Haar null sets and the consistent reflection of nonmeagerness

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Abstract

It will be shown that assuming the existence of Woodin cardinal with a measurable cardinal above it, it is consistent that every non meagre subset of the Erdös-Kakutani group intersects a translate of a small set on a relatively non meagre set.

The Erdös-Kakutani group will be denoted by $\mathbb{EK} = \Pi_{m=2}^{\infty} m$.

Definition 0.1 $\Sigma = \cup_{m=2}^{\infty} \Pi_{i=2}^{m-1} i$,

that is, for $s \in \Sigma$ the sets $[s] = \{x \in \mathbb{EK} : s \subset x\}$ form the usual clopen base of $\mathbb{EK}$.

Definition 0.2 $C_{EK} = \Pi_{m=2}^{\infty}(m \setminus \{0\})$,

Recall that $\forall^{\infty}$ means 'for all but finitely many'.

Definition 0.3 Let $s \in \Sigma$ and $k \in \omega$. Then $S : \omega \setminus 2 \to [\omega]^{<\omega}$ is *a finite k-slalom above s*, if

1. $\forall i \in [2, |s|) \ \ S(i) = \{s(i)\}$,

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2. $\forall i \in \omega \setminus 2 \ \ |S(i)| \leq k$,

3. $\forall^\infty i \in \omega \setminus 2 \ \ S(i) = \emptyset$.

Definition 0.4 $\mathrm{ht}(S) = \min\{i \in \omega \setminus 2 : S(i) = \emptyset\}$.

Definition 0.5 Let S be a finite k-slalom above s and $t \in \Sigma$. Then t *escapes* S if $|t| > \mathrm{ht}(S)$ and $\forall i \in [2, |t|) \ \ t(i) \notin S(i)$.

Definition 0.6 Let $s \in \Sigma$ and $F \subset \{t \in \Sigma : t \supset s\}$. Then F *is k-fat above s*, if for every finite k-slalom S above s there exists $t \in F$ escaping S.

Recall that if $(T, \leq)$ is a tree and $t \in T$ then $\mathrm{succ}_{(T,\leq)}(t)$ is the set of immediate successors of t in T. We simply write $\mathrm{succ}(t)$ when there is no danger of confusion.

Now we define our notion of forcing.

Definition 0.7 Let $p \in \mathbb{P}$ iff

1. $p = (T^p, \leq^p)$,

2. $T^p \subset \Sigma$,

3. $\leq^p$ is a partial ordering of T^p such that $(T^p, \leq^p)$ is a tree,

4. $t \leq^p t' \implies t \subset t'$,

5. $\forall k \in \omega \ \forall^\infty t \in T^p \ \mathrm{succ}(t)$ is k-fat.

Definition 0.8 Let $p, p' \in \mathbb{P}$ then $p = (T^p, \leq^p)$, $p' = (T^{p'}, \leq^{p'})$. Define $p \leq_{\mathbb{P}} p'$ iff

1. $T^p \subset T^{p'}$,

2. $\leq^p = \leq^{p'} \cap (T^p \times T^p)$.

We will usually simply write $\leq$ for $\leq_{\mathbb{P}}$.

It will be necessary, of course, to prove that $\mathbb{P}$ is proper but for the intended iteration to preserve non-meagre sets a stronger condition will be used.

Definition 0.9 A proper forcing notion $\mathbb{Q}$ is said to be *Cohen-preserving* if for every countable elementary submodel $\mathfrak{N}$ of $(H(\chi), \in)$ and every condition $p \in \mathbb{Q}$ and every real x belonging to $\mathfrak{N}$ such that x is a Cohen real over $\mathfrak{N}$, there is an $\mathfrak{N}$-generic condition $q \leq p$ such that $q \Vdash$ "x is a Cohen real over $\mathfrak{N}[G]''$.

Theorem 0.10 *The countable support iteration of proper and Cohen-preserving partial orders that preserve a non-meagre set also preserves the non-meagre set.*

Lemma 0.11 *$\mathbb{P}$ is proper and Cohen-preserving.*

Lemma 0.12 *$\mathbb{P}$ is a Borel set and $\leq_{\mathbb{P}}$ and $\perp$ — being incompatible in $\mathbb{P}$ — are Borel relations (in the appropriate spaces...).*

Proof. ... □

Let $\dot{x}$ be the canonical name for the generic real added by $\mathbb{P}$.

Let $\{\dot{U}_n\}_{n\in\omega}$ be a name for a decreasing sequence of dense open subsets of $C_{EK} + \dot{x}$.

Define

$$R_p = \{r \in \mathbb{EK} : \exists p' \leq p \; p' \Vdash \text{“}r \in \cap_{n\in\omega}\dot{U}_n\text{”}\}.$$

and note that R_p is projective, in fact it is no more complex that Π^1_2.

Lemma 0.13 *R_p is not meagre.*

Proof. Let $W = \bigcap_{n\in\omega} W_n$ be such that each W_n is dense and open. Contruct by induction on n conditions $p_n = (T^n, \leq^n)$ and $r_n \in \Sigma$ and $\{t_i\}_{i\in n}$ such that

1. $p_0 = p$
2. $p_{n+1} \leq p_n$
3. $\{t_i\}_{i\in n} \subseteq T^n$
4. $|r_n| \geq |t_i|$ for $i \in n$
5. $T^{n+1}[t_n] \Vdash$ "$[r_{n+1}] \subseteq \cap_{i\in n+1}\dot{U}_i$" where $T[t]$ is all elements of T comparable with t

6. $[r_n] \subseteq W_n$

7. $\mathrm{succ}_{p_m}(t_n)$ is $(n+1)$-fat for all m

8. $t_n \supseteq t_{J_n}$ and t_n escapes s_n where $\{(J_n, s_n)\}_{n\in\omega}$ is an enumeration of all pairs (J, s) such that s is a J-slalom such that $J_n < n$

9. $t_i \subseteq t_j$ implies that $i \leq j$

10. $r_n(i) \neq t(i)$ for all $i \in |r_n|$ and $t \in T^n$.

If this inductive construction can be completed then let $p' = (T^{p'}, \leq^{p'})$ where $T^{p'} = \{t_i\}_{i\in\omega}$ and let $\leq^{p'} = \leq^p \cap T^{p'}$. Observe first that $\mathrm{succ}_{p'}(t_i)$ contains all t_n such that $J_n = i$ by Conditions 8 and 9. Hence $\mathrm{succ}_{p'}(t_i)$ is i-fat and so $p' \in \mathbb{P}$ and $p' \leq p$.

Let $r = \cup_{n\in\omega} r_n$. Condition 6 guarantees that $r \in W$. To see that $r \in R_p$ it suffices to show that $p' \Vdash$ "$r \in \cap_{n\in\omega}\dot{U}_n$". To this end let $q \leq p'$ and $n \in \omega$. There is some $k \geq n$ such that $t_k \in q$. Condition 5 implies that $T^{k+1}[t_k] \Vdash$ "$[r_{k+1}] \subseteq \dot{U}_n$". Since $q' = (T^q[t_k], \leq^q) \leq (T^{k+1}[t_k], \leq^{k+1})$ it follows that $q' \Vdash$ "$r \in \dot{U}_n$". Since W was arbitrary this shows that R_p is not meagre.

To start the construction choose $p_1 \leq p$ such that $p_1 \Vdash$ "$[r'_0] \subseteq \dot{U}_0$" for some $r'_0 \in \Sigma$. Notice that since $\dot{U}_0$ be a name for a dense open subset of $C_{EK} + \dot{x}$ it follows that $r'_0(i) \neq t(i)$ for all $t \in T^{p_0}$. Choose $t_0 \in T^{p_0}$ such that $\mathrm{succ}_{p_0}(t_0)$ is 1-fat. Extend r'_0 to r''_0 such that $r''_0(i) \neq t_0(i)$ for all $i \in |t_0|$ and $|r''_0| \geq |t_0|$. Then extend r''_0 to r_0 such that $[r_0] \subseteq W_0$.

To continue the construction assume that p_n and $r_n \in \Sigma$ and $\{t_i\}_{i\in n}$ have been constructed. Let $j = J_{n+1}$. Since Inductive Hypothesis 7 implies that $\mathrm{succ}_{p_n}(t_j)$ is $j+1$-fat it is possible to find $t \in \mathrm{succ}_{p_n}(t_j)$ such that $t \not\subseteq t_i$ for $i \leq n$ and such that t escapes the $(j+1)$-slalom s defined by

$$s(i) = \begin{cases} \{r_n(i)\} & \text{if } i \in |t_j| \\ s_n(i) \cup \{r_n(i)\} & \text{if } i \geq |t_j| \end{cases}$$

because $r_n(i) \neq t_j(i)$ for all i.

Any extension of this t will guarantee that Conditions 3, 8 and 9 are satisfied. Choose $r' \supseteq r_n$ such that $|r'| \geq |t|$ and $r'(i) \neq t(i)$ for all $i \in |r'|$ and $j \leq n$. Then find $p_{n+1} \leq (T^n[t], \leq^n)$ such that $p'_{n+1} \Vdash$ "$[r''] \subseteq \cap_{i\in n+1}\dot{U}_i$" for some $r'' \supseteq r'$. Notice that since $\cap_{i\in n+1}\dot{U}_i$ be a name for a dense open subset of $C_{EK} + \dot{x}$ it follows that r'' escapes all $s \in T^{p'_{n+1}}[t]$. Choose $t_{n+1} \in T^{p'_{n+1}}[t]$

such that $\text{succ}_{p'_{n+1}}(t_{n+1})$ is $(n+3)$-fat. Extend r'' to r^* such that $|r^*| = |t_{n+1}|$ and r^* escapes t_{n+1}. Let $r_{n+1} \supseteq r^*$ be such that $[r_{n+1}] \subseteq W_{n+1}$. Then let

$$T^{n+1} = \{s \in T^{p'_{n+1}}[t_{n+1}] \mid s \text{ escapes } r_{n+1}\} \cup \{s \in T^{p_n} \mid s \not\supseteq t\}$$

and let p_{n+1} be the associated condition. $\square$

Lemma 0.14 *Assuming that all Π^1_2 sets have the Property of Baire, if G is $\mathbb{P}$ generic over V and $X \subseteq \mathbb{EK}$ is nowhere meagre then in $V[G]$ there is $x \in \mathbb{EK}$ such that $X \cap C_{EK} + x$ is also not meagre.*

Proof. Let x be the generic real added by $\mathbb{P}$ and suppose that $p \Vdash$ "$X \cap \cap_{n\in\omega}\dot{U}_n = \emptyset$". By Lemma 0.13 it follows that R_p associated with this name for a dense G_δ is non-meagre. It has been noted that R_p is Π^1_2 so it has the Property of Baire and hence is co-meagre. There is then some $x \in X \cap R_p$ and hence $p' \leq p$ be such that $p' \Vdash$ "$x \in \cap_{n\in\omega}\dot{U}_n \cap X$". $\square$

Theorem 0.15 *It is consistent with set theory that for every non-meagre subset $X \subseteq \mathbb{EK}$ there is some $x \in \mathbb{EK}$ such that $X \cap x + C_{EK}$ is also non-meagre.*

Proof. Iterate $\mathbb{P}$ of length ω_2 with countable support over a model V of the Continuum Hypothesis to obtain V_α for $\alpha \leq \omega_2$. If $X \subseteq \mathbb{EK}$ in the generic extension V_{ω_2} then there is an inner model V_α such that $V_\alpha \models V_\alpha \cap X$ is non-meagre. Applying Lemma 0.14 yields that in $V_{\alpha+1}$ there is some $x \in \mathbb{EK}$ such that $X \cap V_\alpha \cap x + C_{EK}$ is also non-meagre. By Theorem 0.10 $X \cap V_\alpha \cap x + C_{EK}$ remains non-meagre in V_{ω_2}.

References

[1] A. S. Kechris, *Classical Descriptive Set Theory*. Springer-Verlag, 1995.