Foreman constructed an amenable discrete group with a unique amenable action on the integers assuming $\mathfrak{p} = \mathfrak{c}$. This note will modify his contruction to yield an amenable topological group with a unique amenable action on the integers without any additional set theoretic assumptions (thus answering a question of Pestov?).

Let $\{\xi_n\}_{n=0}^\infty$ enumerate all finite one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. For any partial partial involution σ let $\overline{\sigma}$ denote the permutation that agrees with σ on its domain and is the identity elsewhere, Construct by induction on n sets $\{A^j_n\}_{j=0}^\infty$ and involutions $\sigma^j_n : A^j_n \to A^{j+1}_n$ such that

(1) $\mathbb{N} \setminus \bigcup_{i=0}^n \bigcup_{j=0}^J A^j_i$ is infinite for each J
(2) $A^j_n \cap A^k_n = \emptyset$ of $j \neq k$.
(3) If $A_n = \bigcup_{j=1}^\infty A^j_n$ then $A_n \cap A^j_m$ is finite for $m < n$ and all j.
(4) If there are $\{B^j\}_{j=0}^\infty$ such that setting $A^j_n = B^j$ yields that (1), (2) and (3) hold and if for some j there is an involution $\theta : B^j \to B^{j+1}$ such that $\xi_n \subseteq \overline{\theta}$ then $\xi_n \subseteq \overline{\sigma}^j_n$.

Let $\mathbb{A}$ be the group generated by $\{\overline{\sigma}^j_n\}_{n,j\in\omega}$.

Extend the ideal generated by $\{A^j_n\}_{n,j\in\omega}$ to a maximal ideal $\mathcal{J}$ and for each $X \in \mathcal{J}$ choose A_X such that $A_X \cap X = \emptyset$ and $A_X \cap A^j_n$ is finite for all n and j. This is possible because $\{A^j_n \setminus X \mid n, j \in \omega\}$ generates a proper ideal on the complement of X. Let $\{A^j_X\}_{j=1}^\infty$ partition A_X into infinite sets and choose $\theta^j_X : A^j_X \to A^{j+1}_X$ to be involutions. Let $\mathbb{G}$ be the group generated by $\mathbb{A} \cup \{\overline{\theta}^j_X\}_{X\in\mathcal{J},j\in\omega}$. Let $\mathbb{G}$ inherit the topology of pointwise convergence from the full symmetric group.

Claim 1. $\mathbb{A}$ is dense in $\mathbb{G}$.

Proof. It suffices to show that a generating set for $\mathbb{G}$ is contained in the closure of $\mathbb{A}$. It may be assumed that a given neighbourhood of $\overline{\theta}^j_X$ is of the from $[\xi] = \{g \in \mathbb{G} \mid \xi \subseteq g\}$. Let n be such that $\xi = \xi_n$. Since $A_X \cap A^0_n$ is finite it follows that letting $X = B^0$ and $B^j = A^j_X$ for $j \geq 1$ yields $\{B^j\}_{j=0}^\infty$ such that setting $A^j_n = B^j$ yields that (1), (2) and (3) hold. Since $\xi \subseteq \overline{\theta}^j_X$ it follows that and $\xi_n \subseteq \overline{\sigma}^j_n$. In other words, $\overline{\sigma}^j_n \in \mathbb{A}$ belongs to the given neighbourhood of $\overline{\theta}^j_X$. ☐

Claim 2. $\mathbb{A}$ is amenable,

Proof. That $\mathbb{A}$ is amenable is just Foreman's argument restricted to $\mathbb{A}$. In fact, $\mathbb{A}$ is locally finite. To see this let $\mathcal{F}$ be a finite subset of $\{\overline{\sigma}^j_n\}_{n,j\in\omega}$. Begin by observing that if the group generated by $\mathcal{F}$ is finite if and only if there is a uniform bound for the size of the orbit $O_{\mathcal{F}}(n)$ of n under $\mathcal{F}$.

Let K be the largest integer such that there is some j such that $\overline{\sigma}^j_K \in \mathcal{F}$. Proceed by induction on K to show that there is a uniform bound for the size of $O_{\mathcal{F}}(n)$ under $\mathcal{F}$. If $K = 0$ it is easy to see that the size of $O_{\mathcal{F}}(n)$ is bounded by $2|\mathcal{F}|$. Let H be the group generated by $\{\overline{\sigma}^j_K\}_{j=0}^\infty \cap \mathcal{F}$. For $K \geq 1$ let $\mathcal{F}^*$ be the set of $\overline{\sigma}^j_m \in \mathcal{F}$ such that $m < K$ and let $W = A_K \cap \cup \left\{A^j_i \;\middle|\; \sigma^j_i \in \mathcal{F}^*\right\}$. Then W is finite and so are $H(W)$ and $\mathcal{F}^*(W)$ the orbits of W under H and $\mathcal{F}^*(W)$. As well, $\mathcal{F}^*(H(W))$ is finite. It suffices, therefore, to consider only $n \notin W \cup \mathcal{F}^*(H(W)) \cup \mathcal{F}^*(W)$.

It furthermore suffices to show that for any such n, if f is in the group generated by $\mathcal{F}$ then $f(n) = h_0 f^* h_1(n)$ where f^* is in the group generated by $\mathcal{F}^*$ and h_0 and h_1 belong to H. The reason this suffices is the fact, already noted, that the orbit of an integer under the group H has size bounded by twice the cardinality of H. All that needs to be checked now is that if $h \in H$ and $\sigma_0 = \overline{\sigma}^{j(0)}_{i(0)}$ and $\sigma_1 = \overline{\sigma}^{j(1)}_{i(1)}$ are in $\mathcal{F}^*$ then $\sigma_0 h \sigma_1(n)$ is equal to one of the following:

- $\sigma_0 h(n)$
- $\sigma_0 \sigma_1(n)$
- $h\sigma_1(n)$.

To see this, first suppose that $\sigma_1(n) = n$. Then $\sigma_0 h \sigma_1(n) = \sigma_0 h(n)$. Otherwise, it must be that $\sigma_1(n) \in A^{j(1)}_{i(1)}$. Since $n \notin \mathcal{F}^*(W)$ it must be that $\sigma_1(n) \notin W$ and, hence, $\sigma_1(n) \notin A_K$. This implies that either $h(\sigma_1(n)) = \sigma_1(n)$ or $h(\sigma_1(n)) \in A_K$. In the first case, $\sigma_0 h\sigma_1(n) = \sigma_0\sigma_1(n)$. In the second, it must be that either $\sigma_0(h(\sigma_1(n))) = h(\sigma_1(n))$ or $h(\sigma_1(n)) \in A^{j(0)}_{i(0)}$. If the first alternative holds, then $\sigma_0 h\sigma_1(n) = h\sigma_1(n)$. However, it is not possible that $h(\sigma_1(n)) \in A^{j(0)}_{i(0)}$ because this would imply that $h(\sigma_1(n)) \in W$ contradicting that $n \notin \sigma_1(H(W))$. □

By Fremlin 449Y (ii) a topological group with a dense amenable subgroup is amenable and so, it follows from Claims 1 and 2 that $\mathbb{G}$ is amenable.

Claim 3. The natural action of $\mathbb{G}$ on $\mathbb{N}$ is uniquely amenable.

Proof. Let μ be an arbitrary $\mathbb{G}$-invariant probability measure and suppose that the null sets of μ do not coincide with $\mathcal{J}$. The maximality of $\mathcal{J}$ implies that there must be some $Z \in \mathcal{J}$ such that $\mu(Z) > 0$. The sets $Z, \theta^0_Z, \theta^1_Z\theta^0_Z, \ldots$ are pairwise disjoint contradicting that μ is a $\mathbb{G}$-invariant probability measure. □