RWDRE (and indeed RWRE for any environment giving rise to such graphs) in Examples 2.4, 2.5, 2.6 visits infinitely many sites almost surely, by choosing e.g. $V = \{\uparrow, \leftarrow\}$ in each case. Likewise this proves the assertion of Example 1.1, that the uniform RWDRE in that environment gets stuck a.s.

From [4] (see also [5] for improvements to some of these values), the following Lemma immediately implies that the RWDRE is transient in the following situations: when $\mathcal{G}_o(\omega) \in \{\downarrow_{\omega}, \downarrow_{\omega}\}$ almost surely (as in Example 2.4), with $\mu(\mathcal{G}_o = \downarrow_{\omega}) > .83270$; when $\mathcal{G}_o(\omega) \in \{\Uparrow_{\omega}, \uparrow\}$ almost surely (as in Example 2.6), with $\mu(\mathcal{G}_o = \uparrow) > .83270$.

**Lemma 3.3.** For any environment $\omega$ such that $|\mathcal{C}_x| = \infty$ and $|\mathcal{M}_x| < \infty$ for every $x \in \mathbb{Z}^d$, the random walk in environment $\omega$ is transient $\mathbb{P}_\omega$-almost surely.

**Proof.** Starting from any $x$, we visit infinitely many sites. Thus the time $T_x$ taken for the walk to exit $\mathcal{M}_x$ is finite, and any vertex reached thereafter is necessarily in $\mathcal{C}_x \setminus \mathcal{M}_x$. Hence the random walk never returns to $x$ after time $T_x$. ■

Lemma 3.3 also suggests that we should be careful about what we mean by transience and recurrence. For some of the most interesting models, such as the uniform RWDRE in Example 2.5, $\mathcal{C}_o$ is infinite almost surely while $\mathcal{M}_o$ can be finite or infinite with non-trivial probability. This means that instead of asking if the origin is visited infinitely often almost surely, one should instead ask if the origin is visited infinitely often, given that $\mathcal{M}_o$ is infinite and $\mathcal{M}_o = \mathcal{C}_o$.

**Lemma 3.4.** For any environment $\omega$ such that $|\mathcal{C}_y| = \infty$ for every $y$ and such that there is a unique infinite mutually connected cluster $\mathcal{M}_\infty$, we have that $\{x : X_n = x \text{ infinitely often}\} \in \{\emptyset, \mathcal{M}_\infty\}$, $\mathbb{P}_\omega$-almost surely.

**Proof.** Let $\mathcal{R} = \{x : X_n = x \text{ infinitely often}\}$ be the recurrent set. If $x \in \mathcal{R}$, then $\mathcal{C}_x \subset \mathcal{R}$ by the argument for (i) of Lemma 3.1. Clearly also $\mathcal{R} \subset \mathcal{M}_x$ (every site in $\mathcal{R}$ must be reachable from $x$ and vice versa), almost surely. Since $\mathcal{M}_x \subset \mathcal{C}_x$ this implies that $\mathcal{C}_x = \mathcal{R} = \mathcal{M}_x$ and also that $|\mathcal{M}_x| = \infty$ (so $\mathcal{M}_x = \mathcal{M}_\infty$). ■

The above proof also shows that if there are multiple distinct infinite $\mathcal{M}_x$ then $\mathcal{R}$ is either empty or is equal to exactly one such infinite $\mathcal{M}_y$. An equally simple result is the following, which applies to Examples 2.4 and 2.6 for some values of $p$.

**Lemma 3.5.** Let $\omega$ satisfy $|\mathcal{C}_y| = \infty$ for every $y$ and suppose there exists $x$ such that $\mathcal{M} = \mathcal{M}_x$ is giant. Then $T_M \equiv \inf\{n \geq 0 : X_n \in \mathcal{M}\} < \infty$ and $X_n \in \mathcal{M}$ for all $n \geq T_M$, $\mathbb{P}_\omega$-almost surely.

We have already obtained transience in a number of examples. We now turn to a result that allows us to improve this to directional transience. Given a vector $v \in \mathbb{R}^d \setminus o$, and $N \geq 1$, define $A_N(v) = \{y : y \cdot v < -N\}$.

**Theorem 3.6.** Assume (3.3), and suppose that there is a vector $v \in \mathbb{R}^d \setminus o$ such that
We claim this is finite. To see this, write $w$ for any record levels $(\nu \sum u)$. Therefore for each $k$, $c$ for every $u$ with $V$ the environment in $T$ the almost-surely finite, since eventually any set of record levels will be surpassed. In other words, there are only finitely many possibilities for each $w$. Thus $\nu(o \in C_v) = \varepsilon$ for some $\varepsilon > 0$.

Proof. Without loss of generality, $|v| = 1$. For $v \in \mathbb{Z}^d$ define $F_x(v) := \{y : (y-x) \cdot v < 0\}$ and $C_v := \{x : C_x \cap F_x(v) = \emptyset\}$. We show that under the hypotheses of the theorem:

(i) $\nu(o \in C_v) = \varepsilon$ for some $\varepsilon > 0$
(ii) $P(\sup_n X_n \cdot v = \infty) = 1$
(iii) for all $M < \infty$, $P(\lim inf_{n \to \infty} X_n \cdot v \geq M) = 1$.

Note that the desired result clearly follows from (iii).

To prove (i), assume that the claim does not hold. Then we must have $\nu(C_x = F_x(v) = \emptyset) = 0$ for every $x$ and therefore $\nu$-almost surely there is a sequence of vertices $o = x_0, x_1, x_2, \ldots$ such that $x_i \in C_{x_{i-1}} \cap F_{x_{i-1}}(v)$ for each $i$, and by induction $x_N \in \cup_{j \geq N} A_j^-(v)$. This implies that $\nu(C_o \cap F_N^-(v) = \emptyset) = 0$ for all $N$, which contradicts assumption (a).

To prove (ii), let $W_n = X_n \cdot v$ and let $U_v$ denote the set of unit vectors $u \in \mathbb{Z}^d$ such that $v \cdot u \geq 0$. For any record levels $(z_u)_{u \in U_v}$ and any $N \geq 0$, consider

$$\{w \in \mathbb{Z}^d : w \cdot v \geq -N \text{ and } w \cdot u \leq z_u \forall u \in U_v\}.$$ 

We claim this is finite. To see this, write $w = \sum w_i u_i$, where we select a basis $u_i$ from $U_v$. If $w_i \cdot v = 0$, then $z_u u_i \leq w_i \leq z_u$. If $w_i \cdot v > 0$ then $-N \leq w \cdot v = \sum w_i u_i \cdot v \leq \sum_j z_u u_j \cdot v + w_i u_i \cdot v$ so in fact

$$z_u u_i \geq w_i \geq \frac{-N - \sum_j z_u u_j \cdot v}{u_i \cdot v}.$$ 

In other words, there are only finitely many possibilities for each $w_i$.

By Lemma 3.1, $X_n$ visits infinitely many sites. Since $\inf_n W_n > -\infty$ (which follows from the fact that $\nu(\cup_{N \geq 1} C_o \cap A_N^-(v) = \emptyset) = 1$), the above argument shows that the random times $T_i = \inf\{n \geq 1 : \exists u \in U_v ; X_n \cdot u > X_m \cdot u \text{ for all } m < n\}$ and

$$T_i = \inf\{n > T_{i-1} : \exists u \in U_v ; X_n \cdot u > X_m \cdot u \text{ for all } m < n\}, \quad i = 2, 3, \ldots$$

are almost-surely finite, since eventually any set of record levels will be surpassed.

Let $V_x = \{y : (y-x) \cdot u \geq 0 \forall u \in U_v\}$. Then each time $T_i$ is the first hitting time of $V_{X_{T_i}}$, so the environment in $V_{X_{T_i}}$ is unexplored. Note that (b) implies that there exists $\epsilon_v > 0$ and $u \in U_v$, with $u \cdot v > 0$ such that $\mu(\omega(u) > \epsilon_v) > \epsilon_v$. Set $U_v^0 = \{u \in U_v : u \cdot v > 0, \mu(\omega(u) > \epsilon_v) > \epsilon_v\}$, and $c_v = \{v \cdot u : u \in U_v^0\} > 0$. Then for each $k \in \mathbb{N}$, there is probability at least $\epsilon_v^k$ of there being an admissible path from $X_{X_{T_i}}$ consisting of $k$ arrows from $U_v^0$. Following this path keeps us in the previously unexplored region $V_{X_{T_i}}$. So given this, there is probability at least $\epsilon_v^k$ that the next $k$ steps of $X_n$ will follow this path. In other words, for each $i, k \in \mathbb{N}$, there is probability at least $\epsilon_v^{2k}$ under $P$, independent of the history of the walk up to time $T_i$, that $\{(X_{T_i+k} - X_{T_i}) \cdot v > c_v k\}$. Therefore for each $k \in \mathbb{N}$ there will eventually be an $i$ such that $\{(X_{T_i+k} - X_{T_i}) \cdot v > c_v k\}$. Recalling that $\inf W_n > -\infty$, we conclude that $\sup X_n \cdot v = \infty$.

To prove (iii), fix $M_1 \geq 1$ and define $T_i$ to be the first hitting time of $M_1$ by $W$, i.e.

$$T_i = \inf\{n > 0 : W_n \geq M_1\},$$
which is $P$-almost surely finite by (ii). Given $\mathcal{T}_i < \infty$, let

$$N_i = \inf\{k \geq 1 : \text{there exists a } \mathcal{G} \text{-admissible path (of length } k) \tilde{\eta}_k : X_{\mathcal{T}_i} \to F_{X_{\mathcal{T}_i}}(v)\}.$$

If $N_i < \infty$ set $M_{i+1} = M_i + N_i$ and

$$\mathcal{T}_{i+1} = \inf\{n > \mathcal{T}_i : W_n \geq M_{i+1}\}.$$

Note that $\{N_i = \infty\} = \{X_{\mathcal{T}_i} \in \mathcal{C}_v\}$ and if $N_i = \infty$ then $W_n \geq M_i$ for all $n \geq \mathcal{T}_i$. Moreover, to determine if $N_i \leq m$, we need only look at the environment within distance $m$ of $X_{\mathcal{T}_i}$, so if $N_i < \infty$ then the walk visits an unexplored environment at time $\mathcal{T}_{i+1}$. In other words, the event that $X_{\mathcal{T}_i} \in \mathcal{C}_v$ depends only on the unexplored environment in $\mathbb{Z}^d \setminus F_{\mathcal{T}_i}(v)$, so that by (i), $\mathcal{I} = \inf\{i : X_{\mathcal{T}_i} \in \mathcal{C}_v\}$ has a geometric distribution with parameter $\varepsilon > 0$.

Thus $P$-almost surely there is a $\mathcal{T}_i < \infty$ such that $W_n \geq M_i \geq M_1$ for all $n \geq \mathcal{T}_i$. Since $M_1$ was arbitrary, this proves (iii).

An elementary corollary of Theorem 3.6 is that (assuming (3.3)) if $\mu(-e_1 \in \mathcal{G}_o) = 0$ but $\mu(e_1 \in \mathcal{G}_o) > 0$ then the RWRE is transient in direction $e_1$, $P$-almost surely. More significantly, we obtain directional transience in settings where previously we only knew transience:

**Corollary 3.7.** Assume (3.3). For each $d \geq 2$ there exists $\varepsilon_d$ such that the following holds: If there exists an orthogonal set $V$ of unit vectors such that $\mu(\mathcal{G}_o \subset V) > 1 - \varepsilon_d$, then the random walk is transient in direction $v = \sum_{u \in V} u$.

**Proof.** The proof of Theorem [4, Theorem 4.2] verifies Theorem 3.6 (a) for $v = \sum_{u \in V} u$, while condition (b) holds with $\varepsilon_v = 1 - \varepsilon_d$ by the assumption that $\mu(\mathcal{G}_o \subset V) > 1 - \varepsilon_d$. ■

Further consequences (see [4, Corollary 4.3]) are that the $(\uparrow \downarrow \downarrow \downarrow)$ model of Example 2.4 is transient in direction $(1,1)$ whenever $\mu(\mathcal{G}_o = \uparrow \downarrow \downarrow) > 0.83270$ and that the $(\uparrow \downarrow \downarrow \downarrow \uparrow)$ model of Example 2.6 is transient in direction $(0,1)$ whenever

(3.4) \[ \mu(\mathcal{G}_o = \uparrow \downarrow \downarrow \uparrow \uparrow) > 0.83270 \]

The former result is improved considerably via the following result.

**Corollary 3.8.** For any model $\mathcal{G}_o \sim (\uparrow \downarrow \downarrow \downarrow \uparrow)$, any RWRE is transient in direction $(1,1)$ when $\mu(\mathcal{G}_o = \uparrow \downarrow \downarrow \downarrow \downarrow) > p_o^{2\varepsilon}$, where $p_o^{2\varepsilon}$ is the critical occupation prob. for oriented site percolation on the triangular lattice. For $\mu(\mathcal{G}_o = \uparrow \downarrow \downarrow \downarrow \uparrow) < 1 - p_o^{2\varepsilon}$, any RWRE is transient in direction $(-1, -1)$.

**Proof.** As in [5], when $p = \mu(\mathcal{G}_o = \uparrow \downarrow \downarrow \downarrow \downarrow) > p_o^{2\varepsilon}$, $\mathcal{C}_o$ has NW and SE boundaries with asymptotic slopes $\rho(p) < -1$ and $1/\rho(p) > -1$ respectively. In particular for each such $p$, the assumptions of Theorem 3.6 hold with $v = (1,1)$.

We believe that a similar argument shows directional transience in the direction $\uparrow$ (as opposed to just transience) for any model $\mathcal{G}_o \sim (\uparrow \downarrow \downarrow \downarrow \uparrow)$, provided $\mu(\mathcal{G}_o = \uparrow \downarrow \downarrow \downarrow \downarrow \uparrow) > p_o^{2\varepsilon}$, where the latter critical percolation threshold is defined in [4]. But we have not checked this in detail.

4. Directional transience

In the previous section we inferred transience results from strong conditions on the laws of the clusters $\mathcal{C}_o$ and $\mathcal{M}_o$. However there are many examples where $\mathcal{M}_o$ is infinite with probability $> 0$ (or even 1), but where the RWRE is still transient. In this section we investigate directional transience properties without the assumption that $\mathcal{M}_o$ is finite.
4.1. Elementary transience results.

**Definition 4.1.** $X_n$ has positive lim inf speed (resp. positive lim sup speed) in the direction $v$ if there exists $\epsilon > 0$ such that $\liminf_{n \to \infty} n^{-1} X_n \cdot v \geq \epsilon$ a.s. (resp. $\limsup_{n \to \infty} n^{-1} X_n \cdot v \geq \epsilon$ a.s.).

Here is one such example.

**Lemma 4.2.** The uniform RWDRE $(\uparrow\downarrow, \uparrow\downarrow)$ is directionally transient for all $p$, with strictly positive lim inf speed.

**Proof.** Let $\theta \in (\frac{\pi}{4}, 0)$, and let $v$ be the unit vector in direction $\theta$, clockwise from the positive $x$-axis. When the random walk is at a $\uparrow\downarrow$ site, it has drift in the direction of $v$ of $\frac{1}{2} [\cos(\theta) - \cos(\frac{\pi}{2} - \theta)] > 0$. When the random walk is at a $\uparrow\downarrow$ site, it has drift in the direction of $v$ of $\frac{1}{3} [\cos(\theta) - \cos(\theta) + \cos(\frac{\pi}{2} - \theta)] = \frac{1}{3} \cos(\frac{\pi}{2} - \theta) > 0$. In the following result, we show that this condition easily implies positive lim inf speed in the direction $v$. In the following subsection we go on to show that directional transience implies the actual existence of the speed.  

The following will largely be subsumed by other results. But when it applies, it gives a short and elementary argument for positive lim inf speeds.

**Lemma 4.3.** Assume (3.3). If $\exists \epsilon > 0$ such that $\sum_{e \in E} e \cdot v_\omega(e) \geq \epsilon$ $\mu$-a.s. then $X_n$ has positive lim inf speed in direction $v$.

**Proof.** Suppose first that $\mu$ is concentrated on $m$ environments $\omega^1, \ldots, \omega^m$ all satisfying the above hypothesis. Let $Y^k_i$ be $(X_{j+1} - X_j) \cdot v$ when $j$ is the $i$th time $X$ encounters an environment of type $k$. The $Y^k_i$ are all independent, and for each $k$ are IID in $i$ with mean $\geq \epsilon$. If $N^k_n$ denotes the number of type $k$ environments encountered up to time $n$, we have

$$\frac{1}{n} X_n \cdot v = \sum_{k=1}^{m} \frac{N^k_n}{n} \times \frac{1}{N^k_n} \sum_{i=1}^{N^k_n} Y^k_i. \tag{4.1}$$

By the strong law of large numbers, each $\frac{1}{n} \sum_{i=1}^{j} Y^k_i$ converges to the mean of $Y^k_1$, which is $\geq \epsilon$. Therefore $\liminf \frac{1}{n} X_n \cdot v \geq \epsilon$ too.

In the general case, cover $\mathcal{P}$ by finitely many disjoint sets $C_1, \ldots, C_m$ of $L^\infty$-diameter less than some small $\delta > 0$. By decreasing the weights given to $e$ with $e \cdot v > 0$ and increasing the weights to $e$ with $e \cdot v \leq 0$ we can find $\omega^k_\delta \in \mathcal{P}$ within $L^\infty$-distance $\delta$ of $C_k$ such that random walk steps $Y'$ chosen from $\omega^k_\delta \in C^k$ and steps $Y$ chosen from $\omega_\delta$ can be coupled so $Y' \cdot v \geq Y \cdot v$. Choosing $\delta$ small, we have

$$\sum e \cdot v_\omega^k(e) \geq \sum e \cdot v_\omega(e) - d\delta \|v\|_{\infty} \geq \epsilon/2$$

whenever $\omega_\delta \in C_k \cap \text{Support}(\mu)$. Therefore $\frac{1}{n} X_n \cdot v$ dominates a sum of the form of the right hand side of (4.1), and the positivity of the lim inf speed of $X_n$ in the direction $v$ now follows as before.  

Later on we will establish the existence of a limiting velocity $v$ for such models. Once that is known, it is simple to show that both coordinates of $v$ cannot simultaneously vanish in the model $(\uparrow\downarrow, \uparrow\downarrow)$ of Example 4.2. It is nevertheless useful to have an elementary proof of this fact, as given above.
4.2. Orthogonal arrows: adapting elliptic approaches to 0 – 1 laws. In this section we prove Theorems 4.8 and 4.10. This involves adapting results obtained by Sznitman and Zerner [10], Zerner [12] and Zerner and Merkl [15] to our non-elliptic setting.

For fixed $d \geq 2$ we define a slab to be a region between any two parallel $d – 1$ dimensional hyperplanes in $\mathbb{R}^d$. Let $\mathcal{H}_j$ be the set of slabs $\mathcal{S}$ for which there exists a constant $H = H(\mathcal{S}) \in \mathbb{N}$ such that $(S + H e_j) \cap \mathcal{S} = \emptyset$. This is the set of slabs with finite width in direction $e_j$. Note that for every $d$-dimensional slab $\mathcal{S}$, there exists some $j \in \{1, 2, \ldots, d\}$ such that $\mathcal{S} \in \mathcal{H}_j$. Moreover, if $v$ is a normal to the hyperplanes defining the slab, then $\mathcal{S}$ has finite width in the direction $u \neq u \cdot v = 0$.

It is convenient to introduce the following condition, which (up to reflections of the directions) says that the RWRE is truly $d$-dimensional:

\begin{equation}
\mu(e_i \in \mathcal{G}_o) > 0 \quad \text{for } i = 1, \ldots, d.
\end{equation}

**Lemma 4.4.** Assume (3.3) and (4.2). Then for each unit vector $v$, $P$-almost surely,

$$
\liminf_{n \to \infty} X_n \cdot v \in \{-\infty, +\infty\}.
$$

**Proof.** Let $\mathcal{A} = \{u \in \mathcal{E} : \mu(u \in \mathcal{G}_o) > 0\}$, $\mathcal{B}_+ = \{u \in \mathcal{E} : u \cdot v > 0\}$, and $\mathcal{B}_- = \{u \in \mathcal{E} : u \cdot v < 0\} = -\mathcal{B}_+$. Note that $\{e_1, \ldots, e_d\} \subset \mathcal{A}$ by (4.2). If $\mathcal{B}_- \subset \mathcal{A}^c$ then the random walk is transient in every direction in $\mathcal{B}_+ \cap \mathcal{A}$ (e.g. by Theorem 3.6, or by the simple argument of Lemma 6.1), hence it is also transient in direction $v$. So assume there exists some $u_- \in \mathcal{B}_- \cap \mathcal{A}$. Without loss of generality we can assume that $u_- \in \{\pm e_1\}$.

Assume that the claim of the lemma is false, i.e. there exists $r \in \mathbb{R}$ such that $\liminf_{n \to \infty} X_n \cdot v = r$. Then the slab $\mathcal{S} = \{x \in \mathbb{R}^d : r - 1 \leq x \cdot v \leq r + 1\}$ is visited infinitely often. Suppose that there are actually infinitely many sites in $\mathcal{S}$ that are visited. Then the set of sites of $\mathcal{S}$ visited is unbounded in the direction of at least one vector $u \in U_1 = \{\pm e_j : j \neq 1\}$. Let $T_1 = 1$, and let

$$
T_{k+1} = \inf\{n > T_k : X_n \in \mathcal{S} \text{ and } \exists u \in U_1 \text{ s.t. } X_n \cdot u > \max\{X_m \cdot u : m < n, X_m \in \mathcal{S}\}\}
$$

be the times the walk reaches a new record level within $\mathcal{S}$. These are all finite, and by definition, the sites $\mathcal{S} \cap (X_{T_k} + Zu_-)$ were not explored prior to time $T_k$. Let $H < \infty$ be the width of $\mathcal{S}$ in the direction $e_1$, defined above. With probability at least $\epsilon^{H+1}$ there is an admissible path, just using $u_-$ arrows, of length at most $H + 1$, that connects $X_{T_k}$ to a point $z$ outside $\mathcal{S}$, with $z \cdot v \leq r - 1$. So with probability at least $\epsilon^{2(H+1)}$, the random walk follows this path and exits $\mathcal{S}$ in at most $H + 1$ steps. This must therefore occur almost surely, for infinitely many $k$. It follows that $\liminf_{n \to \infty} X_n \cdot v \leq r - 1$, which is a contradiction.

Thus in fact there are only finitely many points of $\mathcal{S}$ that get visited. Therefore at least one point gets visited infinitely often. Let $\mathcal{R}$ be the set of sites in $\mathcal{S}$ that are visited infinitely often. We conclude that $\mathcal{R}$ is finite but non-empty. We may therefore choose an element $x$ of $\mathcal{R}$ such that $x \cdot v$ is minimal. Since every $y \in \mathcal{C}_x$ is also visited infinitely often (by the argument for (i) of Lemma 3.1), we must have that $(y - x) \cdot v \geq 0$ for each $y \in \mathcal{C}_x$, i.e. $x \in \mathcal{C}_v$. Since this happens with positive probability, we in fact have $\nu(o \in \mathcal{C}_v) > 0$. The proof of Theorem 3.6 shows that if $\mu(e_1 \in \mathcal{G}_o) > 0$ and (3.3) holds, then on the event that $o \in \mathcal{C}_v$ we have $\liminf X_n \cdot v = \infty$ (a.s.), which contradicts the definition of $r$.

Note that we can similarly get that $\liminf X_n \cdot e_1 \in \{-\infty, +\infty\}$, $P$-almost surely under the weaker assumptions (3.3) and $\mu(e_1 \in \mathcal{G}_o) > 0$. 


For any RWRE, and fixed $\ell \in \mathbb{R}^d \setminus o$, let $A_+ = A_+(\ell)$ be the event that the walk is transient in direction $\ell$, i.e. $A_+ = \{X_n \cdot \ell \to +\infty\}$, and $A_- = A_-(\ell) = \{X_n \cdot \ell \to -\infty\}$. Let $O = (A_+ \cup A_-)^c$ and let $O_m$ be the event that both $X_n \cdot \ell \leq m$ and $X_n \cdot \ell \geq m$ infinitely often. For $k \geq 0$ let $T_k = \inf\{n : X_n \cdot \ell \geq k\}$.

**Lemma 4.5.** Assume (3.3) and (4.2). Then $P(O) = P(\cap_{n \in \mathbb{Z}} O_n)$.

**Proof.** First note that $\cap_{n \in \mathbb{Z}} O_n \subset \cup_{n \in \mathbb{Z}} O_n \subset O$ so in particular if $P(O) = 0$ then the result is trivial. By Lemma 4.4 applied to both $\ell$ and $-\ell$, $P$-almost surely on the event $O$ we have $\liminf X_n \cdot \ell = -\infty$ and $\limsup X_n \cdot \ell = +\infty$. This implies that $\cap_{n \in \mathbb{Z}} O_n$ occurs almost surely on the event $O$ as required. \qed

**Lemma 4.6.** Assume (3.3) and (4.2). If $P(A_+) > 0$ then $P(O) = 0$ and

\[(3.3) \quad P(A_+ \cap \{X_n \cdot \ell \geq 0 \forall n \geq 0\}) > 0.\]

**Proof.** Suppose that $P(A_+) > 0$. Let $P_{\omega,y}$ denote the quenched law of the RW in the environment $\omega$, starting from $X_0 = y$, and recall that $P_y$ is the corresponding annealed law. For any $x$ and $n$, let $A(x,n)$ be the event $A_+ \cap \{X_n = x \text{ and } X_k \cdot \ell \geq x \cdot \ell \forall k \geq n\}$. Since $P(A_+) > 0$, we must have $P(A(x,n)) > 0$ for some $x$ and $n$. By translation invariance, $P_{-x}(A(o,n)) > 0$. By the Markov property, $P_{\omega,-x}(A(o,n) | X_0 = o) = P_{\omega,o}(A(o,0))$ for every environment $\omega$. Therefore $P_{\omega,-x}(A(o,n)) \leq P_{\omega,o}(A(o,0))$. Integrating out $\omega$ we get $0 < P_{-x}(A(o,n)) \leq P(A(o,0))$, which proves (4.3).

To prove that $P(O) = 0$, note that by Lemma 4.5, almost surely on the event $O$, all $O_m$ occur and therefore $\limsup X_n \cdot \ell = \infty$. It is therefore sufficient to show that almost surely $O$ does not occur on the event $\limsup X_n \cdot \ell = \infty$, under the assumptions of the lemma. Let $\delta = P(X_k \cdot \ell \geq 0 \forall k \geq 0) \geq P(A(o,0)) > 0$. Let $T_0 = 0$. Given $T_k < \infty$, let $D_k = \inf\{n > T_k : X_n \cdot \ell < X_{T_k} \cdot \ell\}$. If $D_k < \infty$, let $M_k = \sup\{X_n \cdot \ell : n \leq D_k\}$, and let $T_{k+1} = \inf\{n > D_k : X_n \cdot \ell \geq M_k + 1\}$. Let $K$ be the first value of $k$ such that $T_k = \infty$ or $D_k = \infty$.

At time $T_k < \infty$ the walker has not explored any of the environment in direction $\ell$ from $X_{T_k}$, so $P(D_k = \infty) = P(T_k < \infty) = \delta > 0$. Therefore on the event that $\limsup X_n \cdot \ell = \infty$, we have repeated independent trials, and so eventually will have a $k$ with $D_k = \infty$. In other words, $K < \infty$ a.s. on $\{\limsup X_n \cdot \ell = \infty\}$. But if $K < \infty$, then some $O_m$ does not occur, and by Lemma 4.5 neither can $O$. In other words, $O$ fails a.s. on $\{\limsup X_n \cdot \ell = \infty\}$.

Note the absence of hypotheses in the following two results.

**Lemma 4.7.** $P(O) \in \{0,1\}$ and $P(A_+ \cup A_-) \in \{0,1\}$.

**Proof.** If (3.3) fails, then by Lemma 3.1, the random walk a.s. only visits finitely many sites. Thus $P(O) = 1$. So assume (3.3). Let $B = \{i = 1, \ldots, d : \mu(e_i \in G_o) + \mu(-e_i \in G_o) > 0\}$, and write $\ell = \sum_{i \in B} \ell[i] e_i + \sum_{i \in B^c} \ell[i] e_i = \ell_B + \sum_{i \in B} \ell[i] e_i$.

If $\ell_B = 0$ then $X_n \cdot \ell = 0$ for all $n$ almost surely so that $P(O) = 1$. Otherwise $\ell_B \neq 0$ and since $X_n \cdot \sum_{i \in B} \ell[i] e_i = 0$ for all $n$, we have that $O(\ell) \iff O(\ell_B), A_+(\ell) \iff A_+(\ell_B)$ and $A_- (\ell) \iff A_-(\ell_B)$. This has reduced the problem to a $|B|$-dimensional one, so without loss of generality we may assume that $|B| = d$. Then by considering reflections of the axes, we may
further assume that (4.2) holds. In this case, if \( P(A_+) > 0 \) then \( P(O) = 0 \) by Lemma 4.6. Similarly \( P(A_-) > 0 \) \( \Rightarrow \) \( P(O) = 0 \) by applying Lemma 4.6 to \(-\ell\). The result then follows from the fact that \( P(O) + P(A_+ \cup A_-) = 1 \).

With this preparation, we are ready for the following result, which implies that if the RWRE is transient in direction \( \ell \) almost surely, then the speed exists in that direction almost surely.

**Theorem 4.8.** Fix \( \ell \in \mathbb{R}^d \setminus o \). There exist deterministic \( v_+, v_- \) such that

\[
\lim_{n \to \infty} \frac{X_n \cdot \ell}{n} = v_+ I_{A_+} + v_- I_{A_-}, \quad P - a.s.
\]

**Proof.** We have the same hierarchy of possibilities as in Lemma 4.7. If (3.3) fails, then the random walk a.s. only visits finitely many sites, and \( n^{-1} X_n \to 0 \) a.s. Likewise, if all symmetric versions of (4.2) fail then the problem reduces to a lower dimensional one (where some symmetric version of (4.2) with smaller \( d \) does hold). Therefore without loss of generality we assume that both (3.3) and (4.2) hold. By Lemma 4.7 there are two cases to consider, namely \( P(A_+ \cup A_-) = 1 \) and \( P(A_+ \cup A_-) = 0 \).

The former is addressed in Theorem 3.2.2 of [11], for IID uniformly elliptic environments, drawing on ideas that go back to [8], together with contributions of Zerner and [10]. As pointed out in [12], the proof does not actually require uniform ellipticity, but works simply assuming ellipticity. In fact, even ellipticity is used only to obtain (4.3). In other words, the argument of [11] applies to IID environments satisfying (4.3) and \( P_o(A_+ \cup A_-) = 1 \). In particular, this proves our theorem when \( P(A_+ \cup A_-) = 1 \).

For completeness, we sketch the argument. Adopting notation from the proof of Lemma 4.6, let \( \tau_1 = D_K \). On the event \( A_+ \), \( \tau \) acts as a regeneration time, so conditional on \( A_+ \), the process \( \hat{X}_n = X_{\tau_1 + n} - X_{\tau_1} \) and the environment \( \hat{\omega}_x = \omega_{x + \tau_1} \) (for \( x \cdot \ell \geq 0 \)) are independent of the environment and walk observed up to time \( \tau_1 \). This allows one to construct additional regeneration times \( \tau_1 < \tau_2 < \ldots \) such that (conditional on \( A_+ \)) the \( X_{(\tau_k + n) \wedge \tau_{k+1}} - X_{\tau_k} \) are IID segments of path. If \( E[\tau_1] < \infty \), the strong law of large numbers now implies the existence of a deterministic speed \( v_+ \) on the event \( A_+ \). If \( E[\tau_1] = \infty \), one appeals to a calculation [11, Lemma 3.2.5] due to Zerner, which shows that

\[
E[(X_{\tau_{k+1}} - X_{\tau_k}) \cdot \ell | A_+] \leq \frac{C}{P(D_0 = \infty)} < \infty.
\]

This estimate is enough to give, by the law of large numbers again, that the speed \( v_+ \) exists on \( A_+ \) and \( = 0 \).

Note that the proof of [11, Lemma 3.2.5] is presented with \( \ell = e_1 \), in which case the left side of (4.4) is actually shown to equal \( 1/P(D_0 = \infty) \). With general \( \ell \) we have only been able to verify the inequality given in (4.4), but that certainly suffices for our purposes.

It remains to show the case \( P(A_+ \cup A_-) = 0 \). This is addressed in Theorem 1 of [12]. Again, this result is stated under stronger hypotheses, namely that the environment is IID and elliptic, and \( P(A_+ \cup A_-) = 0 \). But the proof carries over verbatim if ellipticity is replaced by the following weaker condition:

If \( \{ x \in \mathbb{Z}^d : a \leq x \cdot \ell \leq b \} \) is visited by \( X_n \) infinitely often,

then there a.s. exist \( n, m \) with \( X_n \cdot \ell < a \) and \( X_m \cdot \ell > b \).
The latter property holds in our setting, by Lemma 4.4.

**Corollary 4.9.** Assume that \( P(A_\ell) \in \{0, 1\} \) for each \( \ell \in \{e_1, \ldots, e_d\} \). Then there exists a deterministic \( v \in \mathbb{R}^d \) such that
\[
\lim_{n \to \infty} \frac{X_n}{n} = v.
\]

**Proof.** As in Corollary 2 of [12], apply Theorem 4.8 to each coordinate direction. Note that by Lemma 4.7, \( P(A_{e_k}) \in \{0, 1\} \Rightarrow P(A_{-e_k}) \in \{0, 1\} \).

**Theorem 4.10.** When \( d = 2 \), \( P(A_\ell) \in \{0, 1\} \).

**Proof.** Zerner and Merkl prove this (Theorem 1 of [15]), under the assumption of ellipticity. In fact, the proof is valid under the following conditions:
\[
P(A_+ \cup A_-) \in \{0, 1\}; \quad P(A_+) > 0 \Rightarrow P(X_n \cdot \ell \geq 0 \forall n) > 0; \quad P(A_-) > 0 \Rightarrow P(X_n \cdot \ell \leq 0 \forall n) > 0.
\]
The first property holds in our setting, by Lemma 4.7. The second and third properties holds by Lemma 4.6, provided that some symmetric version of (4.2) holds. Cases where all symmetric versions of (4.2) fail can be reduced to lower dimensional problems as in Theorem 4.8.

Note that one explicit apparent use of ellipticity in [15] involves having positive probability of 6 consecutive steps in direction \( e_1 \). In fact this is used only at a record time in direction \( \ell \) with \( \ell \cdot e_1 > 0 \), so the sites that these steps depend on have never been visited before. Thus the condition \( \mu(e_1 \in G_t) > 0 \) is sufficient to reproduce this part of the argument. A similar remark applies when adapting the argument for (4.4) in the proof of Theorem 4.8.

**Corollary 4.11.** Assume \( d = 2 \). There exists a deterministic \( v \in \mathbb{R}^2 \) such that
\[
\lim_{n \to \infty} \frac{X_n}{n} = v.
\]

**Proof.** Corollary 4.9 and Theorem 4.10.

---

5. Properties obtained by coupling

In this section we use coupling methods to prove a number of results, beginning with the monotonicity result Theorem 2.8.

5.1. Monotonicity. In this section we prove Theorem 2.8, via two results giving monotonicity (as a function of \( p \)) of certain quantities for 2-valued environments.

For a RWRE \( X \) in a 2-valued environment \((\gamma^1, \gamma^2)\), we let \( E^i = \{e \in E : \gamma^i(e) > 0\} \) and define
\[
N_n = \{0 \leq m < n : \omega_{X_m} = \gamma^1\}.
\]
We use \( P[p] \) to denote the law of \( X \) (averaged over environments) with parameter \( p = \mu(\omega_o = \gamma^1) \).

**Theorem 5.1.** For any 2-valued model \((\gamma^1, \gamma^2)\) with \( \gamma^1, \gamma^2 \neq \emptyset \), there exists a coupling under which, for all \( 0 \leq p < p' \leq 1 \), \( N_n[p'] \geq N_n[p] \) almost surely. Under this coupling we have \( X_n[p'] \cdot u \geq X_n[p] \cdot u \) for every \( n \geq 0 \) and every \( u \) such that \( u \in E^1 \cap (E^2)^c \) and \( -u \notin E^1 \).
Proof. Let \( \{U_x\}_{x \in \mathbb{Z}^d}, \{Y_i\}_{n \in \mathbb{N}}, \) and \( \{Z_n\}_{n \in \mathbb{N}} \) be independent random variables with distributions \( U[0,1], \gamma^1, \) and \( \gamma^2 \) under \( \mathbb{P} \) respectively. Define \( \omega[p] = (\omega_x[p])_{x \in \mathbb{Z}^d} \) by

\[
\omega_x[p] = \begin{cases} 
\gamma^1 & \text{if } U_x < p \\
\gamma^2 & \text{otherwise.}
\end{cases}
\]

Set \( X_0[p] = 0 \) and given \( X_0[p], \ldots, X_n[p] \) define,

\[
X_{n+1}[p] - X_n[p] = \begin{cases} 
Y_k, & \text{if } \omega_x[p] = \gamma^1, \text{ and } N_n[p] = k - 1 \\
Z_k, & \text{if } \omega_x[p] = \gamma^2, \text{ and } n - N_n[p] = k - 1.
\end{cases}
\]

One can easily check that \( X[p] \) is a RWRE in environment \( \omega[p] \).

Consider the first claim. Let \( p' > p \). Define \( T_1 = \inf \{ n \geq 1 : N_n[p] \neq N_n[p'] \} = \inf \{ n \geq 1 : N_n[p] < N_n[p'] \} \) and \( T_1^* = \inf \{ n \geq T_1 + 1 : N_n[p] = N_n[p'] \} \). Therefore if \( T_1 = \infty \), or if \( T_1 < \infty \) and \( T_1^* = \infty \), then there is nothing to prove. So assume \( T_1, T_1^* < \infty \). We have \( N_{T_1}[p'] = N_{T_1}[p] \). Under the given coupling \( X_{T_1}[p'] \) and \( X_{T_1}[p] \) have therefore taken exactly the same number of steps in each direction so that \( X_{T_1}[p'] = X_{T_1}[p] \), and the walks are recoupled. We can now repeat the above argument with \( n = \inf \{ n > T_1^* + \max(n) : N_n[p] \neq N_n[p'] \} = \inf \{ n \geq 1 : N_n[p] < N_n[p'] \} \) and \( T_2 = \inf \{ n \geq T_2 + 1 : N_n[p] = N_n[p'] \} \), etc. to get the first claim.

Suppose \( u \in E^1 \cap (E^2)^c \) but \( -u \notin E^1 \). Then the number of \( u \)-steps taken by the walk \( X[p] \) up to time \( n \) is \( \# \{ k \leq N_n[p] : Y_k = u \} \). The number of \( -u \) steps taken is \( \# \{ k \leq n - N_n[p] : Z_k = -u \} \).

The second claim now follows since \( N_n[p] \) is increasing in \( p \) for each \( n \) under this coupling. \( \blacksquare \)

We may apply Theorem 5.1 to 2-valued RWRE models \( (\uparrow, \downarrow), (\downarrow, \uparrow), \) and \( (\leftrightarrow, \leftrightarrow) \). In each case it shows that there exists a coupling under which \( X_n[p] \cdot (0,1) \) is almost surely increasing in \( p \) for all \( n \). Applied to the model \( (\uparrow, \downarrow) \) the theorem gives a coupling under which \( X_n[p] \cdot (1,1) \) is almost surely increasing in \( p \) for all \( n \). This in turn implies that for \( p \geq \frac{1}{3} \) the probability that the model is transient in the direction \((1,1)\) is at most \( \frac{1}{2} \), and by Theorem 4.10, it must be 0.

A non-trivial example where the first part of this Theorem applies, but does not imply monotonicity for the position of the walk, is the model \( (\uparrow, \leftrightarrow) \).

For two-valued environments \( \gamma^1, \gamma^2 \), suppose that the local drift for configuration \( \gamma^i \) is \( u_i, \) \( i = 1, 2, \) i.e. \( u_i = \sum_{e \in E} \gamma^i(e) e \). Then

\[(5.1) \quad \mathbb{E}_\omega[X_{n+1} - X_n] = \mathbb{E}_\omega[\mathbb{E}_\omega[X_{n+1} - X_n|X_n]] = \mathbb{E}_\omega\left[ \sum_{i=1}^2 u_i I_{\{\omega_{X_n} = \gamma^i\}} \right] = \sum_{i=1}^2 u_i \mathbb{P}_\omega(\omega_{X_n} = \gamma^i).
\]

Thus

\[(5.2) \quad \mathbb{E}_\omega[X_n] = \sum_{k=0}^{n-1} \mathbb{E}_\omega[X_{k+1} - X_k] = \sum_{i=1}^2 u_i \mathbb{E}_\omega\left[ \sum_{k=0}^{n-1} I_{\{\omega_{X_n} = \gamma^i\}} \right] = \sum_{i=1}^2 u_i \mathbb{E}_\omega[N^i_n],
\]

where \( N^i_n \) is the number of times (up to time \( n \)) that the RWDRE has departed from a site of type \( i \). If \( \omega \) is such that \( \mathbb{E}_\omega[n^{-1}X_n] \to v_\omega \) (which holds for example by dominated convergence if \( n^{-1}X_n \to v_\omega \) almost surely), then

\[(5.3) \quad v_\omega = \lim_{n \to \infty} u_1 \mathbb{E}_\omega[n^{-1}N_n] + u_2 \mathbb{E}_\omega[n^{-1}(N_n - n)] = u_2 + (u_1 - u_2) \lim_{n \to \infty} \mathbb{E}_\omega\left[ \frac{N_n}{n} \right].
\]
If $u_1 \neq u_2$, this implies that $\mathbb{E}_\omega \left[ \frac{N_n}{n} \right]$ converges to some $\rho_\omega$. Moreover, letting $\partial = u_1 - u_2$ and $\partial_\perp$ be such that $\partial_\perp \cdot \partial = 0$ we have $\partial_\perp \cdot u_\perp = \partial_\perp \cdot u_2$. In 2 dimensions, when both components of $\partial_\perp$ and $u_2$ are non-zero this gives a simple linear relationship between $v^{[1]}$ and $v^{[2]}$ that is independent of $p$ (see e.g. Table 1).

Reversing this argument we see that if $\mathbb{E}_\omega \left[ \frac{N_n}{n} \right] \rightarrow \rho_\omega$ for some $\rho_\omega$ then

$$
\mathbb{E}_\omega \left[ \frac{X_n}{n} \right] \rightarrow u_2 + (u_1 - u_2)\rho_\omega.
$$

The following Corollary to Theorem 5.1 immediately implies Theorem 2.8. It improves the monotonicity result of [7] in the case of 2-valued environments. Of course, earlier results do show the existence of deterministic speeds $v[p]$ and $v[p']$ in dimension $d = 2$ (as in the hypotheses).

**Corollary 5.2.** For any 2-valued model $(\gamma^1, \gamma^2)$, let $u_i$ be the local drift of $\gamma^i$, $i = 1, 2$. Suppose that there exist $v[p]$ and $v[p']$ such that $n^{-1}X_n[p] \rightarrow v[p]$, $P_{[p]}$-almost surely, and similarly for $v[p']$. Then for any $u \in \mathbb{R}^2 \setminus \partial$,

- $v[p'] \cdot u \geq v[p] \cdot u$ if $(u_1 - u_2) \cdot u \geq 0$
- $v[p'] \cdot u \leq v[p] \cdot u$ if $(u_1 - u_2) \cdot u \leq 0$.

**Proof.** If $v[p]$ exists almost surely, then from (5.3) we have that, for $\nu$-almost every $\omega[p]$,

$$v[p] = u_2 + (u_1 - u_2) \lim_{n \rightarrow \infty} \mathbb{E}_\omega[p] \left[ \frac{N_n}{n} \right].$$

It follows that also $\nu$-almost surely

$$v[p] \cdot u = u_2 \cdot u + (u_1 - u_2) \cdot u \lim_{n \rightarrow \infty} \mathbb{E}_\omega[p] \left[ \frac{N_n}{n} \right],$$

whence the limit on the right is deterministic and

$$(v[p'] - v[p]) \cdot u = (u_1 - u_2) \cdot u \lim_{n \rightarrow \infty} \left[ \mathbb{E}_\omega[p'] \left[ \frac{N_n}{n} \right] - \mathbb{E}_\omega[p] \left[ \frac{N_n}{n} \right] \right].$$

The deterministic limit on the right hand side is the difference of two $\nu$-almost surely convergent series, and is non-negative (e.g. under the coupling of Theorem 5.1). Let this limit be $a[p] \geq 0$. Then

$$(v[p'] - v[p]) \cdot u = a[p](u_1 - u_2) \cdot u,$$

as required. \[\blacksquare\]

### 5.2. Transience.

In this subsection we begin by stating a trivial coupling criterion, which guarantees that the RWRE is transient when some related walk is transient. We apply this criterion to prove transience results for some of our models.

**Lemma 5.3.** Suppose that a RWRE $\{X_n\}_{n \geq 0}$ can be coupled with a random walk $X'_n$, such that for all $n, m \geq 0$, $X_n = X_m \Rightarrow X'_n = X'_m$. Then

- if $\{X'_n\}_{n \geq 0}$ is transient (almost surely) then so is $\{X_n\}_{n \geq 0}$
- if $\{X'_n\}_{n < m} \cap \{X'_n\}_{n \geq m} = \emptyset$ then $\{X_n\}_{n < m} \cap \{X_n\}_{n \geq m} = \emptyset$ (i.e cut-times for $X'$ are also cut-times for $X$).
A natural application of this result is the following result, that concerns the high-dimensional analogue of the uniform RWDRE of Example 2.4 (which we call the orthant model). Recall that $E^i = \{e : \gamma^i(e) > 0\}$. 

**Corollary 5.4.** Let $X = X(d, p)$ denote the uniform RWDRE $(\gamma_1, \gamma_2)$ with $E^1 = \{e_i, i = 1, \ldots, d\}$ and $E^2 = -E^1$. Then

- when $d \geq 6$, $X$ is transient for all $p$, $P_p$-almost surely, and
- when $d \geq 10$, for each $p$, there exists $v[p]$ with $v[i][p]$ non-decreasing in $p$ for each $i = 1, \ldots, d$ such that $P_p(n^{-1}X_n \rightarrow v[p]) = 1$.

**Proof.** Fix $d \geq 6$ and define a $d'$-dimensional (with $d' = \lfloor d/2 \rfloor$) random walk $\{Y_n\}_{n \geq 0}$ by $Y_0 = X_0 = 0$, and for $n \geq 1$

$$Y_n - Y_{n-1} = \begin{cases} +e_i, & \text{if } X_n - X_{n-1} \in \{+e_{2i-1}, -e_{2i} : 2i \leq d\} \\ -e_i, & \text{if } X_n - X_{n-1} \in \{-e_{2i-1}, +e_{2i} : 2i \leq d\} \\ 0, & \text{otherwise}. \end{cases}$$

Then $(Y_n^{[1]}, \ldots, Y_n^{[d']}) = (X_n^{[1]}, \ldots, X_n^{[d]})A$ where $A^{[k,j]} = I_{j=2i-1} - I_{j=2i}$, i.e.

$$A^d = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Clearly then, $Y_n = Y_m$ whenever $X_n = X_m$. If $d$ is even, then $Y$ is a simple random walk in $d'$ dimensions. If $d$ is odd, then $Y$ is a random walk with nearest neighbour steps but also a $\frac{1}{d}$ probability of $Y$ in place. Thus $Y$ is transient when $d' \geq 3$, so $X$ is transient when $d \geq 6$ by Lemma 5.3. When $d' \geq 5$, $Y$ has well behaved cut-times. Therefore so does $X$, so it is shown [3] (see also [7]) that the velocity $v[p]$ exists (for all $p$). The monotonicity claim now follows from Corollary 5.2.

5.3. **Coupling with 1-d multi-excited random walks.** When there is sometimes a drift in direction $u$ but never a drift in direction $-u$ the walk should be almost surely transient in direction $u$ with positive speed. The transience result can be proved by considering the accumulated drift in direction $u$ (which is non-decreasing in this case) and adapting arguments appearing for example in Zerner [14], while we expect that a proof of the speed result requires the extension of more technical machinery (such as Kalikow’s condition, or methods used for the standard excited random walk) to our non-elliptic setting. The authors have some such proofs in preparation. Instead, in this paper we make stronger assumptions, essentially that with sufficiently large probability we have a sufficiently large local drift. This enables us to give a relatively simple coupling proof of the above claim, by appealing to results from [6, 13, 1].

For convenience, we state the result for the direction $u = e_1 + \cdots + e_d$, and we denote by $\overline{w}_x \in \mathbb{Z}^d$ the vector with entries $\overline{w}_x[i] = \omega_x(e_i) - \omega_x(-e_i)$. Therefore the mean quenched drift in the direction $u$ is $\sum_{e \in E} e \cdot u \omega_o(e) = \overline{w}_x \cdot u$.

**Proposition 5.5.** Fix $d \geq 2$ and let $u = e_1 + \cdots + e_d = (1, \ldots, 1)$. Suppose that $\mu(\overline{w}_x \cdot u \geq 0) = 1$ and $\mu(\overline{w}_x \cdot u \geq a) = b$ for some $a, b > 0$. If $\frac{ab}{1-b} > 1$ (resp. $> 2$) the RWRE $X$ is transient (resp. has positive lim sup speed) in direction $u$, $P$-almost surely.
Proof. If $b = 1$ the walk has a positive lim inf speed in direction $u$, by Lemma 4.3. So suppose that $b \in (0, 1)$. Then the conditional measures $\mu^+(\omega_o \in \cdot) = \mu(\omega_o \in \cdot \mid \omega_x \cdot u \geq a)$ and $\mu^-(\omega_o \in \cdot) = \mu(\omega_o \in \cdot \mid \omega_x \cdot u < a)$ are well defined. Let $\{W^+_j\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ and $\{W^-_j\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ be independent random variables with laws $\mu^+$ and $\mu^-$ respectively.

For $j \in \mathbb{Z}$ let $B_j = \{x \in \mathbb{Z}^d : x \cdot u = j\}$, and let $\{G_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ and $\{U_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{N}}$ be independent random variables with laws $\sim \text{Geometric}(1-b)$ and $\sim U[0, 1]$ respectively. The random variables $G_{j,k}$ will indicate the numbers of previously unvisited vertices in $B_j$ we have to visit before finding the next new site such that $\omega_x \cdot u < a$. Let $(u_1, \ldots, u_{2d}) = (e_1, \ldots, e_d, -e_1, \ldots, -e_d)$.

Let $X_0 = o$. Define $\omega_o = W^+_o$ if $G_{o,1} > 1$ and $\omega_o = W^-_o$ otherwise. Given $\{X_j, \omega_{X_j}\}_{j \leq n}$, let $Y_n = \sum X^{[k]}_n = X_n \cdot u$, and let $L_n(j) = \{|i \leq n : Y_i = j\}$ be the local time of $Y$ at $j$ up to time $n$. Set $L_n = L_n(Y_n)$. Then define

$$X_{n+1} - X_n = u_i, \quad \text{if } \sum_{j=1}^{i-1} \omega_{X_n}(u_j) < U_{Y_n, L_n} \leq \sum_{j=1}^{i} \omega_{X_n}(u_j), \quad i = 1, \ldots, 2d.$$ 

Given $\{X_k\}_{k \leq n+1}$ and $\{\omega_{X_k}\}_{k \leq n}$ let

$$\omega_{X_{n+1}} = \begin{cases} \omega_{X_l}, & \text{if } X_{n+1} = X_l, l < n+1; \\
W^-_{Y_{n+1}, r}, & \text{if } X_{n+1} \notin \{X_0, \ldots, X_n\}, \text{ and} \\
\{X_k\}_{k \leq n+1} \text{ has visited } r \text{ distinct sites in } B_{Y_{n+1}}, \text{ and } r \in \{\sum_{u=1}^{s} G_{Y_{n+1}, u}\}_{s \in \mathbb{N}} \\
W^+_{Y_{n+1}, r}, & \text{if } X_{n+1} \notin \{X_0, \ldots, X_n\}, \text{ and} \\
\{X_k\}_{k \leq n+1} \text{ has visited } r \text{ distinct sites in } B_{Y_{n+1}}, \text{ and } r \notin \{\sum_{u=1}^{s} G_{Y_{n+1}, u}\}_{s \in \mathbb{N}}. 
\end{cases}$$

The reader can check that $X$ is then a random walk in a random environment $\omega$ that is i.i.d. with $\omega_o \sim \mu$.

Note that the increments of $Y$ are in $\{-1, 1\}$ and that $\mathbb{P}(Y_{n+1} - Y_n = 1|\omega_{X_n}) = \sum_{j=1}^{d} \omega_{X_n}(u_i)$. For at least the first $G_{j,1} - 1$ visits of $X$ to $B_j$, the environment seen by the walker has law $\mu^+$ (not necessarily independently, as the same site could be visited more than once). Thus for at least the first $G_{j,1} - 1$ visits of $Y$ to $j$, the next increment of $Y$ has probability at least $\sum_{i=1}^{d} \omega(u_i) = (1 + \omega_x \cdot u)/2 \geq (1 + a)/2$ of being $+1$. On subsequent visits, independent of the history it has probability at least $\frac{1}{2}$ of being $+1$.

Now consider a random walk $Z$ on $\mathbb{Z}$, with $Z_0 = 0$, that evolves as follows. Given that $Z_n = j$ and $|\{k \leq n : Z_k = j\}| = r$,

$$Z_{n+1} - Z_n = \begin{cases} 1, & \text{if } r < G_{j,1} \text{ and } U_{j,r} \leq (1 + a)/2 \\
-1, & \text{if } r < G_{j,1} \text{ and } (1 + a)/2 < U_{j,r} \\
1, & \text{if } r = G_{j,1} \text{ and } U_{j,r} \leq \frac{1}{2} \\
-1, & \text{if } r = G_{j,1} \text{ and } \frac{1}{2} < U_{j,r} 
\end{cases}$$

The reader can check that this has coupled $Z$ and $Y$ together so that for all $j, r$ if $Y$ goes left on its $r$th departure from $j$ then so does $Z$ (if $Z$ visits $j$ at least $r$ times). It now follows from [6, Theorem 1.4] that if $Z$ is transient to the right then so is $Y$, moreover $\limsup_{n \to \infty} \frac{Z_n}{n} \leq \limsup_{n \to \infty} \frac{Y_n}{n}$.

The random walk $Z$ defined in (5.6) is a multi-excited random walk in a random cookie environment $\nu$ such that $\{\nu(i, \cdot)\}_{i \in \mathbb{Z}}$ are.i.i.d. with $\nu(0, r) = (1 + a)/2$ for $r < G_{0,1}$ and $\nu(0, r) = \frac{1}{2}$ for $r \geq G_{0,1}$ (i.e. a Geometric$(1 - b)$ number of cookies at each site). By [13, 1], $Z$ is transient to
the right (resp. has positive speed) if and only if \( \alpha = E[\sum_{k \geq 1}(2v(o,k) - 1)] > 1 \) (resp. > 2). The result now follows since

\[
E \left[ \sum_{i \geq 1} (2v(0,i) - 1) \right] \leq E \left[ \sum_{i=1}^{G_{0,1}} a \right] = \frac{ab}{1-b}.
\]

In particular, from Proposition 5.5 we conclude that the uniform RWDRE has positive speed in direction \( u = (1,1) \) in the following cases: \((\rightarrow, \leftrightarrow)\) and \((\rightarrow, \leftarrow \rightarrow)\) when \( p > \frac{6}{7} \) \((a = \frac{1}{3} \text{ and } b = p)\); and \((\rightarrow, \leftarrow \rightarrow)\) and \((\uparrow \rightarrow, \leftarrow \rightarrow)\) when \( p > \frac{2}{3} \) \((a = 1 \text{ and } b = p)\).

6. Calculation of speeds for uniform RWDRE

In earlier sections, we worked hard to prove transience in non-trivial settings, and in some cases showed existence of asymptotic speeds. It is worth pointing out that there are many uniform RWDRE for which it is actually obvious that transience holds and that speeds exist, due to the presence of a simple renewal structure. In a number of cases, speeds can be calculated explicitly.

We will sketch the argument in the case of Example 1.2, and will give a table, summarizing the results we know of in other 2-valued 2-dimensional models. Readers are referred to the authors’ websites for the detailed calculations in other cases.

**Lemma 6.1.** Assume (3.3) and suppose that \( \mu(\downarrow \in G_0) = 0 \) but \( \mu(\uparrow \in G_0) > 0 \). Then the RWRE is transient in direction \( e_2 \), \( P \)-almost surely. Let \( T \) be the first time the RWRE follows direction \( e_2 \). If \( E[T] < \infty \) then \( X_n \) has an asymptotic speed \( v = (v^{[1]}, \ldots, v^{[d]}) \), in the sense that \( P(n^{-1}X_n \to v) = 1 \). Moreover, \( v^{[i]} = E[X_T^{[i]}]/E[T] \).

**Proof.** The first claim follows from Theorem 3.6 with \( v = e_2 \). But the argument in this case is very simple: The random walk visits infinitely many sites, and at each visit to a new site there is positive (non-vanishing) probability of then taking a step in direction \( e_2 \). Thus the second coordinate of the random walk converges monotonically to \( \infty \).

Let \( \tau_k \) be the \( k \)'th time that \( X_n \) moves in direction \( e_2 \), and \( \tau_0 = 0 \). Let \( Y_k = X_{\tau_k} - X_{\tau_{k-1}} \). Since the environment seen by the random walker is refreshed at every time \( \tau_k \), the \( Y_k \) are IID, and the \( \tau_k \) are sums of IID random variables with distribution that of \( T \). Because \( E[T] < \infty \), it follows that \( E[|Y_k|] < \infty \) as well. By the law of large numbers, \( \tau_k/k \to E[T] \) and \( X_{\tau_k}/k \to E[Y_1] \) almost surely. Moreover \( k^{-1} \max\{|X_n - X_{\tau_{k-1}} : \tau_{k-1} \leq n \leq \tau_k\} \to 0 \).

Thus

\[
\frac{1}{n}X_n \to \frac{1}{E[T]} E[Y_1] = v \quad P\text{-almost surely.}
\]

**Lemma 6.2.** Consider the uniform RWDRE in the \((\uparrow \rightarrow, \uparrow \rightarrow)\) environment of Example 1.2. In other words, \( \mu(\{\uparrow, \rightarrow\}) = p \) and \( \mu(\{\uparrow, \leftarrow\}) = 1 - p \). The asymptotic speed is \((v^{[1]}, v^{[2]})\) with

\[
v^{[1]} = \frac{(2p-1)(p^2 - p + 6)}{6(2-p)(1+p)}, \quad v^{[2]} = \frac{1}{2}.
\]
Proof. For \( n \geq 0 \), let \( \tau_n = \inf\{m \geq 0 : X_m^{[2]} = n\} \). Then for \( i \geq 1 \), \( T_i = \tau_i - \tau_{i-1} \) are i.i.d. Geometric(1/2) random variables (with mean 2), and \( Y_i = X_{\tau_{i-1}}^{[1]} - X_{\tau_{i-1}}^{[1]} \) are i.i.d. random variables, independent of the \( \{T_i\}_{i \geq 1} \). So \( E[T_i] = 2 \) and \( v^{[2]} = 1/2 \). As in Lemma 6.1 we have (almost surely as \( n \to \infty \))

\[
\frac{Y_n^{[1]}}{n} \to \frac{E[Y_1]}{E[T_1]} = \frac{E[Y_1]}{2}.
\]

Letting \( Y = Y_1 \), it remains to calculate \( E[Y] \).

For \( j \geq 1 \), we can have \( Y = j \) three ways – reaching no \( \downarrow \uparrow \) vertex, reaching a \( \downarrow \uparrow \) vertex at \( (j, 0) \), or reaching a \( \uparrow \downarrow \) vertex at \( (j + 1, 0) \). Thus

\[
P(Y = j) = p^j (\frac{1}{2})^{j+1} + p^j (1-p) \sum_{n=0}^{\infty} (\frac{1}{2})^{j+2n+1} + p^{j+1} (1-p) \sum_{n=0}^{\infty} (\frac{1}{2})^{j+2n+3}
\]

\[
= \frac{p^j (4-p^2)}{3 \cdot 2^{j+1}}.
\]

Likewise, we can have \( Y = -j, j \geq 1 \) three ways, depending on where if anywhere \( X_n \) reaches a \( \uparrow \downarrow \) vertex, giving \( P(Y = -j) = \frac{(1-p)^j (4 - (1-p)^2)}{3 \cdot 2^{j+1}} \). The case \( j = 0 \) would be similar, but is not needed. Summing over \( j \) gives that

\[
E[Y] = \frac{p(4-p^2)}{12} \cdot \frac{1}{(1-p/2)^2} - \frac{1}{12} \cdot \frac{(1-p)(4 - (1-p)^2)}{(1-(1-p)/2)^2} = \frac{p(2+p)}{3(2-p)} - \frac{(1-p)(3-p)}{3(1+p)} = \frac{(2p-1)(p^2 - p + 6)}{3(2-p)(1+p)}.
\]

Table 1 summarizes what we know about uniform RWDRE in 2-dimensional 2-valued random environments. Explicit speeds are calculated as in Lemma 6.2. All other conclusions follow immediately from results stated in the paper. Note that many of the conjectures would follow if we knew that speeds were continuous in \( p \) and that monotonicity was strict.

Note that there is a related table in [4], giving percolation properties for the directed graphs \( \mathcal{C} \) and \( \mathcal{M} \). The latter includes 2-valued environments such as \( (\downarrow \rightarrow, \cdot) \) (site percolation), in which one of the possible environments has no arrows. These environments do not appear in the present table, because (as remarked in Section 2), the walk gets stuck on a finite set of vertices (in this case 1 vertex) the RWRE setup we have chosen requires that motion be possible in at least one direction.

Notes to Table 1

1. The authors believe it follows from results of Berger & Deuschel [2] that \( \mathcal{M} \) is recurrent \( \forall p \).
2. Bounds on the critical probability are given in [4]. Improved bounds are in preparation.
3. Improved ranges of values giving transience and speeds are in preparation.

Acknowledgements

Holmes’s research is supported in part by the Marsden fund, administered by RSNZ. Salisbury’s research is supported in part by NSERC. Both authors acknowledge the hospitality of the Fields Institute, where part of this research was conducted.
<table>
<thead>
<tr>
<th>$\gamma^1, \gamma^2$</th>
<th>Random walk</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\uparrow \rightarrow$</td>
<td>$v = (1 - p, p)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\uparrow \downarrow$</td>
<td>Stuck on two vertices.</td>
<td>Lemma 3.2</td>
</tr>
<tr>
<td>$\leftrightarrow \uparrow$</td>
<td>$v = \left(0, \frac{(1-p)^2}{p+1-(1-p)^2}\right)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\leftrightarrow \rightarrow$</td>
<td>$v = \left(\frac{1-p}{1+p}, 0\right)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\leftrightarrow \downarrow$</td>
<td>$v = (0, 0)$.</td>
<td>Symmetry$^1$</td>
</tr>
<tr>
<td>$\uparrow \rightarrow \rightarrow$</td>
<td>$v = \left(\frac{p}{2}, 1 - \frac{p}{2}\right)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\uparrow \rightarrow \uparrow$</td>
<td>$v = \left(\frac{(2p-1)(p^2-p+6)}{6(2-p)(1+p)}, \frac{1}{2}\right)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\uparrow \rightarrow \leftrightarrow$</td>
<td>$v = \left(\frac{1}{p^2} + \frac{(1-p)^2}{2p(1-p^2+p\log p)}\right)^{-1} \cdot (1, 1)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\uparrow \rightarrow \leftrightarrow$</td>
<td>$v = \left(\frac{p(2-p)}{2+3p-2p^2-p^5}\right) \cdot (3, 1) + (-1, 0)$.</td>
<td>As in Lemma 6.2</td>
</tr>
<tr>
<td>$\uparrow \rightarrow \leftrightarrow$</td>
<td>$v^{[1]} = v^{[2]} \uparrow$ in $p$. Transient$^2$ for $p \approx 0, 1$.</td>
<td>Cor. 3.8 / Cor. 5.2</td>
</tr>
<tr>
<td></td>
<td><strong>Conjecture:</strong> $v \neq 0$ for $p \neq \frac{1}{2}$, Recurrent when $p = \frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \downarrow$</td>
<td>$v^{[1]} = 1 - 3v^{[2]}$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \rightarrow$</td>
<td>$v^{[1]} = 0, v^{[2]} \downarrow$ in $p$. Transient$^2$ for $p \approx 0$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Conjecture:</strong> $\exists p(\neq 3/4)$ s.t. $v[p] = 0$. Recurrent for this $p$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \uparrow$</td>
<td>$v^{[1]} = 0, v^{[2]} &lt; 0$ for $p &gt; 0$. $v^{[2]}$ strictly $\downarrow$ in $p$.</td>
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<tr>
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<td></td>
<td>$v^{[1]} = 0, v^{[2]} \downarrow$ in $p$. Transient$^3$ for $p &gt; \frac{3}{4}$, $v^{[2]} &lt; 0$ for $p &gt; \frac{6}{7}$.</td>
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<tr>
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<td></td>
<td><strong>Conjecture:</strong> $v^{[2]} &lt; 0$ for $p &gt; 0$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \rightarrow \rightarrow$</td>
<td>$3v^{[2]} = 5v^{[1]} - 1$. $v^{[1]} \downarrow$ in $p$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$v^{[1]} = 1 + 3v^{[2]}$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \rightarrow \leftrightarrow$</td>
<td>$v \cdot (1, -1) = \frac{1}{3}$; $v \cdot (1, 1) \downarrow$ in $p$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$v^{[1]} = 0, v^{[2]} \downarrow$ in $p$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Conjecture:</strong> $v^{[2]} \neq 0$ for $p \neq \frac{1}{2}$, Recurrent when $p = \frac{1}{2}$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \leftrightarrow \uparrow$</td>
<td>$v^{[1]} = 0, v^{[2]} \downarrow$ in $p$. Transient$^3$ for $p &lt; \frac{1}{2}$, $v^{[2]} &gt; 0$ for $p &lt; \frac{1}{3}$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Conjecture:</strong> $v^{[2]} &gt; 0$ for $p &lt; 1$.</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \leftrightarrow \uparrow$</td>
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</tr>
<tr>
<td></td>
<td>$\leftrightarrow \leftrightarrow \downarrow$</td>
<td>$v = (0, 0)$</td>
</tr>
<tr>
<td></td>
<td>$\leftrightarrow \leftrightarrow \uparrow$</td>
<td>$v^{[1]} = 0, v^{[2]} \uparrow$ in $p$. Transient$^3$ for $p &lt; \frac{1}{4}$, $v^{[2]} &lt; 0$ for $p &lt; \frac{1}{7}$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td><strong>Conjecture:</strong> $v^{[2]} &lt; 0$ for $p &lt; 1$.</td>
</tr>
</tbody>
</table>

Table 1. Table of results for RW in 2-dimensional 2-valued degenerate random environments, where the first configuration occurs with probability $p \in (0, 1)$ and the other with probability $1 - p$. 

$$1 \leq k \leq 5$$
REFERENCES


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