Continuity of convolution and SIN groups

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Abstract

Let the measure algebra of a topological group $G$ be equipped with the topology of uniform convergence on bounded right uniformly equicontinuous sets of functions. Convolution is separately continuous on the measure algebra, and it is jointly continuous if and only if $G$ has the SIN property. On the space $\text{LUC}(G)^*$ which includes the measure algebra, convolution is also jointly continuous if and only if the group has the SIN property, but not separately continuous for many subgroups of the infinite symmetric group.

1 Introduction

When $G$ is a topological group, the set of all continuous right-invariant pseudometrics on $G$ induces the topology of $G$ and its right uniformity [6, sec.3.2] [8, 7.4]. In what follows, we denote by $G$ not only $G$ with its topology but also $G$ with its right uniformity. Since we do not consider other uniform structures on $G$, this convention will not lead to any confusion.

Say that a pseudometric on $G$ is bi-invariant iff it is both left- and right-invariant. A topological group $G$ is a SIN group iff its topology (equivalently, its right uniformity) is induced by the set of all continuous bi-invariant pseudometrics [8, 7.12].

Throughout the paper we assume that linear spaces are over the field $\mathbb{R}$ of real numbers, and functions are real-valued. Our results hold also when scalars are the complex numbers, with essentially the same proofs.

The space $\text{LUC}(G) = \text{U}_b(G)$ of bounded uniformly continuous functions on $G$ has a prominent role in abstract harmonic analysis. It is a Banach space with the sup norm. Its dual $\text{LUC}(G)^*$ is a Banach algebra in which the multiplication is the convolution operation $\star$, defined as follows. When $\varphi$ is an expression with several parameters, $\backslash_x \varphi$ denotes $\varphi$ as a function of $x$. Define

$$n \bullet f(x) := n(\backslash_y f(xy)) \quad \text{for } n \in \text{LUC}(G)^*, f \in \text{LUC}(G), x \in G$$

$$m \star n(f) := m(n \bullet f) \quad \text{for } m, n \in \text{LUC}(G)^*, f \in \text{LUC}(G)$$

Here $(n, f) \mapsto n \bullet f$ is the canonical left action of $\text{LUC}(G)^*$ on $\text{LUC}(G)$. 

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We identify every finite Radon measure $\mu$ on $G$ with the functional $m \in \text{LUC}(G)^*$ for which $m(f) = \int f \, d\mu$, $f \in \text{LUC}(G)$. That way the space $M_r(G)$ of finite Radon (a.k.a. tight) measures on $G$ is identified with a subspace of $\text{LUC}(G)^*$. With convolution, this is the measure algebra of $G$, often denoted simply $M(G)$.

Along with the norm topology, another topology on $\text{LUC}(G)^*$ and $M_r(G)$ commonly considered is the weak* topology $w(\text{LUC}(G)^*, \text{LUC}(G))$. Questions about separate weak* continuity of convolution on $\text{LUC}(G)^*$ lead to the problem of characterizing the weak* topological centre of $\text{LUC}(G)^*$ — see [3], [4] and Chapter 9 of [6]. Joint weak* continuity of convolution on $\text{LUC}(G)^*$ was studied by Salmi [9], who showed that convolution need not be jointly weak* continuous even on bounded subsets of $M_r(G)$.

Here we consider the UEB topology on the space $\text{LUC}(G)^*$. This topology, finer than the weak* topology, arises naturally in the study of continuity properties of convolution. When restricted to the space $M_r(G)$, the UEB topology and the weak* topology $w(M_r(G), \text{LUC}(G))$ are closely related: It follows from general results in Chapter 6 of [6] that these two topologies on $M_r(G)$ have the same dual $\text{LUC}(G)$ and the same compact sets (hence the same convergent sequences), and they coincide on the positive cone of $M_r(G)$.

The UEB topology may be defined independently of the group structure of $G$, for a general uniform space; for the details of the general theory we refer the reader to [6]. In our current setting of the right uniformity on a topological group $G$, the UEB topology is defined as follows. As in [6], for a continuous right-invariant pseudometric $\Delta$ on $G$ and $m \in \text{LUC}(G)^*$ let

$$\text{BLip}_b(\Delta) := \{f : G \to [-1,1] \mid |f(x) - f(y)| \leq \Delta(x,y) \text{ for all } x,y \in G\}$$

$$\|m\|_\Delta := \sup \{m(f) \mid f \in \text{BLip}_b(\Delta)\}$$

The UEB topology on $\text{LUC}(G)^*$ is the locally convex topology defined by the seminorms $\|\cdot\|_\Delta$ where $\Delta$ runs through continuous right-invariant pseudometrics on $G$.

In [5] the UEB topology is defined as the topology of uniform convergence on equi-$\text{LUC}$ subsets of $\text{LUC}(G)$. That definition is equivalent to the one given here, since by Lemma 3.3 in [6] for every equi-$\text{LUC}$ set $\mathcal{F} \subseteq \text{LUC}(G)$ there are $r \in \mathbb{R}$ and a continuous right-invariant pseudometric $\Delta$ on $G$ such that $\mathcal{F} \subseteq r\text{BLip}_b(\Delta)$.

When the group $G$ is locally compact and $M_r(G)$ is identified with the algebra of right multipliers of $L_1(G)$, the UEB topology on $M_r(G)$ coincides with the right multiplier topology [5, Th.3.3]. If $G$ is discrete then $\text{LUC}(G) = \ell_\infty(G)$ and the UEB topology on $\text{LUC}(G)^*$ is simply its norm topology. If $G$ is compact then $\text{LUC}(G)$ is the space of continuous functions on $G$ and the UEB topology is the topology of uniform convergence on norm-compact subsets of $\text{LUC}(G)$.

When the group $G$ is metrizable by a right-invariant metric $\Delta$, the seminorm $\|\cdot\|_\Delta$ on $\text{LUC}(G)^*$ is a particular case of the Kantorovich–Rubinshtein norm, which has many uses in topological measure theory and in the theory of optimal transport [1, 8.3] [10, 6.2]. In this case the topology of $\|\cdot\|_\Delta$ coincides with the UEB topology on bounded subsets of $\text{LUC}(G)^*$ [6, sec.5.4] but typically not on the whole space $\text{LUC}(G)^*$. As we show in section 3, when
considered on the whole space $\text{LUC}(G)^*$ or even $\text{M}_t(G)$, convolution behaves better in the UEB topology than in the $||\cdot||_\Delta$ topology.

Our results in this paper complement those in [5]. By Corollary 4.6 and Theorem 4.8 in [5], convolution is jointly UEB continuous on bounded subsets $\text{LUC}(G)^*$ when $G$ is a SIN group, and jointly UEB continuous on the whole space $\text{LUC}(G)^*$ when $G$ is a locally compact SIN group. Our main result (Theorem 3.2 in section 3) states that convolution is jointly UEB continuous on $\text{LUC}(G)^*$ if and only if it is jointly UEB continuous on $\text{M}_t(G)$ if and only if $G$ is a SIN group. In section 4 we prove that for every topological group $G$ convolution is separately continuous on $\text{M}_t(G)$. In section 5 we describe examples of topological groups $G$ for which convolution is not separately continuous on $\text{LUC}(G)^*$.

2 Preliminaries

In this section we establish several properties of SIN groups that are needed in the proof of the main theorem in section 3.

We specialize the notation of [6], where it is used for functions and measures on general uniform spaces, to the case of a topological group $G$. For every $x \in G$ we denote by $\partial(x)$ the point mass at $x$, the functional in $\text{LUC}(G)^*$ defined by $\partial(x)(f) = f(x)$ for $f \in \text{LUC}(G)$.

$\text{Mol}(G) \subseteq \text{LUC}(G)^*$ is the space of molecular measures; that is, finite linear combinations of point masses. Obviously $\text{Mol}(G) \subseteq \text{M}_t(G)$. For the molecular measure of the special form $m = \partial(x) - \partial(y)$, $x, y \in G$, and for any continuous right-invariant pseudometric $\Delta$ on $G$ we have $||m||_\Delta = \min(2, \Delta(x, y))$, by Lemma 5.12 in [6].

The UEB closure of $\text{Mol}(G)$ in $\text{LUC}(G)^*$ is the space $\text{M}_u(G) \supseteq \text{M}_t(G)$ of uniform measures on the uniform space $G$. In this paper we do not deal with the space $\text{M}_u(G)$; we only point out where a result that we prove for $\text{M}_t(G)$ holds more generally for $\text{M}_u(G)$. The reader is referred to [6] for the theory of uniform measures.

We start with a characterization of SIN groups which is one part of [8, 2.17].

Lemma 2.1. A topological group $G$ with identity element $e$ is a SIN group if and only if for every neighbourhood $U$ of $e$ there exists a neighbourhood $V$ of $e$ such that $xVx^{-1} \subseteq U$ for all $x \in G$.

Lemma 2.2. Let $G$ be a SIN group and $\Delta$ a bounded continuous right-invariant pseudometric on $G$. Then there is a continuous bi-invariant pseudometric $\Theta$ on $G$ such that $\Theta \geq \Delta$.

Proof. The proof mimics that of Lemma 3.3 in [6]. It is enough to consider the case $\Delta \leq 1$. As $G$ is a SIN group, there are continuous bi-invariant pseudometrics $\Theta_j$ for $j = 0, 1, \ldots$, such that

$$\forall x, y \in S \ [ \Theta_j(x, y) < 1 \Rightarrow \Delta(x, y) < \frac{1}{2j+1} ].$$

Define $\Theta$ by

$$\Theta(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \min(\Theta_j(x, y), 1).$$
If $x, y \in X$ and $j$ are such that $\Theta(x, y) < 1/2^j$ then $\Theta_j(x, y) < 1$, whence $\Delta(x, y) < 1/2^{j+1}$. It follows that $\Theta \geq \Delta$.

\[ \text{Corollary 2.3. Let } G \text{ be a SIN group. Then the UEB topology on } \text{LUC}(G)^* \text{ is defined by the seminorms } \| \cdot \|_\Delta \text{ where } \Delta \text{ runs through continuous bi-invariant pseudometrics on } G. \]

If $\Delta$ is a continuous or left- or right-invariant pseudometric on $G$, then so is the pseudometric $\sqrt{\Delta}$ defined by $\sqrt{\Delta}(x, y) := \sqrt{\Delta(x, y)}$ for $x, y \in G$.

In the sequel we deal with functions of the form $f/\sqrt{\| f \|}$ where $f \in \text{LUC}(G)$. To simplify the notation, we adopt the convention that $f/\sqrt{\| f \|} = f$ when $f$ is identically 0.

\[ \text{Lemma 2.4. Let } \Delta \text{ be a pseudometric on } G \text{ and } f \in \text{BLip}_b(\Delta). \text{ Then } f/\sqrt{\| f \|} \in \text{BLip}_b(2\sqrt{\Delta}). \]

\[ \text{Proof. Take any } x, y \in G. \text{ If } \| f \| \leq \Delta(x, y) \text{ then } \left| f(x)/\sqrt{\| f \|} - f(y)/\sqrt{\| f \|} \right| \leq \sqrt{\Delta(x, y)}, \]

hence

\[ \left| \frac{f(x)}{\sqrt{\| f \|}} - \frac{f(y)}{\sqrt{\| f \|}} \right| \leq 2\sqrt{\Delta(x, y)}. \]

If $\| f \| > \Delta(x, y) > 0$ then

\[ \left| \frac{f(x)}{\sqrt{\| f \|}} - \frac{f(y)}{\sqrt{\| f \|}} \right| \leq \frac{|f(x) - f(y)|}{\sqrt{\Delta(x, y)}} \leq \sqrt{\Delta(x, y)}. \]

The following lemma is a key ingredient in the proof of Theorem 3.2.

\[ \text{Lemma 2.5. Let } G \text{ be a topological group, } m, n \in \text{LUC}(G)^*, \text{ and let } \Delta \text{ be a continuous bi-invariant pseudometric on } G. \text{ Then} \]

\[ \| m \ast n \|_\Delta \leq \sqrt{2} \| m \|_{\sqrt{\Delta}} \| n \|_{\sqrt{\Delta}}. \]

\[ \text{Proof. Take any } f \in \text{BLip}_b(\Delta). \text{ As } \Delta \text{ is left-invariant, we have } \{ f(x) \in \text{BLip}_b(\Delta) \text{ for every } x \in G, \text{ and } \| n \cdot f \| \leq \| n \|_\Delta. \text{ Now } \text{BLip}_b(\Delta) \subseteq \text{BLip}_b(\sqrt{2\Delta}) \subseteq \text{BLip}_b(2\sqrt{\Delta}) \text{ because } \sqrt{2t} \geq t \text{ for } 0 \leq t \leq 2, \text{ and thus } \| f \|_\Delta \leq \| f \|_{\sqrt{2\Delta}} \leq \| f \|_{2\sqrt{\Delta}}. \text{ It follows that} \]

\[ \| n \cdot f \| \leq \| n \|_\Delta \leq \| n \|_{2\sqrt{\Delta}}. \text{ (1)} \]

For any $x, y \in G$ we have $g := \frac{1}{2} \{ f(xz) - f(yz) \} \in \text{BLip}_b(\Delta)$, hence $g/\sqrt{\| g \|} \in \text{BLip}_b(2\sqrt{\Delta})$ by Lemma 2.4. Moreover $2\| g \| \leq \Delta(x, y)$ because $\Delta$ is right-invariant, so that

\[ |n \cdot f(x) - n \cdot f(y)| = 2|n(g)| = 2\sqrt{\| g \|} \left| n \left( \frac{g}{\sqrt{\| g \|}} \right) \right| \leq \sqrt{2} \sqrt{\Delta(x, y)} \| n \|_{2\sqrt{\Delta}}. \text{ (2)} \]

Putting (1) and (2) together, we get $n \cdot f \in \sqrt{2\| n \|_{2\sqrt{\Delta}}} \text{BLip}_b(\sqrt{\Delta})$. Hence

\[ |m \ast n(f)| = |m(n \cdot f)| \leq \sqrt{2} \| m \|_{\sqrt{\Delta}} \| n \|_{2\sqrt{\Delta}}. \]
3 Joint UEB continuity

For any topological group $G$ the operation $\star$ is jointly UEB continuous on bounded subsets of $M_t(G)$ [5, 4.5]; in fact, even on bounded subsets of $M_u(G)$ [6, Cor.9.36]. However, as we shall see in this section, convolution need not be jointly UEB continuous on the whole space $M_t(G)$.

The UEB topology is defined by certain seminorms $\|\cdot\|_\Delta$. As a warm-up exercise, consider the continuity with respect to a single such seminorm: Let $G$ be a metrizable topological group whose topology is defined by a right-invariant metric $\Delta$. As we pointed out in the introduction, the topology of the norm $\|\cdot\|_\Delta$ coincides with the UEB topology on bounded subsets of $LUC(G)^*$. Hence $\star$ is jointly $\|\cdot\|_\Delta$ continuous on bounded subsets of $M_t(G)$. However, $\star$ is not jointly $\|\cdot\|_\Delta$ continuous on the whole space $M_t(G)$ or even $M_\Omega(G)$ for $G = \mathbb{R}$:

**Example 3.1.** Let $G$ be the additive group $\mathbb{R}$ with the usual metric $\Delta(x, y) = |x - y|$. For $j = 1, 2, \ldots$ let $m_j := n_j := j \left( \vartheta(1/j^2) - \vartheta(0) \right)$ and $f_j(x) := \min(1, |x - (1/j^2)|)$ for $x \in \mathbb{R}$. Then $f_j \in BLip_b(\Delta)$ and

$$m_j \star n_j = j^2 \left( \vartheta(2/j^2) - 2\vartheta(1/j^2) + \vartheta(0) \right)$$

$$\|m_j \star n_j\|_\Delta \geq m_j \star n_j(f_j) = 2$$

but $\lim_j \|m_j\|_\Delta = \lim_j \|m_j\|_\Delta = 0$.

Note that although the sequence $\{m_j\}$ converges in the norm $\|\cdot\|_\Delta$, it does not converge in the UEB topology; in fact, $\|m_j\|_{\sqrt{\Delta}} = 1$ for all $j$.

Next we shall see that the situation changes when we move from the topology defined by a single seminorm $\|\cdot\|_\Delta$ to the topology defined by all such seminorms, i.e. the UEB topology.

**Theorem 3.2.** The following properties of a topological group $G$ are equivalent:

(i) Convolution is jointly UEB continuous on $LUC(G)^*$.

(ii) Convolution is jointly UEB continuous on $M_t(G)$.

(iii) Convolution is jointly UEB continuous on $M_\Omega(G)$.

(iv) $G$ is a SIN group.

**Proof.** Obviously (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

To prove (iii)$\Rightarrow$(iv), assume that convolution is jointly UEB continuous on $M_\Omega(G)$. Take any neighbourhood $U$ of the identity element $e$. There is a continuous right-invariant pseudometric $\Theta$ such that $\{z \in G \mid \Theta(z, e) < 1\} \subseteq U$. By the UEB continuity there are a continuous right-invariant pseudometric $\Delta$ and $\varepsilon > 0$ such that if $m, n \in M_\Omega(G)$, $\|m\|_\Delta, \|n\|_\Delta \leq \varepsilon$ then $\|m \star n\|_\Theta < 1$. To conclude that $G$ is a SIN group, in view of Lemma 2.1 it is enough to
show that $xVx^{-1} \subseteq U$ for all $x \in G$, where $V := \{v \in G \mid \Delta(v, e) < \varepsilon^2\}$. To that end, take any $x \in G$ and $v \in V$ and define

$$
m := \varepsilon \partial(x)
$$

$$
n := (\partial(v) - \partial(e))/\varepsilon
$$

Then $\|m\|_\Delta = \varepsilon$ and $\|n\|_\Delta = \min(2, \Delta(v, e))/\varepsilon < \varepsilon$, hence

$$
\min(2, \Theta(xv, x)) = \|\partial(xv) - \partial(x)\|_\Theta = \|m \ast n\|_\Theta < 1
$$

and therefore $\Theta(xvx^{-1}, e) = \Theta(xv, x) < 1$ and $xvx^{-1} \in U$. That completes the proof of (iii)$\Rightarrow$(iv).

To prove (iv)$\Rightarrow$(i), assume that $G$ is SIN. Take any continuous bi-invariant pseudometric $\Delta$ on $G$. By Lemma 2.5, if $m, m_0, n, n_0 \in \text{LUC}(G)\ast$ are such that $\|m - m_0\|_{\sqrt{\Delta}} < \varepsilon$ and $\|n - n_0\|_{2\sqrt{\Delta}} < \varepsilon$ then

$$
\|m \ast n - m_0 \ast n_0\|_\Delta \leq \|(m - m_0) \ast n\|_\Delta + \|m_0 \ast (n - n_0)\|_\Delta
$$

$$
\leq \sqrt{2} \varepsilon \|n\|_{2\sqrt{\Delta}} + \sqrt{2} \|m_0\|_{\sqrt{\Delta}} \varepsilon
$$

$$
\leq \sqrt{2} \varepsilon (\varepsilon + \|n_0\|_{2\sqrt{\Delta}} + \|m_0\|_{\sqrt{\Delta}})
$$

which along with Corollary 2.3 proves that $\ast$ is jointly UEB continuous at $(m_0, n_0)$. \qed

## 4 Separate UEB Continuity

By Theorem 3.2, convolution is jointly UEB continuous on $\text{LUC}(G)\ast$, and therefore also separately UEB continuous, whenever $G$ is a SIN group. On the other hand, as we explain in section 5, there are topological groups $G$ for which convolution is not separately UEB continuous on $\text{LUC}(G)\ast$. Nevertheless, we now prove that convolution is separately UEB continuous on $M_b(G)$ for every topological group $G$. The same proof may be used to show that convolution is separately UEB continuous even on $M_0(G)$.

**Lemma 4.1.** Let $G$ be a topological group, $m \in M_c(G)$, and let $\Delta$ be a continuous right-invariant pseudometric on $G$. Then there exists a continuous right-invariant pseudometric $\Delta_m$ such that $\nabla y m(\nabla x f(xy)) \in \|m\|_{\text{BLip}_b(\Delta_m)}$ for every $f \in \text{BLip}_b(\Delta)$.

**Proof.** Evidently $\|\nabla y m(\nabla x f(xy))\| \leq \|m\|$ for every $f \in \text{BLip}_b(\Delta)$. To prove that the function $\nabla y m(\nabla x f(xy))$ is Lipschitz for a suitable $\Delta_m$, first note that if $m = \sum_j c_j m_j$, $m_j \in \text{LUC}(G)^\ast$, is a finite linear combination such that $|m_j(\nabla x f(xy)) - m_j(\nabla x f(xz))| \leq \Delta_j(y, z)$ for every $j$ and $y, z \in G$, then $|m(\nabla x f(xy)) - m(\nabla x f(xz))| \leq \Delta'(y, z)$ where $\Delta' = \sum_j |c_j| \Delta_j$. Thus it is enough to prove the lemma assuming that $m \geq 0$.

We may also assume that $\Delta$ is bounded, as replacing $\Delta$ by $\min(\Delta, 2)$ does not change $\text{BLip}_b(\Delta)$. For $m \geq 0$ and a bounded $\Delta$, define $\Delta_m$ by

$$
\Delta_m(y, z) := m(\nabla x \Delta(xy, xz)) \text{ for } y, z \in G.
$$
Clearly $\Delta_m$ is a right-invariant pseudometric. To see that it is continuous, first apply the estimate

$$|\Delta(xy, x) - \Delta(wy, w)| \leq \Delta(xy, wy) + \Delta(x, w) = 2\Delta(x, w)$$

which shows that $\Delta_m(y, e) \subseteq 2\textup{BLip}_b(\Delta)$ for every $y \in G$. Since $m$ is continuous on $2\textup{BLip}_b(\Delta)$ with the topology of pointwise convergence, it follows that $\Delta_m(y, e)$ is continuous on $G$.

For any $f \in \textup{BLip}_b(\Delta)$ we have

$$|m(\langle x f(xy) \rangle) - m(\langle x f(xz) \rangle)| \leq m(\langle |f(xy) - f(xz)| \rangle) \leq m(\langle \Delta(xy, xz) \rangle) = \Delta_m(y, z)$$

for $y, z \in G$.

**Theorem 4.2.** For every topological group $G$ convolution is separately UEB continuous on $M_t(G)$.

*Proof.* For every $n \in \textup{LUC}(G)^*$ the mapping $m \mapsto m \ast n$ is UEB continuous — this is a special case of [6, Cor.9.21].

For $m \in M_t(G)$ and $n \in \textup{LUC}(G)^*$ we may reverse the order of applying $m$ and $n$ in the definition of convolution:

$$m \ast n(f) = n(\langle y m(\langle x f(xy) \rangle) \rangle) \quad \text{for } f \in \textup{LUC}(G).$$

This is a consequence of a variant of Fubini’s theorem; see [6, sec.9.4] for a proof and discussion.

The UEB continuity of the mapping $n \mapsto m \ast n$ for every $m \in M_t(G)$ now follows from Lemma 4.1. \qed

In analogy with the commonly studied weak$^*$ topological centre of $\textup{LUC}(G)^*$, we may also consider its UEB topological centre $\Lambda_{UEB}$, the set of those $m \in \textup{LUC}(G)^*$ for which the mapping $n \mapsto m \ast n$ is UEB continuous on $\textup{LUC}(G)^*$. Then $\Lambda_{UEB} = \textup{LUC}(G)^*$ for every SIN group $G$ by Theorem 3.2. In the next section we describe examples of topological groups $G$ for which $\Lambda_{UEB} \neq \textup{LUC}(G)^*$. It would be useful to have a tractable description of $\Lambda_{UEB}$ for general topological groups, or at least for locally compact ones.

## 5 Counterexamples for separate UEB continuity

In this section we work in the uniform compactification $G^\text{LUC} = \hat{p}G$ of a topological group $G$ (with its right uniformity). As in [6, sec.6.5], we identify each $x$ of $G$ with the point mass $\delta(x) \in \text{LUC}(G)^*$. Then the compactification $G^\text{LUC}$ is simply the weak$^*$ closure of $G$ in $\text{LUC}(G)^*$. In $\text{LUC}(G)^*$ with the convolution operation, $G^\text{LUC}$ is a subsemigroup.

By Example 4.7 in [5] (which is also Example 9.39 in [6]), convolution is not separately UEB continuous on $\text{LUC}(G)^*$ when $G$ is the group of homeomorphisms of the interval $[0, 1]$ onto itself: There are $m, n \in G^\text{LUC}$ and a sequence $\{x_j\}_{j \in \omega}$ in $G$ such that $\lim_j x_j = n$ in the
UEB topology but the sequence $m \ast x_j$ does not converge to $m \ast n$ even in the weak$^*$ topology. In this section we show that, for the same group and for many other topological groups, such an example may constructed with the sequence $\{x_j\}_{j \in \omega}$ converging to the identity element of $G$.

We shall carry out the construction in certain subgroups of the infinite symmetric group $S_\infty$, i.e. the group of bijections of the countable infinite set $\omega$ onto itself with the topology of pointwise convergence [7, 2.4]. Then we use embeddings of such subgroups to produce examples of discontinuity for other groups as well.

The topology of $S_\infty$ is defined by the metric

$$\Delta(\pi,\sigma) := \sum \{2^{-n} \mid n \in \omega, \pi(n) \neq \sigma(n)\}, \quad \pi, \sigma \in S_\infty.$$ 

To be consistent with the left–right choices in the previous sections, we shall study subgroups of $S_\infty$ with their right uniformities. To make $\Delta$ right-invariant on $S_\infty$, we write the composition of mappings in $S_\infty$ somewhat non-traditionally left to right:

$$\pi \sigma(n) = \sigma(\pi(n)) \text{ for } \pi, \sigma \in S_\infty, n \in \omega.$$ 

Denote by $\text{id}$ the neutral element of $S_\infty$ (the identity mapping from $\omega$ to $\omega$).

For every $\pi \in S_\infty$ denote by $\text{fix}(\pi)$ the set of points fixed by $\sigma$:

$$\text{fix}(\pi) := \{n \in \omega \mid \pi(n) = n\}.$$ 

As in [2], when $G$ is a subgroup of $S_\infty$ and $A \subseteq \omega$, denote by $G_{(A)}$ the pointwise stabiliser of $A$:

$$G_{(A)} := \{\pi \in G \mid A \subseteq \text{fix}(\pi)\}.$$ 

We consider subgroups $G$ of $S_\infty$ with these two properties:

(IO) There exists $a \in \omega$ such that the orbit $G \cdot a := \{\pi(a) \mid \pi \in G\}$ is infinite.

(FF) For every finite $A \subseteq \omega$ there exists $\pi \in G_{(A)}$ for which the set $\text{fix}(\pi)$ is finite.

Note that if $G$ has property (FF) then its topology is not discrete.

Property (FF) is often easy to verify for specific subgroups of $S_\infty$, such as those in the examples at the end of this section. Moreover, it is not difficult to prove (FF) for every automorphism group of a countable Fraïssé structure whose age has the strong amalgamation property (defined in section 2.7 of [2]).

**Theorem 5.1.** Let $G$ be a subgroup of $S_\infty$ that has properties (IO) and (FF). Then there are $m \in \text{GLUC}^*$ and a sequence $\{\sigma_j\}_{j \in \omega}$ in $G$ such that $\lim_j \sigma_j = \text{id}$ and the sequence $\{m \ast \sigma_j\}_j$ does not converge to $m$ in the compact topology of $\text{GLUC}^*$.

When the conclusion of the theorem holds for $G$, convolution is not separately UEB continuous on $\text{LUC}(G)^*$ (not even on $\text{GLUC}^*$). In fact, if a topological group $G$ is identified with a subspace of $\text{LUC}(G)^*$ by means of the mapping $\partial: G \to \text{LUC}(G)^*$, then the topology of $G$ coincides with the UEB topology restricted to $G$ [6, sec.6.5].
Proof. By (FF) for every \( j \in \omega \) there is \( \sigma_j \in G \) such that \( \sigma_j \in G_{\{0,1,\ldots,j\}} \) and \( \text{fix}(\sigma_j) \) is finite. By (IO) there are \( a \in \omega \) and \( \pi_i \in G \), \( i \in \omega \), such that \( \pi_i(a) \neq \pi_j(a) \) for \( i \neq j \). We shall construct a sequence of indices \( i_0 < i_1 < i_2 < \ldots \) in \( \omega \) such that for every \( m \in \omega \) we have

\[
\sigma_j(\pi_{i_k}(a)) \neq \pi_{i_{k'}}(a) \quad \text{for all } j \leq k \leq m \text{ and } k' \leq m.
\] (3)

To satisfy (3) for \( m = 0 \) it is enough to choose \( i_0 \) so that \( \pi_{i_0}(a) \notin \text{fix}(\pi_0) \).

Suppose that \( i_0 < i_1 < \ldots < i_m \) for some \( m \geq 0 \) have been constructed so that (3) holds. As the set \( A_m := \{ \sigma_j(\pi_{i_k}(a)) \mid j \leq k \leq m \} \) is finite, we have \( \pi_i(a) \notin A_m \cup \bigcup_{j=m+1} \text{fix}(\sigma_j) \) for all \( i \) large enough. Similarly, the set \( B_m := \{ \pi_i(a) \mid \ell \leq m \} \) is finite and for every \( j \in \omega \) the elements of the sequence \( \{ \sigma_j(\pi_i(a)) \}_{i \in \omega} \) are pairwise distinct. Therefore \( \sigma_j(\pi_i(a)) \notin B_m \) for all \( i \) large enough and all \( j \leq m + 1 \).

It follows that there is \( i_{m+1} > i_m \) such that \( \pi_{i_{m+1}}(a) \notin A_m \cup \bigcup_{j=m+1} \text{fix}(\sigma_j) \) and \( \sigma_j(\pi_{i_{m+1}}(a)) \notin B \) for all \( j \leq m + 1 \). That yields (3) for \( m + 1 \) in place of \( m \).

Since (3) holds for all \( m \), we have

\[
\sigma_j(\pi_{i_k}(0)) \neq \pi_{i_{k'}}(0) \quad \text{for all } j, k, k' \in \omega \text{ such that } j \leq k.
\]

In other words, \( \sigma_j(\pi_{i_k}(0)) \notin Z \) for all \( j \leq k \), where \( Z := \{ \pi_{i_k}(0) \mid k \in \omega \} \).

Define \( f \in \text{LUC}(G) \) by

\[
f(\pi) := \begin{cases} 
1 & \text{when } \pi(a) \in Z \\
0 & \text{when } \pi(a) \notin Z 
\end{cases}
\]

Fix a free ultrafilter \( \mathcal{U} \) on \( \omega \) and let \( m := \lim_{k \to \mathcal{U}} \pi_{i_k} \in G^{\text{LUC}} \). As \( \lim_{k \to \mathcal{U}} f(\pi_{i_k}) = 0 \) for every \( j \),

\[
\lim_{k \to \mathcal{U}} m * \sigma_j(f) = \lim_{k \to \mathcal{U}} \lim_{k \to \mathcal{U}} f(\pi_{i_k}) = 0 \neq 1 = \lim_{k \to \mathcal{U}} f(\pi_{i_k}) = m(f).
\]

At the same time \( \lim_{j} \sigma_j = \text{id} \) because \( \sigma_j \in G_{\{0,1,\ldots,j\}} \). \( \square \)

To exhibit a discontinuity such as that in Theorem 5.1 for a topological group \( H \), it is enough to show that the theorem applies to a subgroup \( G \) of \( H \). Indeed, if \( G \) is a topological subgroup of \( H \) then \( G \) is a uniform subspace of \( H \) when both are considered with their right uniformities [8, 3.24]. Hence \( G^{\text{LUC}} \) is embedded in \( H^{\text{LUC}} \), both topologically and algebraically (with the convolution operation).

Examples 5.2. Theorem 5.1 applies to groups of automorphisms of many structures. For each such group \( G \) convolution is not separately UEB continuous on \( \text{LUC}(G)^* \). For example:

Infinite symmetric group. The group \( S_\infty \) itself has properties (IO) and (FF).

Automorphisms of rationals with linear order. \( \text{Aut}(\mathbb{Q}, \leq) \), the group of order-preserving mappings of the set \( \mathbb{Q} \) of rational numbers onto itself with the topology of pointwise convergence [7, 6.5.20]. \( \text{Aut}(\mathbb{Q}, \leq) \) as a subgroup of \( S_\infty \) has properties (IO) and (FF).
Automorphisms of the infinite random graph. The group of automorphisms of the infinite random graph \([7, 6.5.21]\) has properties (IO) and (FF).

Homeomorphisms of the Cantor set. \(\text{Homeo}_+(2^\omega)\), the group of autohomeomorphisms of the compact space \(2^\omega\) that preserve the lexicographic order on \(2^\omega\), with the sup topology. The countable set \(D \subseteq 2^\omega\) of the sequences that are eventually constant is dense in \(2^\omega\) and mapped to itself by every element of \(\text{Homeo}_+(2^\omega)\). It then follows that \(\text{Homeo}_+(2^\omega)\) is isomorphic to \(\text{Aut}(D, \leq)\) with the topology of pointwise convergence, which as a subgroup of \(S_\infty\) has properties (IO) and (FF).

Homeomorphisms of \([0,1]\). \(\text{Homeo}([0,1])\), the group of autohomeomorphisms of the interval \([0,1]\) with the sup topology. The group \(\text{Homeo}_+(2^\omega)\) is isomorphic to a subgroup of \(\text{Homeo}([0,1])\), hence from the previous example we again obtain an instance of UEB discontinuity of convolution on \(\text{LUC}(\text{Homeo}([0,1]))^*\).

We do not know any simple characterization of the subgroups \(G\) of \(S_\infty\) for which convolution is not separately UEB continuous on \(\text{LUC}(G)^*\); nor of those subgroups that admit a more special discontinuity constructed in Theorem 5.1.

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