1. New stuff

Definition 1.1. Let $\mathbb{A}$ be a Banach algebra and define the *amenability character of* $\mathbb{A}$ to be the least cardinal of a famlly of a directed set $\mathcal{D}$ such that

(1) there is a mapping $\Delta : \mathcal{D} \to \mathbb{A} \otimes \mathbb{A}$ such that for each $a \in \mathbb{A}$ $\lim_{d\in\mathcal{D}} a\Delta(d) - \Delta(d)a = 0$
(2) if $\mu : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is the natural multiplication functional then $\{\mu(\Delta(d))\}_{d\in\mathcal{D}}$ is an approximate identity for $\mathbb{A}$

provided that such a directed set exists at all.

Lemma 1.1 (B. Johnston)**.** *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be a Banach algebras and* $u \in \mathbb{A}\otimes\mathbb{B}$*. If there are* $\{\psi_i\}_{i=1}^k \subseteq \mathbb{A}^*$ *and* $\{\varphi_i\}_{i=1}^k \subseteq \mathbb{B}^*$ *such that*

$$\psi_n \otimes \varphi_m(u) = \begin{cases} 0 & \text{if } n > m \\ 1 & \text{if } n \leq m \end{cases}$$

then $\|u\|_p \geq \ln(2k-1)/8\pi$.

Proposition 1.1. *A compact Hausdorff space* X *is metrizable if and only if the amenability character of* $C(X)$ *is countable.*

Proof. If X is metrizable then it is second countable and a partition of unity argument yields that the amenability character of $C(X)$ is countable.

On the other hand, suppose that X is not metrizable but that there is a mapping $\Delta : \mathcal{D} \to \mathbb{A} \otimes \mathbb{A}$ witnessing that the amenability character of $C(X)$ is countable. Let

$$\mathcal{F} = \bigcup_{d\in\mathcal{D}} \left\{ \{f_i\}_{j=1}^k \cup \{g_i\}_{j=1}^k \;\middle|\; \Delta(d) = \sum_{j=1}^k f_i \otimes g_j \right\}$$

and let $\mathcal{B} = \{ f^{-1}B_r(q) \mid r \in \mathbb{Q} \text{ and } q \in \mathbb{Q}[\sqrt{-1}] \}$. Since $\mathcal{B}$ is not a base for X and X is compact, there must be some distinct x_0 and x_1 in X such that $x_0 \in U$ if and only if $x_1 \in U$ for all $U \in \mathcal{B}$.

Let $\psi_j \in C(X)^*$ be defined by $\psi_j(f) = f(x_j)$. Let $\varphi_0 = \psi_1$ and $\varphi_1 = \psi_0 + \psi_1$ Now, using the fact that X is regular it is possible to choose a continuous function $w : X \to \mathbb{R}$ such that $w(x_j) = j$. It suffices to check that

$$\lim_{d\in\mathcal{D}} \psi_n \otimes \varphi_m(w\Delta(d) - \Delta(d)w) = \begin{cases} 0 & \text{if } n = 1 \text{ and } m = 0 \\ 1 & \text{otherwise.} \end{cases}$$

To see this let $\epsilon > 0$ and choose $d_0 \in \mathcal{D}$ such that $\|\mu(\Delta(d)) - 1\|_{\sup} < \epsilon$ for all $d \geq d_0$.

Now for $d \geq d_0$ supose that $\Delta(d) = \sum_{j=1}^k f_j \otimes g_j$. Then

$$\psi_m \otimes \psi_n(w\Delta(d) - \Delta(d)w) = \psi_m \otimes \psi_n \left(\sum_{j=1}^k wf_j \otimes g_j - \sum_{j=1}^k f_j \otimes wg_j \right) = \sum_{j=1}^k \psi_m(wf_j)\psi_n(g_j) - \sum_{j=1}^k \psi_m(f_j)\psi_n(wg_j)$$

and this implies that

$$\psi_0 \otimes \psi_0(w\Delta(d) - \Delta(d)w) = \sum_{j=1}^k f_j(x_0)g_j(x_0) - \sum_{j=1}^k f_j(x_0)g_j(x_0) = 0$$

and similarly $\psi_1 \otimes \psi_1(w\Delta(d) - \Delta(d)w) = 0$. Also

$$\psi_0 \otimes \psi_1(w\Delta(d) - \Delta(d)w) = \sum_{j=1}^k f_j(x_0)g_j(x_1) - \sum_{j=1}^k f_j(x_0)0 = \sum_{j=1}^k f_j(x_0)g_j(x_0)$$

because $g_j(x_0) = g_j(x_1)$ for all j. Similarly $\psi_1\otimes\psi_0(w\Delta(d)-\Delta(d)w) = \sum_{j=1}^k f_j(x_1)g_j(x_1)$. Note that $\sum_{j=1}^k f_j(x_j)g_j(x_j) = \mu(\Delta(d)(x_j)$ and $\lim_{d\in\mathcal{D}} \mu(\Delta(d)(x_j) = 1$. Hence

$$\lim_{d\in\mathcal{D}} \psi_0 \otimes \varphi_0(w\Delta(d) - \Delta(d)w) = \lim_{d\in\mathcal{D}} \psi_0 \otimes \psi_1(w\Delta(d) - \Delta(d)w) = 1$$
$$\lim_{d\in\mathcal{D}} \psi_0 \otimes \varphi_1(w\Delta(d) - \Delta(d)w) = \lim_{d\in\mathcal{D}} \psi_0 \otimes (\psi_0 + \psi_1)(w\Delta(d) - \Delta(d)w) = 1$$
$$\lim_{d\in\mathcal{D}} \psi_1 \otimes \varphi_1(w\Delta(d) - \Delta(d)w) = \lim_{d\in\mathcal{D}} \psi_1 \otimes (\psi_0 + \psi_1)(w\Delta(d) - \Delta(d)w) = 1$$
$$\lim_{d\in\mathcal{D}} \psi_1 \otimes \varphi_0(w\Delta(d) - \Delta(d)w) = \lim_{d\in\mathcal{D}} \psi_1 \otimes \psi_1(w\Delta(d) - \Delta(d)w) = 0$$

as required. □

2. Old stuff

Question 2.1. Given a Banach space B and the natural embedding of B into B^{**}, does the hypothesis that every element of B^{**} is in the weak*-closure of a countable subset of B imply that B is separable?

Note that the converse is immediate since B is weak* dense in B^{**}. Also, I guess that non-separable, reflexive B are not of interest.

Proposition 2.1. *If κ is uncountable, then $L_1(2^\kappa)^{**}$ contains elements not in the closure of any countable subset of $L_1(2^\kappa)$.*

Proof. Recall that $L_1(2^\kappa)^* = L_\infty(2^\kappa)$. Let $\mathcal{F}$ be any ultrafilter on 2^κ containing the complement of any null set and let $\Phi_{\mathcal{F}} \in L_\infty(2^\kappa)^*$ be defined by

$$\{\langle \Phi_{\mathcal{F}}, [f]\rangle\} = \bigcap_{A\in\mathcal{F}} \overline{f(A)}$$

where $[f]$ denotes the $L_\infty(2^\kappa)$ equivalence class of the function f. Note that $\langle \Phi_{\mathcal{F}}, [f]\rangle$ does not depend on the choice of f since $\mathcal{F}$ contains no null set.

Now suppose that $\mathcal{L}$ is a countable subset of $L_1(2^\kappa)$ whose weak*-closure contains $\Phi_{\mathcal{F}}$. It is possible to find a countable $\Gamma \subseteq \kappa$ and a countable family $\mathcal{L}'$ of measurable functions on 2^Γ such that $\mathcal{L} = \{[g\circ\pi_\Gamma] \mid g \in \mathcal{L}'\}$ where π_Γ is the projection from 2^κ to 2^Γ.

Choose $\xi \in \kappa\setminus\Gamma$ and let $U^i_\xi = \{h \in 2^\kappa \mid h(\xi) = i\}$. Let H^i be the characteristic function of U^i_ξ. Without loss of generality $\langle \Phi_{\mathcal{F}}, H^i\rangle = i$. However

$$\langle H_i, [g\circ\pi_\Gamma]\rangle = \int_{x\in U^i_\xi} g(x)d\lambda(x) = \int_{x\in 2^\Gamma} g(x)d\lambda(x)/2$$

and so $\langle H_0, [g\circ\pi_\Gamma]\rangle = \langle H_1, [g\circ\pi_\Gamma]\rangle$. In other words, the weak* neighbourhood of $\Phi_{\mathcal{F}}$ consisting of all Ψ such that $|\langle \Psi, H_i\rangle - i| < 1/3$ contains no memebr of $\mathcal{L}$. □

Proposition 2.2. *If X is a compact Hausdorff space then the following are equivalent:*

1. *X is metrizable*
2. *every element of $C(X)^{**}$ is in the weak*-closure of a countable subset of $C(X)$.*

Proof. If X is metrizable then $C(X)$ is separable in the norm topology and, hence, also in the weak*-topology. Since B is always weak*-dense in B^{**} it follows that (1) implies (2).

For the other direction, begin by recalling that $C(X)^*$ is the space of all countably additive, Borel measures on X and this can be decomposed in the space of atomic measures and the diffuse measures. The space of atomic measures is the same as $\ell_1(X)$ where X is considered as a discrete set. Hence $C(X)^* = \ell_1(X)\oplus D$ and so $C(X)^{**} = \ell_1(X)^*\oplus D^* = \ell_\infty(X)\oplus D^*$. Hence is it suffices to find a subsets $A\subseteq X$ such that χ_A is not in the weak*-closure of any countable subset of $C(X)$.

Note that for χ_A to be in the weak* closure of a subset $W\subseteq C(X)$ it must be that for any finitely many functions $\mu_1, \mu_2, \ldots \mu_k$ in $\ell_1(X)$ and any $\epsilon > 0$ there is $w \in W$ such that $|\langle \chi_A, \mu_i\rangle - \langle \mu_i, w\rangle| < \epsilon$. But $\langle \chi_A, \mu_i\rangle = \sum_{a\in A}\mu_i(a)$ and $\langle \mu_i, w\rangle = \sum_{x\in X}\mu_i(x)w(x)$. Since the functions μ_i are in $\ell_1(X)$ and $\epsilon > 0$ it must be that all but finitely many values of μ_i are much less than $\epsilon/\|w\|$. Hence the following are equivalent:

- χ_A is in the weak* closure of $W \subseteq C(X)$
- any finitely many points $x_1, x_2, \ldots x_n$ in X

$$|\sum_{i=1}^n\sum_{j=1}^k \mu_i(x_j)w(x_j) - \sum_{i=1}^n\sum_{j=1}^k \mu_i(x_j)\chi_A(x_j)| < \epsilon$$

- (letting $r_j = \sum_{i=1}^k \mu_i(x_j)$ in the above) for finitely many points $x_1, x_2, \ldots x_n$ in X and scalars $r_1, r_2, \ldots r_k$ the following holds

$$|\sum_{j=1}^n r_j(w(x_j) - \chi_A(x_j))| = |\sum_{j=1}^n r_j w(x_j) - \sum_{j=1}^n r_j\chi_A(x_j)| < \epsilon$$

- χ_A is in the pointwise closure of W.

It will be shown that there is an $A\subseteq X$ for which this is not true. First suppose that X is not first countable and let $x\in X$ have no countable neighbourhood base. Given a countable family $\mathcal{C}\subseteq C(X)$ choose for each n and $f\in\mathcal{C}$ a neighbourhood $U_{n,f}$ of x such that $|f(y) - f(x)| < 1/n$ for each $y \in U_{n,f}$. Since X is compact and $\{U_{n,f}\}_{n\in\mathbb{N}, f\in\mathcal{C}}$ is not a base at x is must be that $\bigcap_{n\in\mathbb{N},f\in\mathcal{C}} U_{n,f}$ is infinite (in fact, uncountable). Let A be any partition of $\bigcap_{n\in\mathbb{N},f\in\mathcal{C}} U_{n,f}$ into two infinite pieces. Then χ_A is not in the pointwise closure of $\mathcal{C}$.

By Arhangleskii's Theorem on Lindelof spaces it must be that $X \leq 2^{\aleph_0}$ and hence $[0,1]^X$ has a countable dense subset $\mathcal{F}$. If for each $F\in\mathcal{F}$ there is a countable family $\mathcal{C}_F\subseteq C(X)$ containing F in its pointwise closure then let $\mathcal{C} = \bigcup_{F\in\mathcal{F}}\mathcal{C}_F$

and note that $\mathcal{C}$ is a countable dense subset of $[0,1]^X$. Since X is compact, it follows that $\{f^{-1}(p,q) \mid f \in \mathcal{C}, p, q \in \mathbb{Q}\}$ is a countable base for X, contradicting that it is not metrizable. ☐

Corollary 2.1. *For every C^* algebra $\mathbb{A}$ the following are equivalent:*

1. *$\mathbb{A}$ is separable*
2. *every element of $\mathbb{A}^{**}$ is in the weak*-closure of a countable subset of $\mathbb{A}$.*

Proposition 2.3. *There is a nonseparable Banach algebra $\mathbb{B}$ such that every element of $\mathbb{B}^{**}$ is in the weak*-closure of a countable subset of $\mathbb{B}$.*

Proof. Let $T : [\omega_1]^2 \to \mathbb{N}$ be a function such that for every uncountable $X \subseteq \omega_1$ and $n, k \in \mathbb{N}$ there is $Y \subseteq X$ such that $|Y| = n$ and $T(\{\alpha, \beta\}) = k$ for all $\{\alpha, \beta\} \in [Y]^2$. (This is the celebrated negative partition relation of Todorcevic.) Now for any $f : \omega_1 \to \mathbb{C}$ define

$$\|f\| = \max \left\{ \left(\sum_{y \in Y} |f(y)|^k \right)^{1/k} \;\middle|\; Y \subseteq \omega_1 \text{ and } T(\{\alpha, \beta\}) = k \text{ for all } \{\alpha, \beta\} \in [Y]^2 \right\}$$

and let $\mathbb{B} = \{f : \omega_1 \to \mathbb{C} \mid \|f\| < \infty\}$. It is routine to check that $(\mathbb{B}, \| \, \|)$ is an abelian Banach algebra.

Claim 1. If $f \in \mathbb{B}$ then the support of f is countable.

Proof. If not then choose a rational $p > 0$ and uncountable $X \subseteq \omega_1$ such that, without loss of generality, $\Re(f(\xi)) > p$ for each $\xi \in X$. Let $Y \subseteq X$ be such that T has constant value 1 on $[Y]^2$ and $|Y| > \|f\|/p$. Then $\|f\| \geq |Y|p > \|f\|$. ☐

It is immediate that $\mathbb{B}$ is not separable. However, let $\mathbb{B}_\xi = \{f \in \mathbb{B} \mid (\forall \eta > \xi) f(\eta) = 0\}$ and note that each $\mathbb{B}_\xi$ is a closed, separable subspace of $\mathbb{B}$ and $\mathbb{B} = \bigcup_{\xi \in \omega_1} \mathbb{B}_\xi$.

For $\xi \in \omega_1$ let $\psi_\xi \in \mathbb{B}^*$ be defined by $\langle \psi_\xi, f \rangle = f(\xi)$.

Claim 2. The span of $\{\psi_\xi \mid \xi \in \omega_1\}$ is norm dense in $\mathbb{B}^*$.

Proof. Each $f \in B_\xi$ can be approximated in norm by some $f' \in \mathbb{B}_\xi$ with finite support. ☐

Claim 3. If $\Psi \in \mathbb{B}^{**}$ then $\langle \Psi, \psi_\xi \rangle = 0$ for all but countably many $\xi \in \omega_1$.

☐