**Theorem 0.1.** Martin's Axiom implies that the closure of any subset of $L_1^+(\lambda)$ in $L_{1*}^*(\lambda)$ of cardinality less than $2^{\aleph_0}$ has closure contained in $L_1^+(\lambda)$ in $L_{1*}^*(\lambda)$.

**Proof.** Before proceeding further the following lemma about non-singular functionals in $L_1^*$ will be established.

**Lemma 0.1.** If $\psi \in L_{1*}^*(\lambda) \setminus L_1^+(\lambda) = L_{1*}^*(\lambda) \setminus L_1^+$ is positive then there is $f \in L_{1*}^*(\lambda)$ and $X$ such that $\lambda(X) > 0$ and $\psi(f \upharpoonright Y) > 0$ for all $Y \subseteq X$ such that $\lambda(Y) > 0$.

**Proof.** Let $\psi = \psi_1 \oplus \psi_2$ be the decomposition such that $\psi_1$ is countably complete and $\psi_2$ is purely finitely additive. Since $\psi \notin L_1^+(\lambda)$ there must be some $f \in L_{1*}^*(\lambda)$ such that $\psi_1(f) \neq 0$. Without loss of generality $f$ is positive and $\psi_1(f) > 0$. Let $X'$ be the support of $f$. Let $\mathcal{B}$ be a maximal disjoint family of sets such that if $B \in \mathcal{B}$ then $\lambda(B) > 0$ and $\psi_1(f \upharpoonright B) = 0$. Then $\mathcal{B}$ is countable and, since $\psi_1$ is countably additive, it follows that if $X'' = X' \setminus \cup \mathcal{B}$ then $\lambda(X'') > 0$. Using Losert’s Lemma let $A$ be such that $\lambda(A) < \lambda(X'')$ and $\psi_2$ is concentrated on $A$. Then $\psi_2(f \upharpoonright Y) = 0$ for any $Y \subseteq X'' \setminus A$. Hence $X = X'' \setminus A$ and $f$ satisfy the lemma.

Now let $\kappa < 2^{\aleph_0}$ and suppose that $\{\psi_\xi\}_{\xi \in \kappa}$ is a family of singular functionals, in other words, purely finitely additive measures. Suppose that $\psi$ is in the weak* closure of $\{\psi_\xi\}_{\xi \in \kappa}$ and that $\psi$ is not singular. Let $f$ and $X$ be as guaranteed by the Lemma for $\psi$ and suppose that $\lambda(X) = z > 0$. Using Losert’s Lemma let $\{A_{\xi,m}\}_{m \in \omega}$ be sets such that $\lambda(A_{\xi,m}) < 1/m$ and $\psi_\xi$ is concentrated on $A_{\xi,m}$. Let $\mathbb{P}$ be the partial order consisting of finite partial functions $h$ from $\kappa$ to $\omega$ such that $\lambda(\bigcup_{\xi \in \text{domain}(h)} A_{\xi,h(\xi)}) < z$. Since $\lim_{m \to \infty} \lambda(A_{\xi,m}) = 0$ it is clear that for each $\xi \in \kappa$ the set $D_\xi = \{h \in \mathbb{P} \mid \xi \in \text{domain}(h)\}$ is dense in $\mathbb{P}$.

Moreover, the argument that the amoeba algebra for random reals is ccc shows that $\mathbb{P}$ is ccc (easy, but I am too lazy to write this down). Hence there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for each $\xi$. In other words, $\cup G = H$ is a function from $\kappa$ to $\omega$ and $\lambda^*(Z) \leq z$ where $Z = \bigcup_{\xi \in \kappa} A_{\xi,H(\xi)}$. Hence there is a measurable $Y \subseteq X \setminus Z$ such that $\lambda(Y) > 0$ and $\psi(f \upharpoonright Y) > 0$. But $f \upharpoonright Y \in L_{1*}^*(\lambda)$ and $\psi_\xi(f \upharpoonright Y) = 0$ for each $\xi$. This contradicts that $\psi$ is in the weak* closure of $\{\psi_\xi\}_{\xi \in \kappa}$. 

\[\square\]