Theorem 0.1. *Martin's Axiom implies that the closure of any subset of $L_1^{\perp}(\lambda)$ in $L_1^{**}(\lambda)$ of cardinality less than $2^{\aleph_0}$ has closure contained in $L_1^{\perp}(\lambda)$ in $L_1^{**}(\lambda)$.*

Proof. Before proceeding further the following lemma about non-singular functionals in L_1^{**} will be established.

Lemma 0.1. *If $\psi \in L_1^{**}(\lambda) \setminus L_1^{\perp}(\lambda) = L_\infty^*(\lambda) \setminus L_1^{\perp}$ is positive then there is $f \in L_\infty(\lambda)$ and X such that $\lambda(X) > 0$ and $\psi(f \restriction Y) > 0$ for all $Y \subseteq X$ such that $\lambda(Y) > 0$.*

Proof. Let $\psi = \psi_1 \oplus \psi_2$ be the decomposition such that ψ_1 is countably complete and ψ_2 is purely finitely additive. Since $\psi \notin L_1^{\perp}(\lambda)$ there must be some $f \in L_\infty$ such that $\psi_1(f) \neq 0$. Without loss of generality f is positive and $\psi_1(f) > 0$. Let X' be the support of f. Let $\mathcal{B}$ be a maximal disjoint family of sets such that if $B \in \mathcal{B}$ then $\lambda(B) > 0$ and $\psi_1(f \restriction B) = 0$. Then $\mathcal{B}$ is countable and, since ψ_1 is countably additive, it follows that if $X'' = X' \setminus \cup\mathcal{B}$ then $\lambda(X'') > 0$. Using Losert's Lemma let A be such that $\lambda(A) < \lambda(X'')$ and ψ_2 is concentrated on A. Then $\psi_2(f \restriction Y) = 0$ for any $Y \subseteq X'' \setminus A$. Hence $X = X'' \setminus A$ and f satisfy the lemma. □

Now let $\kappa < 2^{\aleph_0}$ and suppose that $\{\psi_\xi\}_{\xi\in\kappa}$ is a family of singular functionals, in other words, purely finitely additive measures. Suppose that ψ is in the weak* closure of $\{\psi_\xi\}_{\xi\in\kappa}$ and that ψ is not singular. Let f and X be as guaranteed by the Lemma for ψ and suppose that $\lambda(X) = z > 0$. Using Losert's Lemma let $\{A_{\xi,m}\}_{m\in\omega}$ be sets such that $\lambda(A_{\xi,m}) < 1/m$ and ψ_ξ is concentrated on $A_{\xi,m}$. Let $\mathbb{P}$ be the partial order consisting of finite partial functions h from κ to ω such that $\lambda(\bigcup_{\xi\in\mathbf{domain}(h)} A_{\xi,h(\xi)}) < z$. Since $\lim_{m\to\infty} \lambda(A_{\xi,m}) = 0$ it is clear that for each $\xi \in \kappa$ the set $D_\xi = \{h \in \mathbb{P} \mid \xi \in \mathbf{domain}(h)\}$ is dense in $\mathbb{P}$.

Moreover, the argument that the amoeba algebra for random reals is ccc shows that $\mathbb{P}$ is ccc (easy, but I am too lazy to write this down). Hence there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for each ξ. In other words, $\cup G = H$ is a function from κ to ω and $\lambda^*(Z) \leq z$ where $Z = \bigcup_{\xi\in\kappa} A_{\xi,H(\xi)}$. Hence there is a measurable $Y \subseteq X \setminus Z$ such that $\lambda(Y) > 0$ and $\psi(f \restriction Y) > 0$. But $f \restriction Y \in L_\infty(\lambda)$ and $\psi_\xi(f \restriction Y) = 0$ for each ξ. This contradicts that ψ is in the weak* closure of $\{\psi_\xi\}_{\xi\in\kappa}$. □