1. New Stuff

**Definition 1.1.** Let $\mathbb{A}$ be a Banach algebra and define the amenability character of $\mathbb{A}$ to be the least cardinal of a family of a directed set $D$ such that

1. There is a mapping $\Delta : D \to \mathbb{A} \otimes \mathbb{A}$ such that for each $a \in \mathbb{A}$ $\lim_{d \in D} a\Delta(d) - \Delta(d)a = 0$

2. if $\mu : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is the natural multiplication functional then $\{\mu(\Delta(d))\}_{d \in D}$ is an approximate identity for $\mathbb{A}$.

Provided that such a directed set exists at all.

**Lemma 1.1** (B. Johnston). Let $\mathbb{A}$ and $\mathbb{B}$ be a Banach algebras and $u \in \mathbb{A} \otimes \mathbb{B}$. If there are $\{\psi_i\}_{i=1}^k \subseteq \mathbb{A}^*$ and $\{\varphi_j\}_{j=1}^k \subseteq \mathbb{B}^*$ such that

$$\psi_n \otimes \varphi_m(u) = \begin{cases} 0 & \text{if } n > m \\ 1 & \text{if } n \leq m \end{cases}$$

then $\|u\| \geq \ln(2k - 1)/8\pi$.

**Proposition 1.1.** A compact Hausdorff space $X$ is metrizable if and only if the amenability character of $C(X)$ is countable.

**Proof.** If $X$ is metrizable then it is second countable and a partition of unity argument yields that the amenability character of $C(X)$ is countable.

On the other hand, suppose that $X$ is not metrizable but that there is a mapping $\Delta : D \to \mathbb{A} \otimes \mathbb{A}$ witnessing that the amenability character of $C(X)$ is countable. Let

$$\mathcal{F} = \bigcup_{d \in D} \left\{ \{f_i\}_{i=1}^k \cup \{g_i\}_{i=1}^k \right| \Delta(d) = \sum_{j=1}^k f_i \otimes g_j \right\}$$

and let $\mathcal{B} = \{f^{-1}B_r(q) \mid r \in \mathbb{Q} \text{ and } q \in \mathbb{Q}\sqrt{-1}\}$. Since $\mathcal{B}$ is not a base for $X$ and $X$ is compact, there must be some distinct $x_0$ and $x_1$ in $X$ such that $x_0 \in U$ if and only if $x_1 \in U$ for all $U \in \mathcal{B}$.

Let $\psi_j \in C(X)^*$ be defined by $\psi_j(f) = f(x_j)$. Let $\varphi_0 = \psi_1$ and $\varphi_1 = \psi_0 + \psi_1$. Now, using the fact that $X$ is regular it is possible to choose a continuous function $w : X \to \mathbb{R}$ such that $w(x_j) = j$. It suffices to check that

$$\lim_{d \in D} \psi_n \otimes \varphi_m(w\Delta(d) - \Delta(d))w = \begin{cases} 0 & \text{if } n = 1 \text{ and } m = 0 \\ 1 & \text{otherwise}. \end{cases}$$

To see this let $\epsilon > 0$ and choose $d_0 \in D$ such that $\|\mu(\Delta(d)) - 1\|_{\sup} < \epsilon$ for all $d \geq d_0$.

Now for $d \geq d_0$ suppose that $\Delta(d) = \sum_{j=1}^k f_j \otimes g_j$. Then

$$\psi_m \otimes \psi_n(w\Delta(d) - \Delta(d))w = \psi_m \otimes \psi_n \left( \sum_{j=1}^k w f_j \otimes g_j - \sum_{j=1}^k f_j \otimes wg_j \right) = \sum_{j=1}^k \psi_m(wf_j)\psi_n(g_j) - \sum_{j=1}^k \psi_m(f_j)\psi_n(wg_j)$$

and this implies that

$$\psi_0 \otimes \psi_0(w\Delta(d) - \Delta(d))w = \sum_{j=1}^k f_j(x_0)g_j(x_0) - \sum_{j=1}^k f_j(x_0)g_j(x_0) = 0$$

and similarly $\psi_1 \otimes \psi_1(w\Delta(d) - \Delta(d))w = 0$. Also

$$\psi_0 \otimes \psi_1(w\Delta(d) - \Delta(d))w = \sum_{j=1}^k f_j(x_0)g_j(x_1) - \sum_{j=1}^k f_j(x_0)g_j(x_0)$$

because $g_j(x_0) = g_j(x_1)$ for all $j$. Similarly $\psi_1 \otimes \psi_0(w\Delta(d) - \Delta(d))w = \sum_{j=1}^k f_j(x_1)g_j(x_1)$. Note that $\sum_{j=1}^k f_j(x_j)g_j(x_j) = \mu(\Delta(d)(x_j))$ and $\lim_{d \in D} \mu(\Delta(d)(x_j)) = 1$. Hence

$$\lim_{d \in D} \psi_0 \otimes \psi_0(w\Delta(d) - \Delta(d))w = \lim_{d \in D} \psi_0 \otimes \psi_1(w\Delta(d) - \Delta(d))w = 1$$

$$\lim_{d \in D} \psi_0 \otimes \psi_1(w\Delta(d) - \Delta(d))w = \lim_{d \in D} (\psi_0 + \psi_1)(w\Delta(d) - \Delta(d))w = 1$$

$$\lim_{d \in D} \psi_1 \otimes \psi_1(w\Delta(d) - \Delta(d))w = \lim_{d \in D} (\psi_0 + \psi_1)(w\Delta(d) - \Delta(d))w = 1$$

$$\lim_{d \in D} \psi_1 \otimes \psi_0(w\Delta(d) - \Delta(d))w = \lim_{d \in D} \psi_1 \otimes \psi_1(w\Delta(d) - \Delta(d))w = 0$$

as required. \qed
2. Old stuff

**Question 2.1.** Given a Banach space \( B \) and the natural embedding of \( B \) into \( B^{**} \), does the hypothesis that every element of \( B^{**} \) is in the weak*-closure of a countable subset of \( B \) imply that \( B \) is separable?

Note that the converse is immediate since \( B \) is weak* dense in \( B^{**} \). Also, I guess that non-separable, reflexive \( B \) are not of interest.

**Proposition 2.1.** If \( \kappa \) is uncountable, then \( L_1(2^\kappa)^{**} \) contains elements not in the closure of any countable subset of \( L_1(2^\kappa) \).

*Proof.* Recall that \( L_1(2^\kappa)^* \) is not separable. Let \( \mathcal{F} \) be any ultrafilter on \( 2^\kappa \) containing the complement of any null set and let \( \Phi_F \in L_1(2^\kappa)^* \) be defined by

\[
\{ \langle \Phi_F, [f] \rangle \} = \bigcap_{A \in \mathcal{F}} f(A)
\]

where \([f]\) denotes the \( L_1(2^\kappa) \) equivalence class of the function \( f \). Note that \( \langle \Phi_F, [f] \rangle \) does not depend on the choice of \( f \) since \( \mathcal{F} \) contains no null set.

Now suppose that \( \mathcal{F} \) is a countable subset of \( L_1(2^\kappa) \) whose weak*-closure contains \( \Phi_F \). It is possible to find a countable \( \Gamma \subseteq \kappa \) and a countable family \( \mathcal{L} \) of measurable functions on \( 2^\kappa \) such that \( \mathcal{L} = \{ g \circ \pi \} \) where \( \pi \) is the projection from \( 2^\kappa \) to \( 2^\Gamma \).

Choose \( \xi \in \kappa \setminus \Gamma \) and let \( U^i_\xi = \{ h \in 2^\kappa \ | \ h(\xi) = i \} \). Let \( H^i \) be the characteristic function of \( U^i_\xi \). Without loss of generality \( \langle \Phi_F, H^i \rangle = 0 \). However

\[
\langle H^i, [g \circ \pi] \rangle = \int_{x \in U^i_\xi} g(x) d\lambda(x) = \int_{x \in 2^\kappa} g(x) d\lambda(x)/2
\]

and so \( \langle H^0, [g \circ \pi] \rangle = \langle H^1, [g \circ \pi] \rangle \). In other words, the weak* neighbourhood of \( \Phi_F \) consisting of all \( \Psi \) such that \( |\langle \Psi, H^i \rangle \rangle - i| < 1/3 \) contains no member of \( \mathcal{L} \).

**Proposition 2.2.** If \( X \) is a compact Hausdorff space then the following are equivalent:

1. \( X \) is metrizable
2. Every element of \( C(X)^{**} \) is in the weak*-closure of a countable subset of \( C(X) \).

*Proof.* If \( X \) is metrizable then \( C(X)^* \) is separable in the norm topology and, hence, also in the weak*-topology. Since \( B \) is always weak* dense in \( B^{**} \) it follows that (1) implies (2).

For the other direction, begin by recalling that \( C(X)^* \) is the space of all countably additive, Borel measures on \( X \) and this can be decomposed in the space of atomic measures and the diffuse measures. The space of atomic measures is the same as \( \ell_1(X) \) where \( X \) is considered as a discrete set. Hence \( C(X)^* = \ell_1(X) \oplus D \) and so \( C(X)^{**} = \ell_1(X)^* \oplus D^* = \ell_\infty(X) \oplus D^* \). Hence it is sufficient to find a subsets \( A \subseteq X \) such that \( \chi_A \) is not in the weak* closure of any countable subset of \( \mathcal{C}(X) \).

Note that for \( \chi_A \) to be in the weak* closure of a subset \( W \subseteq C(X) \) it must be that for any finitely many functions \( \mu_1, \mu_2, \ldots, \mu_k \in \ell_1(X) \) and any \( \epsilon > 0 \) there is \( w \in W \) such that \( |\langle \chi_A, \mu_i \rangle - \langle \mu_i, w \rangle| < \epsilon \). But \( \langle \chi_A, \mu_i \rangle = \sum_{A \in E} \mu_i(a) \) and \( \langle \mu_i, w \rangle = \sum_{x \in X} \mu_i(x) w(x) \). Since the functions \( \mu_i \) are in \( \ell_1(X) \) and \( \epsilon > 0 \) it must be that all but finitely many values of \( \mu_i \) are much less than \( \epsilon/\|w\| \). Hence the following are equivalent:

- \( \chi_A \) is in the weak* closure of \( W \)
- Any finitely many points \( x_1, x_2, \ldots, x_n \) in \( X \)

\[
| \sum_{i=1}^n \sum_{j=1}^k \mu_i(x_j) w(x_j) - \sum_{i=1}^n \sum_{j=1}^k \mu_i(x_j) \chi_A(x_j) | < \epsilon
\]

- (letting \( r_j = \sum_{i=1}^k \mu_i(x_j) \) in the above) for finitely many points \( x_1, x_2, \ldots, x_n \) in \( X \) and scalars \( r_1, r_2, \ldots, r_k \) the following holds

\[
| \sum_{j=1}^n r_j (w(x_j) - \chi_A(x_j)) | < \epsilon
\]

- \( \chi_A \) is in the pointwise closure of \( W \).

It will be shown that there is an \( A \subseteq X \) for which this is not true. First suppose that \( X \) is not first countable and let \( x \in X \) have no countable neighbourhood base. Given a countable family \( C \subseteq C(X) \) choose for each \( n \) and \( f \in C \) a neighbourhood \( U_{n,f} \) of \( x \) such that \( |f(y) - f(x)| < 1/n \) for each \( y \in U_{n,f} \). Since \( X \) is compact and \( \{ U_{n,f} \}_{n \in \mathbb{N}, f \in C} \) is not a base at \( x \) it must be that \( \cap_{n \in \mathbb{N}, f \in C} U_{n,f} \) is infinite (in fact, uncountable). Let \( A \) be any partition of \( \cap_{n \in \mathbb{N}, f \in C} U_{n,f} \) into two infinite pieces. Then \( \chi_A \) is not in the pointwise closure of \( C \).

By Arhangleski"i's Theorem on Lindelof spaces it must be that \( X \leq 2^{\aleph_0} \) and hence \([0,1]^X \) has a countable dense subset \( \mathcal{F} \). If for each \( F \in \mathcal{F} \) there is a countable family \( C_F \subseteq C(X) \) containing \( F \) in its pointwise closure then let \( \mathcal{C} = \bigcup_{F \in \mathcal{F}} C_F \).
and note that $C$ is a countable dense subset of $[0, 1]^X$. Since $X$ is compact, it follows that $\{ f^{-1}(p, q) \mid f \in C, p, q \in \mathbb{Q} \}$ is a countable base for $X$, contradicting that it is not metrizable.

**Corollary 2.1.** For every $C^*$ algebra $\mathcal{A}$ the following are equivalent:

1. $\mathcal{A}$ is separable
2. every element of $\mathcal{A}^{**}$ is in the weak$^*$-closure of a countable subset of $\mathcal{A}$.

**Proposition 2.3.** There is a nonseparable Banach algebra $B$ such that every element of $B^{**}$ is in the weak$^*$-closure of a countable subset of $B$.

**Proof.** Let $T : [\omega_1]^2 \to \mathbb{N}$ be a function such that for every uncountable $X \subseteq \omega_1$ and $n, k \in \mathbb{N}$ there is $Y \subseteq X$ such that $|Y| = n$ and $T(\{\alpha, \beta\}) = k$ for all $\{\alpha, \beta\} \in [Y]^2$. (This is the celebrated negative partition relation of Todorcevic.) Now for any $f : \omega_1 \to \mathbb{C}$ define

$$
\|f\| = \max \left\{ \left( \sum_{y \in Y} |f(y)|^k \right)^{1/k} \mid Y \subseteq \omega_1 \text{ and } T(\{\alpha, \beta\}) = k \text{ for all } \{\alpha, \beta\} \in [Y]^2 \right\}
$$

and let $B = \{ f : \omega_1 \to \mathbb{C} \mid \|f\| < \infty \}$. It is routine to check that $(B, \|\|)$ is an abelian Banach algebra.

**Claim 1.** If $f \in B$ then the support of $f$ is countable.

**Proof.** If not then choose a rational $p > 0$ and uncountable $X \subseteq \omega_1$ such that, without loss of generality, $\Re(f(\xi)) > p$ for each $\xi \in X$. Let $Y \subseteq X$ be such that $T$ has constant value 1 on $[Y]^2$ and $|Y| > \|f\|/p$. Then $\|f\| \geq |Y|p > \|f\|$. \qed

It is immediate that $B$ is not separable. However, let $B_\xi = \{ f \in B \mid (\forall \eta > \xi) f(\eta) = 0 \}$ and note that each $B_\xi$ is a closed, separable subspace of $B$ and $B = \bigcup_{\xi \in \omega_1} B_\xi$.

For $\xi \in \omega_1$ let $\psi_\xi \in B^*$ be defined by $\langle \psi_\xi, f \rangle = f(\xi)$.

**Claim 2.** The span of $\{ \psi_\xi \mid \xi \in \omega_1 \}$ is norm dense in $B^*$.

**Proof.** Each $f \in B_\xi$ can be approximated in norm by some $f' \in B_\xi$ with finite support. \qed

**Claim 3.** If $\Psi \in B^{**}$ then $\langle \Psi, \psi_\xi \rangle = 0$ for all but countably many $\xi \in \omega_1$.

\qed