Map operations and k-orbit maps

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Abstract

A k-orbit map is a map with k flag-orbits under the action of its automorphism group. We give a basic theory of k-orbit maps and classify them up to $k \leqslant 4$. "Hurwitz-like" upper bounds for the cardinality of the automorphism groups of 2-orbit and 3-orbit maps on surfaces are given. Furthermore, we consider effects of operations like medial and truncation on k-orbit maps and use them in classifying 2-orbit and 3-orbit maps on surfaces of small genus.

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1 Introduction

A map, as defined in Section 2, is essentially a tiling of a compact closed surface. In this paper we explore some basic properties of highly symmetric maps and in particular those which are not regular.

The barycentric subdivision of a map produces a set of triangles which we call flags. When the symmetry group of the map is transitive on the flags (that is, when there is only one orbit of flags under the automorphism group) we say that the map is regular, or 1-orbit map. Furthermore, if the symmetry group of a map is transitive on the vertices, edges of faces we say that the map is vertex-, edge-, or face-transitive, respectively.

While regular and chiral maps have been studied extensively ([5], [6], [21], [22], [23]) very little work has been done on other symmetric maps, with the notable exception of edge-transitive maps ([18], [20]).

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In Section 2 we define the concept of k-orbit map as a map that has k distinct orbits of flags under the action of its automorphism group. For example, chiral maps are examples of 2-orbit maps. Clearly, the larger k corresponds to the less "symmetric" maps. One should not confuse this with the concept of k-transitive tiling, as defined in for example [15], which specifies that the faces of a given rank belong to exactly k orbits under the automorphism group of the tiling. For example, a tiling obtained by truncation of the tessellation of the Euclidean plane by squares, replacing each square by an octagon and each vertex by a square, is a 3-orbit (infinite) map, but it is edge 2-transitive, tile 2-transitive and vertex 1-transitive.

Classification of 2-orbit maps has been done in [13]. Furthermore, in [11] and [12] Hubard characterizes the automorphism groups of 2-orbit polyhedra. In his dissertation [9], using an algebraic approach, Duarte provides the list of 2-orbit hypermaps on the sphere, projective plane, torus and double torus.

In this paper we determine all classes of k-orbit maps for $k \leq 4$ and list them in Table 1. To do this we make use of holey maps defined in Section 2. For each class of k-orbit maps, the corresponding holey map (see Proposition 3.4) on k flags is essentially the Delaney graph described in [7] and [8]. The classes in [13] were obtained by considering all possible local flag arrangements.

In Sections 5 and 6 we classify 2- and 3-orbit maps on sphere, projective plane, torus and Klein bottle. Our approach to the classification is quite geometric by way of using the operations of truncation and medial. We give algebraic and geometric descriptions of these operations on maps in Section 4. In order to use the operations we employ the concepts of compatibility and admissibility of classes of k-orbit maps.

Furthermore we provide Hurwitz-like upper bounds for the cardinality of the automorphism group of 2- and 3-orbit maps on surfaces. Such upper bounds were used by Conder (see [5] and [4]), where all chiral and orientably regular maps on surfaces of genus 1 to 101 and non-orientably regular maps on surfaces of genus 1 to 202 are determined.

2 Maps and f-graphs

The action of a group G on the set Z is an operation $\cdot : Z \times G \to Z$, such that $z \cdot id = z$ and $(z \cdot g) \cdot h = z \cdot (gh)$, for every $z \in Z$ and $g, h \in G$. We denote it by the triple $(Z, G, \cdot)$ and when the action is clear from the context, we abreviate it with (Z, G). Let $(Z, G, \cdot)$ and $(Z', G', *)$ be two actions. A pair (p, q) consisting of a surjective mapping $p : Z \to Z'$ and a group epimorphism $q : G \to G'$ is called an *action epimorphism* if for every $z \in Z$ and every $g \in G$ it follows that $p(z \cdot g) = p(z) * q(g)$. If both p and q are one-to-one we refer to it as an *action isomorphism*. Denote by $\mathrm{Orb}(Z, G)$ the set of orbits of the action. We will need the following elementary lemma.

Lemma 2.1. *Let $(p, q) : (Z_1, G_1) \to (Z_2, G_2)$ be an action epimorphism, where p is a bijection. Then p maps orbits to orbits and as such defines the bijection $p : \mathrm{Orb}(Z_1, G_1) \to$*

$\mathrm{Orb}(Z_2, G_2)$. *In particular, cardinalities of* $\mathrm{Orb}(Z_1, G_1)$ *and* $\mathrm{Orb}(Z_2, G_2)$ *are equal, and the cardinalities of any two orbits in correspondence are the same.*

Next we provide algebraic, topological and combinatorial definitions of maps.

Let $\mathcal{C} = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0 s_2)^2 \rangle$ be the Coxeter group $\mathcal{C} = [\infty, \infty]$. A *holey map* is a transitive action $(\mathcal{X}, \mathcal{C})$ of the group $\mathcal{C}$ on a set $\mathcal{X}$ (see [1] for further details).

For a topological interpretation of a holey map, assume that the elements of $\mathcal{X}$ are triangles (homeomorphic to closed Euclidean discs) with vertices labeled 0, 1 and 2. Let $T, T' \in \mathcal{X}$ such that $Ts_i = T'$. We define an s_i*-identification* of T and T' as an identification over the side with vertices labeled by $\{0, 1, 2\} \setminus i$ in such a way that the vertices with the same labels are identified. The set $\mathcal{X}$ together with the identifications determined by the action of $\mathcal{C}$ defines a triangulated connected surface with an embedded graph in the following way. The vertices of the graph are combinatorially represented by the orbits of $\langle s_1, s_2 \rangle$. Topologically, each orbit is a set of triangles, which have the same common point labeled by 0, and that point is considered as the embedding of a vertex. Combinatorially, the edges are the orbits of $\langle s_0, s_2 \rangle$, where their topological representation consists of unions of one or two segments used for s_2-identifications within the orbits. Combinatorially, the faces are the orbits of $\langle s_0, s_1 \rangle$. The triangles are also called *flags*.

For a holey map M we denote by $\mathcal{F}(M)$ the set of all flags of M. For any given flag $\Phi \in \mathcal{F}(M)$ we denote the vertex (0-face), edge (1-face) and face (2-face) of Φ by Φ_0, Φ_1 and Φ_2, respectively. For $i = 0, 1, 2$ by $\mathcal{F}_i(M)$ we denote the set of all i-faces of M.

A subclass of holey maps are also *polyhedra* or 3-dimensional abstract polytopes (see [17] for definitions). Polyhedra are usually combinatorially represented with partially ordered sets of rank 3, whose elements are vertices, edges and faces, and the incidences are defined in the obvious way (see also [13]). In this case the flags are the triples (v, e, f) where v, e and f are a vertex, an edge and a face incident to each other. Some holey maps can be recovered from their corresponding partially ordered sets. For those maps we shall also use this combinatorial definition.

Consider the case where none of the elements of $S = \{s_0, s_1, s_2, s_0 s_1, s_0 s_2, s_1 s_2\}$ stabilizes any flag. Then the induced triangulated surface is locally homeomorphic to a disk at any point of the surface. In the case when the number of flags is finite, the surface described in the topological approach is a compact closed surface. The embedded graph is connected, with each edge between two distinct vertices, each vertex of degree at least two and each face of co-degree at least two. The number of flags at each edge is four and the number of flags at each vertex and at each face is even. Such an action is called a *map on a compact closed surface*, or in this paper, simply a *map*. We note that the polyhedra are those maps which satisfy the following two conditions.

- Each edge contains two vertices and belongs to two faces. Furthermore, at each vertex of a face there are precisely two edges belonging to that face. This is commonly referred as *diamond condition*.

- *Strong flag-connectivity* in the sense of [17] Chapter 2A.

Choosing a base flag Φ, let $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$. As the action of $\mathcal{C}$ is transitive, all stabilizers are of the form $w^{-1}Nw$, $w \in \mathcal{C}$. Hence, a map is a holey map with the property that $S \cap w^{-1}Nw = \emptyset$, for every $w \in \mathcal{C}$. A morphism between two maps $(\mathcal{X}_1, \mathcal{C})$ and $(\mathcal{X}_2, \mathcal{C})$ is a surjection $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ respecting both actions, or in other words, the map morphisms are exactly all possible action morphisms of the form (ϕ, Id), where $Id : \mathcal{C} \to \mathcal{C}$ is the identity homomorphism. Since $(\mathcal{X}, \mathcal{C}) \cong (\mathcal{C}/N, \mathcal{C})$, we may assume that all maps are of the form $(\mathcal{C}/N, \mathcal{C})$, where $N \leqslant \mathcal{C}$ and $[\mathcal{C} : N]$ is finite. In this setting the flags can be considered as cosets in $\mathcal{C}/N$ where the coset N corresponds to the base flag Φ. As a consequence we give the following lemmas.

Lemma 2.2. *Two actions $(\mathcal{C}/N, \mathcal{C})$ and $(\mathcal{C}/K, \mathcal{C})$ are isomorphic if and only if N and K are conjugate in $\mathcal{C}$. There is a morphism from $(\mathcal{C}/N, \mathcal{C})$ to $(\mathcal{C}/K, \mathcal{C})$ if and only if there exists $w \in \mathcal{C}$ such that $N \leqslant w^{-1}Kw$.*

Lemma 2.3. *All transitive actions $(Z, \mathcal{C})$ for which an action epimorphism of the form $(\phi, Id) : (\mathcal{C}/N, \mathcal{C}) \to (Z, \mathcal{C})$ exists, are isomorphic to an action of the form $(\mathcal{C}/K, \mathcal{C})$ where $N \leqslant K \leqslant \mathcal{C}$.*

The last lemma essentially tells us that all the quotients of $(\mathcal{C}/N, \mathcal{C})$ are of the form $(\mathcal{C}/K, \mathcal{C})$ where $N \leqslant K \leqslant \mathcal{C}$.

A map $M = (\mathcal{C}/N, \mathcal{C})$ with a chosen base flag Φ defined by a subgroup N of $\mathcal{C}$ of finite index, is an object called an F-action (see [18]). From the theory of F-actions we extract the following. Applying $w = s_{i_1} s_{i_2} \cdots s_{i_k} \in \mathcal{C}$ on $\Phi = N$ corresponds to the sequence of s_{i_j}-identifications describing a path from Φ to $\Phi \cdot w$ through adjacent flags. The path brings us back to Φ if and only if $w \in N$. Denote by $\mathcal{N}$ the normalizer $\mathrm{Norm}_{\mathcal{C}}(N)$. The set $\{Nw \mid w \in \mathcal{N}\} = \mathcal{N}/N$ corresponds to the orbit of N under the action of $\mathrm{Aut}(M)$, while other cosets $\mathcal{N}v$, $v \in \mathcal{C} \setminus \mathcal{N}$, viewed as subsets of $\mathcal{C}/N$, correspond to other orbits. For each $w \in \mathcal{N}$, there exists an automorphism α_w taking Φ to $\Phi \cdot w$, and therefore $\mathrm{Aut}(M) = \{\alpha_w \mid w \in \mathcal{N}\}$. Observe that, for every coset $Nw \in \mathcal{C}/N$ and $v \in \mathcal{N}$ we have $\alpha_v(Nw) = vNw = Nvw$ implying that the image of a single flag completely describes any automorphism, that is, the action of $\mathrm{Aut}(M)$ on the flags is semi-regular. For $w, v \in \mathcal{N}$, $\alpha_w = \alpha_v$ if and only if $wv^{-1} \in N$. The mapping $w \mapsto \alpha_w$ is an epimorphism with kernel N and induces the isomorphism $\mathcal{N}/N \cong \mathrm{Aut}(M)$. For a group K such that $N \leqslant K \leqslant \mathcal{N}$, the map $(\mathcal{C}/K, \mathcal{C})$ represents the quotient obtained by identifying orbits of the subgroup $\langle \alpha_w \mid w \in K \rangle \leqslant \mathrm{Aut}(M)$. The automorphism α_w projects from $(\mathcal{C}/N, \mathcal{C})$ to $(\mathcal{C}/K, \mathcal{C})$, for $N \leqslant K$, if and only if w normalizes K.

Denote by $\mathrm{Orb}(M)$ the set of orbits under the action of $\mathrm{Aut}(M)$ on the flags. It is elementary to see that the orbits of any subgroup of the automorphism group are blocks of imprimitivity for the action of $\mathcal{C}$. In particular, the action $(\mathrm{Orb}(M), \mathcal{C})$ is well defined. Also, the orbits of any subgroup of $\mathcal{C}$ are blocks of imprimitivity for the action of $\mathrm{Aut}(M)$. All orbits in $\mathrm{Orb}(M)$ have the same cardinality and $|\mathrm{Orb}(M)| = [\mathcal{C} : \mathcal{N}]$ (see also [13]).

Denote by $\mathrm{Sym}(Z)$ the symmetric group on elements of a finite set Z. As usual, S_n will stand for the symmetric group of permutations of n elements. The action of the map $M = (\mathcal{C}/N, \mathcal{C})$ is defined by a homomorphism $\chi : \mathcal{C} \to \mathrm{Sym}(\mathcal{C}/N)$. The image of χ together with the generators $\chi(s_i)$ is called the *monodromy group* $\mathcal{M}(M)$ of the map M. When there it is no ambiguity, we will use the labels s_i for $\chi(s_i)$ and denote $\mathcal{M}(M) = \langle s_0, s_1, s_2 \rangle$. Conversely, given a permutation group $\mathcal{M}$ with involutory generators r_0, r_1 and r_2 where r_0 and r_2 commute, and given the homomorphism $\chi : \mathcal{C} \to \mathcal{M}$, $\chi(s_i) = r_i$ for $i = 0, 1, 2$, we define the induced map $M = (\mathcal{C}/\chi^{-1}(\mathcal{M}), \mathcal{C})$, for which $\mathcal{M}$ is its monodromy group. In Section 4 we define the medial and the truncation of a map in this way, that is by constructing its monodromy group.

Note that any set of three involutions $s_0, s_1, s_2 \in S_n$, where s_0 and s_2 commute and the group $\langle s_0, s_1, s_2 \rangle$ is transitive, represents a permutation representation of some map. Therefore, a monodromy group is a permutation group with labeled generators and it will be denoted by the triple $[s_0, s_1, s_2]$, for $s_i \in S_n$ as above. Two such monodromy groups $[s_0, s_1, s_2]$ and $[s'_0, s'_1, s'_2]$ represent isomorphic maps if and only if there is $\tau \in S_n$ such that $[s_0, s_1, s_2]^\tau = [s_0^\tau, s_1^\tau, s_2^\tau] = [s'_0, s'_1, s'_2]$, where s^τ denotes $\tau^{-1} s \tau$. We call $\{[s_0, s_1, s_2]^\tau \mid \tau \in S_n\}$ the *conjugacy class* of the triple. In the context of F-actions, a map $M = (\mathcal{C}/N, \mathcal{C})$ corresponds to the $\mathcal{C}$-action $(\chi, \mathcal{C}/N, \chi(\mathcal{C}), N)$ [18].

We define the *f-graph* $\Gamma(M)$ of any given map $M = (\mathcal{C}/N, \mathcal{C})$ as a (multi)graph labeled on its edges with vertex set equal to the flag set of M, and edge set

$$\{\{g, gs_i\} \mid g \text{ is a flag of } M,\ i = 0, 1, 2\}.$$

The edge $\{g, gs_i\}$ is labeled s_i. The edge $\{g, gs_i\}$ is a *link* when $g \neq gs_i$, and if a *semi-edge* if $g = gs_i$. Each walk in $\Gamma(M)$ (defined in the usual way) can be made *reduced* by recursively deleting sections of two consecutive appearances of the same edge.

Choosing a vertex v of $\Gamma(M)$ we denote by $\pi(\Gamma(M), v)$ the fundamental group containing all reduced walks with initial and terminal vertex v. The group operation is joining the walks, the unit is the trivial walk and the inverse walk is defined by reversing the sequence of edges. Note that with such definition, this concept of a (combinatorial) fundamental group slightly differs from the fundamental group on the underlying topological graph in the classical topological sense, as a walk on a semiedge is not reducible to a trivial walk.

In this setting, the *voltage assignment* on a graph is a mapping ζ from the edge set of $\Gamma(M)$ to a group G, where each edge is mapped into an involution. Such a voltage naturally extends to walks by taking product of voltages on consecutive edges. Note that such an assignment induces a homomorphism $\pi(\Gamma(M), v) \to G$, for every $v \in V(\Gamma(M))$.

Let $\mathcal{C}_0 = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2 \rangle$ and $f_0 : \mathcal{C}_0 \to \mathcal{C}$ the epimorphism defined by taking $s_i \in \mathcal{C}_0$ to $s_i \in \mathcal{C}$. Define the voltage ζ of the f-graph $\Gamma(M)$ of a map M to be determined by the labeling of the edges. The graphs just defined are a special case of so called *flag graphs* (see for instance [19]).

Theorem 2.4. *Let $M = (\mathcal{C}/N, \mathcal{C})$ a map and $\Gamma(M)$ its f-graph. Then the group $f_0^{-1}(N)$ is isomorphic to $\pi(\Gamma(M), N)$.*

Proof. Note that each element in $\mathcal{C}_0$ can be uniquely represented as a reduced word in s_i, $i = 0, 1, 2$. Each such reduced word $w \in \mathcal{C}_0$ uniquely defines a reduced walk in $\Gamma(M)$ starting in N. Note that different words induce different walks, and the words in $f_0^{-1}(N)$ are exactly the words inducing the reduced walks ending in N. This clearly defines the bijection, which is an isomorphism of the groups. □

3 Classes of k-orbit maps

A map $M = (\mathcal{C}/N, \mathcal{C})$ is said to be a *k-orbit map* if $|\mathrm{Orb}(M)| = k$. By the discussion above this corresponds to the case when $[\mathcal{C} : \mathcal{N}] = k$, where we recall that $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$. Let C be a conjugacy class of subgroups of index k in $\mathcal{C}$. We will say that M is in *class* C if $\mathcal{N} \in C$.

The choice of a new base flag $\Psi = \Phi \cdot w$ corresponds, as noted above, to the choice of the new stabilizer $w^{-1}Nw = \mathrm{Stab}_{\mathcal{C}}(\Psi)$, and the normalizer of $\mathrm{Stab}_{\mathcal{C}}(\Psi)$ becomes $w^{-1}\mathcal{N}w = \mathrm{Norm}_{\mathcal{C}}(w^{-1}Nw)$. The automorphism α_v, defined with respect to Φ, taking $\Phi \mapsto \Phi \cdot v$, and thus $\Psi = \Phi \cdot w \mapsto \Phi \cdot vw = \Psi \cdot w^{-1}vw$, becomes $\alpha_{w^{-1}vw}$ with respect to Ψ, taking the new base flag Ψ to $\Psi \cdot w^{-1}vw$. Therefore, if a map M is in C, then each group in C represents the normalizer of the stabilizer of some flag of M. Quotients of such normalizers by the corresponding stabilizers always yield the same automorphism group. Note that classes are disjoint and therefore a map can only belong to one class.

To simplify the classification of k-orbit maps, we will always choose a distinguished representative for a class in the following way. If $\mathcal{T} \in C$ is such a representative, saying that "a map M is in class $\mathcal{T}$" will mean that we have chosen the the base flag Φ of M in such a way that $\mathcal{T}$ is exactly the normalizer of the stabilizer of Φ.

Definition 3.1. *For a class $\mathcal{T}$, we will say that a map $M = (\mathcal{C}/N, \mathcal{C})$ is $\mathcal{T}$-*admissible *if and only if* $\mathrm{Norm}_{\mathcal{C}}(N) \geqslant \mathcal{T} \geqslant N$. *A map M is $\mathcal{T}$-*compatible *if $N \leqslant \mathcal{T}$.*

Let $M = (\mathcal{C}/N, \mathcal{C})$ be a $\mathcal{T}$-admissible map, then $\mathrm{Aut}(M) \cong \mathrm{Norm}_{\mathcal{C}}(N)/N$ contains the *$\mathcal{T}$-admissible subgroup* that corresponds to $\mathcal{T}/N \leq \mathrm{Norm}_{\mathcal{C}}(N)$. In [3] this concept is referred to as $\mathcal{T}$-regular, and $\mathcal{T}$-compatible is referred to as $\mathcal{T}$-conservative.

From the combinatorial point of view, a map M is $\mathcal{T}$-compatible if and only if we can label the flags of M with the orbits of flags of any map M' in class $\mathcal{T}$ in the following way. Each flag of M labeled k is i-adjacent to a flag labeled k' whenever the flags of M' in the orbit k are i-adjacent to the flags in the orbit k' (in M'). Moreover, a $\mathcal{T}$-compatible map M with flags labeled in this way is $\mathcal{T}$-admissible whenever every two flags with the same label belong to the same flag orbit of the map M.

For example, a map $M = (\mathcal{C}/N, \mathcal{C})$ is *regular* (or sometimes called *reflexible*), if the normalizer of every flag is the whole group $\mathcal{C}$. Let $\mathcal{C}^+$ denote the index 2 subgroup of $\mathcal{C}$ containing all even length words. A $\mathcal{C}^+$-admissible map is called *orientably regular*. This includes reflexible maps on orientable surfaces (when the normalizer is $\mathcal{C}$) and chiral maps

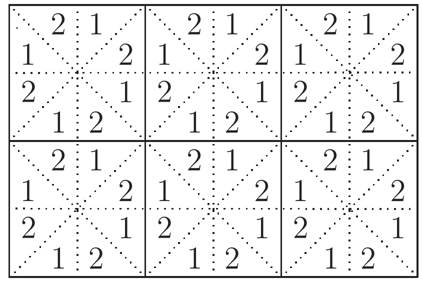


Figure 1: 2-compatible but not 2-admissible map.

(the normalizer is $\mathcal{C}^+$). In this paper we shall refer to $\mathcal{C}^+$-admissible maps as 2-admissible (see Table 1). For an orientably regular map M the 2-admissible subgroup of $\mathrm{Aut}(M)$ is exactly the orientation preserving subgroup.

The 2-compatible maps are called *orientable.* As an example, the labels 1 and 2 in the toroidal map M in Figure 1 show that M is 2-compatible. However, M is not 2-admissible because Φ and Φ^{01} belong to two different flag orbits for any flag Φ in M.

This properties ($\mathcal{T}$-admissibility and $\mathcal{T}$-compatibility) will play an important role in Section 4, when we consider medials and truncations of maps.

Let $G \leqslant \mathcal{C}$ and $M = (\mathcal{C}/N, \mathcal{C})$ in class $\mathcal{T}$. The orbit of N under the action of the automorphism group corresponds to the set of cosets $\mathcal{T}N = N\mathcal{T} = \{Nt \mid t \in \mathcal{T}\}$, while for any $w \in \mathcal{C}$, the orbit of Nw corresponds to $\mathcal{T}Nw = \{Ntw \mid t \in \mathcal{T}\}$. As we noted, the orbits of G (for action $(\mathcal{C}/N, G)$, henceforth called *G-orbits*) are blocks of imprimitivity for the action of the automorphism group. We define a map to be *G-orbit transitive*, if the automorphism group is transitive on G-orbits. Therefore M is G-orbit transitive if and only if $\mathcal{T}NG = \{tNg \mid g \in G, t \in \mathcal{T}\} = \mathcal{C}/N$.

For example, if $G = \langle s_0, s_2\rangle$, then the orbits of G are exactly the edges of a map. Hence, the G-transitive maps are exactly the edge-transitive maps. Similarly the $\langle s_0, s_1\rangle$-transitive maps are the face-transitive maps and the $\langle s_1, s_2\rangle$-transitive maps are the vertex-transitive maps.

A class $\mathcal{T}$ is said to be a *G-orbit transitive* class if $\mathcal{T}$ is transitive on $\mathcal{C}/G$. We proceed to show that G-orbit transitivity does not depend on any particular choice of $\mathcal{T}$ for a representative of the class.

Lemma 3.2. *Let K and H be sugroups of a group L such that the natural action of K on the factor set L/H is transitive. Then any conjugate $w^{-1}Kw$, $w \in L$, acts transitively on L/H.*

Proof. A subgroup K acts transitively on L/H if an only if $L = HK$. Assuming $L = HK$

and $x \in L$, we have $x = hk$, for some $h \in H$ and $k \in K$. The conjugation in L by any $u \in H$ is an (inner) automorphism taking an element $x = hk$ to $(u^{-1}h'u)(u^{-1}k'u)$. This implies that $L = H(u^{-1}Ku)$ for any $u \in H$. As every $w \in L$ is of the form $w = hk$, for some $h \in H$ and $k \in K$, it follows that $wKw^{-1} = hkKk^{-1}h^{-1} = hKh^{-1}$. Hence, for every $w \in L$, $L = H(wKw^{-1})$. □

Note that in this context, if $L = HK$, then $L = KH$. This follows because the inversion of all elements in any group is a bijection.

The lemma immediately implies the following.

Corollary 3.3. *Let $\mathcal{T}$ be a class of k-orbit maps and $G \leqslant \mathcal{C}$. The following are equivalent.*

1. *$\mathcal{T}$ is G-orbit transitive.*
2. *$\mathcal{C} = G\mathcal{T} = \mathcal{T}G$.*
3. *Every map in class $\mathcal{T}$ is G-orbit transitive.*

Proof. A map $M = (\mathcal{C}/N, \mathcal{C})$ in class $\mathcal{T}$ is G-orbit transitive if $\mathcal{C} = \mathcal{T}NG = \mathcal{T}G$. On the other hand, if $\mathcal{C} = \mathcal{T}G$, than for any N such that $\mathrm{Norm}_{\mathcal{C}}(N) = \mathcal{T}$ it follows that $\mathcal{C} = \mathcal{T}NG$. □

Theorem 3.4. *The classes of k-orbit maps are in one-to-one correspondence with the isomorphism classes of holey maps on k flags. The correspondence is given by $\Theta : \mathcal{T} \mapsto (\mathcal{C}/\mathcal{T}, \mathcal{C})$. Furthermore, the class $\mathcal{T}$ is G-orbit transitive if and only if G is transitive on $(\mathcal{C}/\mathcal{T}, \mathcal{C})$.*

Proof. It is clear that $(\mathcal{C}/\mathcal{T}, \mathcal{C})$ is a holey map on $[\mathcal{C} : \mathcal{T}]$ flags. Note also that different representatives $\mathcal{T}$ of the same class are conjugate and therefore correspond to isomorphic holey maps (see Lemma 2.2). Moreover, $\mathcal{T}$ is G-orbit transitive if and only if G is transitive on $\mathcal{C}/\mathcal{T}$, but this is equivalent to $\mathcal{T}G = \mathcal{C}$. □

To determine all classes of k-orbit maps it is sufficient to determine all non-isomorphic holey maps on k flags. This is equivalent to finding all conjugacy classes of monodromy groups $[r_0, r_1, r_2]$, $s_i \in S_k$. We proceed to do this in the following steps.

1) Determine all cyclic structures of conjugacy classes of involutions or *id* in S_k. For each cyclic structure we assign one representative to r_0. For each of these choices we proceed to step 2.

2) We determine the centralizer $Z_0 = Z_{S_k}(r_0)$ and the conjugacy classes of involutions or *id* in Z_0 under conjugation by Z_0. For each such class we assign one representative to r_2 and for each of these choices we continue to step 3.

3) We compute the centralizer $Z_{02} = Z_{Z_0}(r_2)$, and for each conjugacy class of involutions or id in S_k under conjugation by Z_{02} we pick a representative and assign it to r_1.

4) If the choice makes the group $\langle r_0, r_1, r_2 \rangle$ transitive, the triple $[r_0, r_1, r_2]$ is one of the required representatives.

The algorithm returns exactly one representative for each conjugacy class of triples.

As an example, we show how to determine the classes of 3-orbit maps. We first find representatives of conjugacy classes of the triples $[r_0, r_1, r_2]$. There are three conjugacy classes of elements in S_3: $A = \{id\}$, $B = \{(1,2),(1,3),(2,3)\}$ and $C = \{(1,2,3),(1,3,2)\}$. By the algorithm we have two choices for r_0, namely id and $(1,2)$. If $s_0 = id$, then $Z_0 = S_3$. There are two conjugacy classes for involutions in Z_0, namely A and B. First we choose $r_2 = id$. Then $Z_{02} = S_3$ and we have the same two conjugacy classes for choices of r_1. However, none of them provides a choice of r_1 which would make $\langle r_0, r_1, r_2 \rangle$ transitive. Assuming $r_2 = (1,2)$ we have $Z_{02} = \{id, (1,2)\}$. The conjugacy classes of involutions of S_3 under the conjugation by Z_{02} are $A' = \{id\}$, $B' = \{(1,2)\}$ and $C' = \{(1,3),(2,3)\}$. Only the choices $r_1 \in C'$ give us transitive groups. So far we only have one triple $3^0 = [id, (1,3), (1,2)]$.

Assume now that $s_0 = (1,2)$. Then $Z_0 = \{id, (1,2)\}$. The conjugacy classes of involutions in Z_0 under conjugation by itself are A', B' as above. Chosing $r_2 = id$ we have $Z_{02} = Z_0$ and the corresponding conjugacy classes for the choice of r_1 are A', B' and C'. Again, only $r_1 \in C'$ gives the triple $3^2 = [(1,2),(1,3),id]$ with the transitive group. Proceeding with the alternative choice $s_2 = (1,2)$ gives third possible triple $3^{02} = [(1,2),(1,3),(1,2)]$.

By Theorem 3.4, none of them is edge-transitive (which is no surprise by [10]). The class 3^0 is vertex-transitive, 3^2 is face-transitive and 3^{02} is vertex- and face- transitive. In Figure 2 the flag graphs for all three classes are given.

For example, to determine a set of generators for a representative of the class 3^0 we proceed as follows. We choose a base flag in the corresponding holey map M_{3^0}. In our case, this will be the flag numbered 3 (the choice of the flag corresponds to the choice of a conjugate in the class). To determine the generators of the fundamental group $\pi(\Gamma(M_{3^0}), 3)$ we choose any spanning tree in the f-graph (in our case the path 3-1-2). Each edge e not covered by the tree uniquely determines (up to inverse) the closed walk starting in the vertex 3, traversing the unique path in the tree from 3 to one endpoint of e, then traversing e and returning from the other endpoint of e to the vertex 3 using the unique path in the tree. It is easy to see that the set of all closed walks obtained in this way generates $\pi(\Gamma(M_{3^0}), 3)$. In terms of voltages on edges, the set of generating walks is $S = \{s_0, s_2, s_1s_0s_1, s_1s_2s_1s_2s_1, s_1s_2s_0s_2s_1\}$. By Theorem 2.4, the voltages of generating walks of the fundamental group are the generators of $f_0^{-1}(N)$. By using the relations in $\mathcal{C}$ we further reduce the set S of generators to obtain the generating set for N, which is $\{s_0, s_2, s_1s_0s_1, s_1s_2s_1s_2s_1\}$.

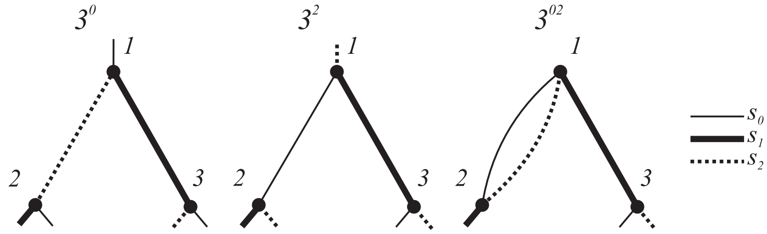


Figure 2: The flag graphs for the classes 3^0, 3^2 and 3^{02}.

The procedures described above produce the entries in the third and the fourth column of Table 1 below.

The alternative approach for determining types of k-orbit maps would be simply to use `LowIndexSubgroups` algorithm in MAGMA[2]. In this view, the proof of Graver and Watkins [10] can be interpreted as a kind of an enumeration of subgroups of $\mathcal{C}$ up to index 4 determining edge-transitive types.

A similar analysis can be carried out for 2-orbit and 4-orbit classes. The results are presented in Table 1. The first and the second column of the table contain the names of classes and the number of orbits under the action of the automorphism group for the type, respectively. The generators for the monodromy group of the corresponding holey map and the generators for a chosen representative of the class are given in the third and the fourth column. Note that in the fourth column we use the abbreviated notation where $i_1 i_2 \dots i_k$ stands for$s_{i_1} s_{i_2} ... s_{i_k}$. The fifth column specifies the ranks of faces of the map, on which the automorphism group for the type is transitive. The last two columns give dual and Petrie dual types.

4 Operations on maps

In this section we will investigate operations on maps producing new maps on the same surface. There are several approaches to these map operations. The most common one (and the most intuitive), the *geometric* approach, is by introducing local transformations on flags where we substitute each flag with a set of new flags. For example, Figures 3 and 6 show the way how each flag Φ is subdivided to obtain two new flags for the operation medial and three flags for the truncation.

An example of an operation dividing each flag into five new flags is described in Figure 4. Here the numbers indicate the face rank corresponding to the vertices of the new flags.

The second approach, the *combinatorial* one, which is mainly used with polyhedra (abstract 3-polytopes), consists of considering the operations as transformations on the corresponding posets (for example, see [13]). The third approach, the *algebraic* approach,

Class	#orb.	Involutions			Generators of N	Trans. ranks	dual	Petrie
		s_0	s_1	s_2				
1	1	id	id	id	s_0, s_1, s_2	0,1,2	1	1
2_0		id	$(1,2)$	$(1,2)$	0, 12	0,1,2	2_2	2
2_2		$(1,2)$	$(1,2)$	id	2, 01	0,1,2	2_0	2_2
2	2	$(1,2)$	$(1,2)$	$(1,2)$	01, 12	0,1,2	2	2_0
2_{01}		id	id	$(1,2)$	1, 0, 212	0,1	2_{12}	2_1
2_1		$(1,2)$	id	$(1,2)$	1, 010, 02	0,1,2	2_1	$2_{0,1}$
2_{12}		$(1,2)$	id	id	1, 2, 010	1,2	2_{01}	2_{12}
2_{02}		id	$(1,2)$	id	0, 2, 101, 121	0,2	2_{02}	2_{02}
3^0	3	id	$(1,3)$	$(1,2)$	0, 2, 101, 12121	0	3^2	3^{02}
3^2		$(1,2)$	$(1,3)$	id	0, 2, 121, 10101	2	3^0	3^2
3^{02}		$(1,2)$	$(1,3)$	$(1,2)$	0, 2, 1021, 10101	0,2	3^{02}	3^0
4_A	4	id	$(1,3)(2,4)$	$(1,2)$	0, 2, 101, 1210121, $(12)^3 1$	0	4_{Ad}	4_{Ap}
4_{Ad}		$(1,2)$	$(1,3)(2,4)$	id		2	4_A	4_{Ad}
4_{Ap}		$(1,2)$	$(1,3)(2,4)$	$(1,2)$		0,2	4_{Ap}	4_A
4_B		id	$(1,3)$	$(1,2)(3,4)$	0, 1, 21012, $(21)^3 2$	0	4_{Bd}	4_{Bp}
4_{Bd}		$(1,2)(3,4)$	$(1,3)$	id		2	4_B	4_{Bd}
4_{Bp}		$(1,2)(3,4)$	$(1,3)$	$(1,2)(3,4)$		0,2	4_{Bp}	4_B
4_C		id	$(1,3)(2,4)$	$(1,2)(3,4)$	0, 101, 1212	0	4_{Cd}	4_{Cp}
4_{Cd}		$(1,2)(3,4)$	$(1,3)(2,4)$	id		2	4_C	4_{Cd}
4_{Cp}		$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,2)(3,4)$		0,2	4_{Cp}	4_C
4_D		$(1,2)(3,4)$	$(1,3)$	$(1,2)$	1,2, 010210, $(01)^3 0$	2	4_{Dd}	4_{Dp}
4_{Dd}		$(1,2)$	$(1,3)$	$(1,2)(3,4)$		0	4_D	4_{Dd}
4_{Dp}		$(1,2)$	$(1,3)$	$(3,4)$			4_{Dp}	4_D
4_E		$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,2)$	2, 1010, 1210	0,2	4_{Ed}	4_{Ep}
4_{Ed}		$(1,2)$	$(1,3)(2,4)$	$(1,2)(3,4)$		0,2	4_E	4_{Ed}
4_{Ep}		$(1,2)$	$(1,3)(2,4)$	$(3,4)$		0,2	4_{Ep}	4_E
4_F		$(1,2)(3,4)$	id	$(1,3)(2,4)$	1,212, 010, 02120	1	4_F	4_F
4_G		$(1,2)(3,4)$	$(1,2)(3,4)$	$(1,3)(2,4)$	01, 2102	0,1	4_{Gd}	4_{Gp}
4_{Gd}		$(1,2)(3,4)$	$(1,3)(2,4)$	$(1,3)(2,4)$		1,2	4_G	4_{Gd}
4_{Gp}		$(1,2)(3,4)$	$(1,4)(2,3)$	$(1,3)(2,4)$		0,1,2	4_{Gp}	4_G
4_H		$(1,2)(3,4)$	$(1,2)$	$(1,3)(2,4)$	1, 010, 2102	0,1	4_{Hd}	4_{Hp}
4_{Hd}		$(1,2)(3,4)$	$(1,3)$	$(1,3)(2,4)$		1,2	4_H	4_{Hd}
4_{Hp}		$(1,2)(3,4)$	$(1,4)$	$(1,3)(2,4)$	6	0,1,2	4_{Hp}	4_H

Table 1: Classes of k-orbit maps, $k \leq 4$.

is by viewing the operations as transformations induced by certain automorphisms of $\mathcal{C}$ [13], or monomorphisms from $\mathcal{C}$ to certain subgroups of quotients of $\mathcal{C}$.

Not all approaches can be used for all maps. For instance, the combinatorial approach can be used only when the map is faithfully represented by its poset (i.e. maximal chains are in one-to-one correspondence with flags). However, on maps as defined above and the operations we will discuss in this paper, geometric and algebraic (and sometimes also combinatorial) approaches will turn out to be equivalent.

It is convenient to simplify the notation in the following way. We define $s_{ij} := s_i s_j$ and $r_{ij} := r_i r_j$, and recursively for any sequence I of elements in $\{0, 1, 2\}$, we define $s_{Ij} := s_I s_j$ and $r_{Ij} := r_I r_j$.

4.1 Medial

Let $\mathcal{M}(M)$ be the monodromy group of a map $M = (\mathcal{C}/N, \mathcal{C})$. We define the medial $\mathrm{Me}(M)$ of M by constructing its monodromy group.

Considering the subdivision of the flag in Figure 3, to derive the monodromy group

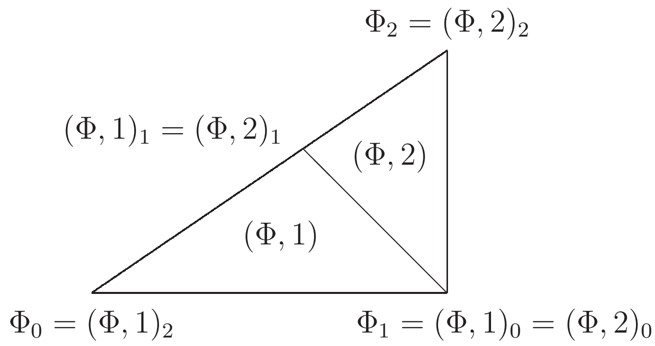


Figure 3: Operations on maps

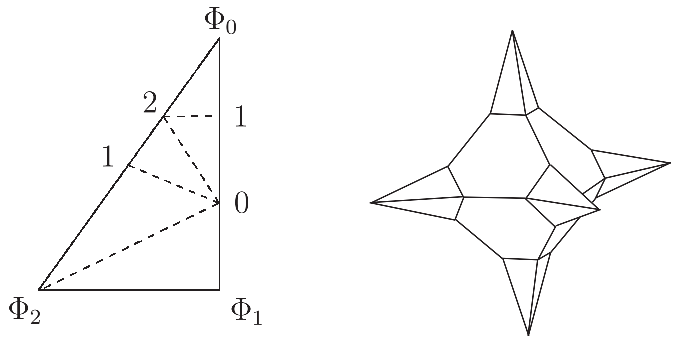


Figure 4: Piñata operation Pñ and Pñ($\{3, 4\}$)

of the *medial* $\mathrm{Me}(M)$ acting on the new set of flags $\mathcal{F}(M) \times \{1, 2\}$, we first define the generators

$$\begin{array}{llllllllllll} r_0: & (\Phi,1) & \mapsto & (\Phi^{s_1},1) & r_1: & (\Phi,1) & \mapsto & (\Phi^{s_2},1) & r_2: & (\Phi,1) & \mapsto & (\Phi,2) \\ & (\Phi,2) & \mapsto & (\Phi^{s_1},2) & & (\Phi,2) & \mapsto & (\Phi^{s_0},2) & & (\Phi,2) & \mapsto & (\Phi,1). \end{array}$$

Note that there is the action isomorphism $(p, q) : (\mathcal{F}(M), [s_0, s_1, s_2]) \to (\mathcal{F}(M) \times \{2\}, [r_1, r_0, r_{212}])$, where $p : \Phi \mapsto (\Phi, 2)$ and $q : (s_0, s_1, s_2) \mapsto (r_1, r_0, r_2r_1r_2)$. Let $\mathcal{C}_4 = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0s_2)^2, (s_1s_2)^4 \rangle$ be the quotient of $\mathcal{C}$ and $f_4 : \mathcal{C} \to \mathcal{C}_4$ be the corresponding epimorphism. For any map M, the monodromy group $\mathcal{M}(\mathrm{Me}(M))$ is a quotient of $\mathcal{C}_4$, where the epimorphism is defined on the generators by $f_M : (s_0, s_1, s_2) \mapsto (r_0, r_1, r_2)$. Consider the subgroup $W = \langle r_1, r_0, r_{212} \rangle$ of $\mathcal{C}$. Note that $f_M(f_4(2_{01})) = T$ (see Table 1 and the diagram below).

Considering the fact, that $2_{01,4} := f_4(2_{01}) = \langle s_1, s_0, s_{212} \rangle$, we have that $\varphi : \mathcal{C} \to 2_{01,4}$ defined on generators by $\varphi : (s_0, s_1, s_2) \mapsto (s_1, s_0, s_{212})$ is an isomorphism. For any flag N' of $\mathrm{Me}(M)$, the stabilizer $\mathrm{Stab}_{\mathcal{M}(\mathrm{Me}(M))}(N') \leqslant W$.

Hence, for any map $M = (\mathcal{C}/N, \mathcal{C})$ we can define its medial as the map $\mathrm{Me}(M) = (\mathcal{C}/f_4^{-1}(\varphi(N)), \mathcal{C}) \cong (\mathcal{C}_4/\varphi(N), \mathcal{C}_4)$, giving us the algebraic definition of the medial.

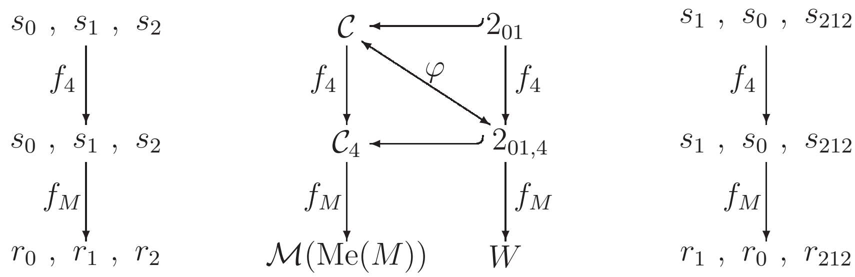


Figure 5: Algebraic definition of the medial operation

According to [13], the medial of the map can be also defined in an combinatorial way. Here the medial $\mathrm{Me}(M)$ of a map M is a poset with the set of i-faces $\mathcal{F}_i(\mathrm{Me}(M))$, $i = 0, 1, 2$, defined as follows

$$\begin{aligned}
&\mathcal{F}_0(\mathrm{Me}(M)) = \mathcal{F}_1(M),\\
&\mathcal{F}_1(\mathrm{Me}(M)) = \{\{(\Phi)_0, (\Phi)_2\} \mid \Phi \in \mathcal{F}(M)\},\\
&\mathcal{F}_2(\mathrm{Me}(M)) = \mathcal{F}_0(M) \cup \mathcal{F}_2(M).
\end{aligned}$$

The partial order on the faces $G_i \in \mathcal{F}_i(\mathrm{Me}(M))$ is given by

$$\begin{aligned}
&G_0 \leqslant_{\mathrm{Me}(M)} G_1 \Leftrightarrow \{G_0\} \cup G_1 \in \mathcal{F}(M),\\
&G_1 \leqslant_{\mathrm{Me}(M)} G_2 \Leftrightarrow G_2 \in G_1.
\end{aligned}$$

In [13] we show that the medial of a polyhedron is a polyhedron.

Any flag Φ of M induces in a natural way (see Figure 3) exactly two (adjacent) flags of $\mathrm{Me}(M)$ related to Φ, namely,

$$\Psi^0 := \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_0\} \quad \text{and} \quad \Psi^2 = \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_2\},$$

called *vertex-flag* and *face-flag* of Φ.

Conversely, if Ψ is a flag of $\mathrm{Me}(M)$, then $\{(\Psi)_0\} \cup (\Psi)_1$ is a flag of M. Denoting it by Φ, we have that $(\Psi)_0 = (\Phi)_1$, $(\Psi)_1 = \{(\Phi)_0, (\Phi)_2\}$ and $(\Psi)_2 = (\Phi)_0$ or $(\Phi)_2$. From connectivity of $\mathrm{Me}(M)$ it follows that any automorphism of $\mathrm{Me}(M)$ either preserves the sets of vertex-flags (therefore also face-flags) or interchanges vertex- and face-flags.

Note that $\mathrm{Aut}(M)$ is a subgroup of $\mathrm{Sym}(\mathcal{F}(M))$ containing all permutations π such that for every $w \in \mathcal{C}$ and every $\Phi \in \mathcal{F}(M)$ $(\Phi \cdot w)\pi = \Phi\pi \cdot w$. Clearly, every automorphism of M induces an automorphism of $\mathrm{Me}(M)$ that preserves vertex-flags and face-flags in the medial.

Let $d \in \mathrm{Aut}(\mathcal{C})$ be the automorphism fixing s_1 and interchanging s_0 and s_2. A *duality* $\delta \in \mathrm{Sym}(\mathcal{F}(M))$ is a permutation such that for every $w \in \mathcal{C}$ and every $\Phi \in \mathcal{F}(M)$ $(\Phi \cdot w)\delta = \Phi\pi \cdot d(w)$. A map M is *self-dual* if and only if has a duality. Any duality of M interchanges vertex-flags and face-flags in $\mathrm{Me}(M)$. The *extended group* $\mathcal{D}(M)$ is the subgroup of $\mathrm{Sym}(\mathcal{F}(M))$ containing all automorphisms and all dualities. Note that $[\mathcal{D}(M) : \mathrm{Aut}(M)] \leq 2$, and $[\mathcal{D}(M) : \mathrm{Aut}(M)] = 2$ if and only if M is self-dual.

Lemma 4.1. *If M is a map and $\mathrm{Me}(M)$ its medial, then $\mathrm{Aut}(\mathrm{Me}(M)) = \mathcal{D}(M)$.*

Proof. From the definition it follows that any automorphism or duality of M induces an automorphism of $\mathrm{Me}(M)$.

Let $\alpha \in \mathrm{Aut}(\mathrm{Me}(M))$. Since α maps 2-adjacent flags of $\mathrm{Me}(M)$ to 2-adjacent flags, α induces a permutation of $\mathcal{F}(M)$ which either preserves flag types or reverses them. Therefore α induces an automorphism or duality of M. □

The following theorem is an immediate consequence of the definition of medial operation.

Theorem 4.2. *The medial $\mathrm{Me}(M)$ of a k-orbit map M is either a k-orbit or a $2k$-orbit map. Furthermore, $\mathrm{Me}(M)$ is k-orbit if and only if M is self-dual. In particular $\mathrm{Me}(M)$ is regular if and only if M is regular and self-dual.*

In Table 2 we extend this to characterize the medials of 2-orbit maps. The class of the medial of a map in any class $\mathcal{T}$ can be easily derived from the local arrangements of the flags and Table 1. For a class containing a self-dual map M we can get two different classes depending on whether M is properly or improperly self-dual. A map M is said to be *properly self-dual* if it contains a duality which preserves all flag orbits of M, otherwise we say that M is *improperly self-dual*. For additional properties for dualities see [13].

Class of M	**Class of $\mathrm{Me}(M)$**		
	Proper	Improper	Not self-dual
1	1	—	2_{01}
2	2_2	2	4_G
2_0	—	—	4_H
2_1	2_{02}	2_0	4_C
2_2	—	—	4_H
2_{01}	—	—	4_A
2_{02}	2_{12}	2_1	4_F
2_{12}	—	—	4_A

Table 2: Classes of k-orbit maps, $k \leq 4$

As we can easily see from the description given in the following section of the maps on class 3^{02} (the only class containing self-dual 3-orbit maps), the medials of any self-dual 3-orbit map is a map in class 3^2.

In [13] we investigated properties of self-dual 2-orbit maps yielding 2-orbit medials (In fact, that has been done for polyhedra but can easily be extended to maps.)

We now explore some other interesting connections between two-orbit maps and medials.

Theorem 4.3. *A map with vertices of degree four is 2_{01}-admissible if and only if it is the medial of a regular map.*

Proof. Let M be a 2_{01}-admissible map and $G \leqslant \mathrm{Aut}(M)$ the 2_{01}-admissible subgroup. Let O be one of the two flag-orbits under the action of G. For any flag $\Psi \in O$ consider the pair Ψ and Ψ^2 (clearly $\Psi^2 \notin O$). The group G acts regularly on the pairs of 2-adjacent flags. Each such pair can be assembled into a single flag (see Figure 3). Clearly the map with so assembled flags is regular with M being its medial.

Conversely, for any regular map M, the group $\mathrm{Aut}(M)$ interpreted on the flags of $\mathrm{Me}(M)$ acts as the 2_{01}-admissible subgroup. □

From the proof of the theorem we derive the following corollary.

Corollary 4.4. *The medial of any map M is 2_{01}-compatible.*

4.2 Truncation

Given the monodromy group $\mathcal{M}(M) = \langle s_0, s_1, s_2\rangle$ of a map $M = (\mathcal{C}/N, \mathcal{C})$ we now derive the monodromy group of $\mathrm{Tr}(M)$ (see Figure 6). The new set of flags is the set $\mathcal{F}(M) \times \{1, 2, 3\}$ and the new generators are defined as follows.

$$\begin{array}{llll} r_0: & (\Phi,1) \mapsto (\Phi^{s_0},1) & r_1: (\Phi,1) \mapsto (\Phi,2) & r_2: (\Phi,1) \mapsto (\Phi^{s_2},1) \\ & (\Phi,2) \mapsto (\Phi^{s_1},2) & \quad\;\; (\Phi,2) \mapsto (\Phi,1) & \quad\;\; (\Phi,2) \mapsto (\Phi,3) \\ & (\Phi,3) \mapsto (\Phi^{s_1},3) & \quad\;\; (\Phi,3) \mapsto (\Phi^{s_2},3) & \quad\;\; (\Phi,3) \mapsto (\Phi,2) \end{array}$$

It is clear that each r_i is an involution and that $(r_0r_2)^2 = (r_1r_2)^3 = id$. Consider the action of the subgroup $T = \langle r_0, r_{101}, r_2\rangle$ of $\mathcal{M}(\mathrm{Tr}(M)) = \langle r_0, r_1, r_2\rangle$ on the subset of flags of the form $(\Phi, 1)$. Note that for any $\Phi \in \mathcal{F}(M)$ it follows $(\Phi, 1)^{r_0} = (\Phi^{s_0}, 1)$, $(\Phi, 1)^{r_{101}} = (\Phi^{s_1}, 1)$ and $(\Phi, 1)^{r_2} = (\Phi^{s_2}, 1)$ and therefore the monodromy groups $\mathcal{M}(M) = [s_0, s_1, s_2]$ and $[r_0, r_{101}, r_2]$ represent the same map.

Let $\mathcal{C}_3 = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0s_2)^2, (s_1s_2)^3\rangle$ be the quotient of $\mathcal{C}$ and $f_3 : \mathcal{C} \to \mathcal{C}_3$ the corresponding epimorphism. For any map M, the monodromy group $\mathcal{M}(\mathrm{Tr}(M))$ is a quotient of $\mathcal{C}_3$ where the epimorphism is defined on the generators by $f_M : (s_0, s_1, s_2) \mapsto (r_0, r_1, r_2)$. Consider the subgroup $T = \langle r_0, r_{101}, r_2\rangle$ of $\mathcal{C}$. Note that $f_M(f_3(3^0)) = T$ (see

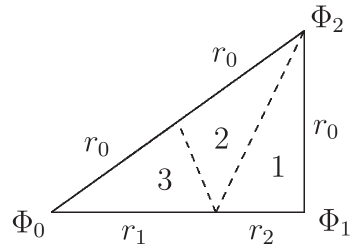


Figure 6: Adjacencies in $\mathrm{Tr}(M)$

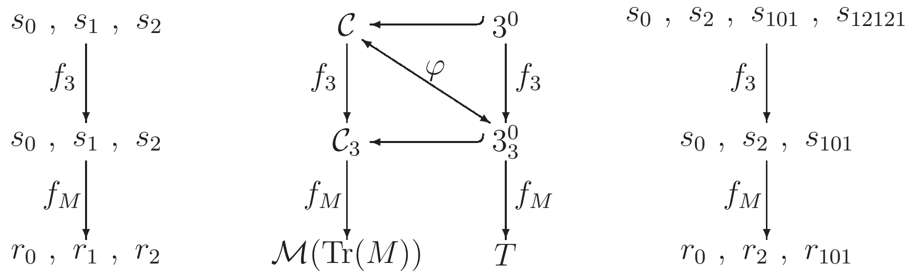


Figure 7: Algebraic definition of the truncation operation

Table 1 and the diagram below).

Considering the fact, that $3_3^0 = f_3(3^0) = \langle s_0, s_{101}, s_2 \rangle$, we have the isomorphism $\varphi : \mathcal{C} \to 3_3^0$ defined on generators by $\varphi : (s_0, s_1, s_2) \mapsto (s_0, s_{101}, s_2)$. For any flag N' of $\mathrm{Tr}(M)$, the stabilizer $\mathrm{Stab}_{\mathcal{M}(\mathrm{Tr}(M))}(N') \leqslant T$. Therefore, for any map $M = (\mathcal{C}/N, \mathcal{C})$ and its truncation $\mathrm{Tr}(M) = (\mathcal{C}/N', \mathcal{C})$, it follows that there exists a conjugate $wN'w^{-1}$ of N' in $\mathcal{C}$ such that $wN'w^{-1} \leqslant 3^0$.

Hence, for any map $M = (\mathcal{C}/N, \mathcal{C})$ we can define its truncation as the map $\mathrm{Tr}(M) = (\mathcal{C}/f_3^{-1}(\varphi(N)), \mathcal{C}) \cong (\mathcal{C}_3/\varphi(N), \mathcal{C}_3)$, giving us the algebraic definition of the truncation.

From the geometric definition of the truncation we derive the following combinatorial definition (when it applies). The truncation $\mathrm{Tr}(M)$ of a map M is the poset with i-faces $\mathcal{F}_i(\mathrm{Tr}(M))$ defined as

$$
\begin{aligned}
\mathcal{F}_0(\mathrm{Tr}(M)) &= \{\{(\Phi)_0, (\Phi)_1\} \mid \Phi \in \mathcal{F}(M)\}, \\
\mathcal{F}_1(\mathrm{Tr}(M)) &= \mathcal{F}_1(M) \cup \{\{(\Phi)_0, (\Phi)_2\} \mid \Phi \in \mathcal{F}(M)\}, \\
\mathcal{F}_2(\mathrm{Tr}(M)) &= \mathcal{F}_0(M) \cup \mathcal{F}_2(M).
\end{aligned}
$$

The partial order on the faces $G_i \in \mathcal{F}_i(\mathrm{Tr}(M))$ is given by

$$G_0 \leqslant_{\mathrm{Tr}(M)} G_1 \Leftrightarrow \begin{cases} G_1 \in G_0 \text{ or} \\ G_0 \cup G_1 \in \mathcal{F}(M), \end{cases}$$

$$G_1 \leqslant_{\mathrm{Tr}(M)} G_2 \Leftrightarrow \begin{cases} G_2 \in G_1 \text{ or} \\ G_1 \leq_M G_2. \end{cases}$$

In what follows we consider the impact of the truncation operation on the number of orbits when applied to a k-orbit map.

Proposition 4.5. *Let $M = (\mathcal{C}/N, \mathcal{C})$ be a k-orbit map. Then $\mathrm{Tr}(M)$ is either a k-orbit map, a $\frac{3k}{2}$-orbit map or a $3k$-orbit map.*

Proof. There are three cosets in $\mathcal{C}_3/3_3^0$, namely $3_3^0 a$, $a \in A = \{id, s_1, s_{121}\}$. Therefore, every element $x \in \mathcal{C}_3$ is of the form $x = ta$, $t \in 3_3^0$, $a \in A$.

Let $N' = \varphi(N)$ (see diagram in Figure 7). For any $a \in A$ let $x, y \in 3_3^0 a$, with $x = ta$, $y = sa$, for some $t, s \in 3_3^0$. If both x and y normalize N', it follows that $t^{-1}N't = aN'a^{-1} = s^{-1}N's$. Hence t and s must be in the same coset in $3_3^0/\mathcal{N}$, where $\mathcal{N} = \mathrm{Norm}_{3_3^0}(N') \leqslant 3_3^0$. This implies that $\mathrm{Norm}_{\mathcal{C}_3}(N')$ consists of cosets in $\mathcal{C}_3/\mathcal{N}$, where at most one such coset can be contained in any of the cosets $3_3^0 a$, $a \in A$. Since $[\mathcal{C}_3 : 3_3^0] = 3$, it follows that $\mathrm{Norm}_{\mathcal{C}_3}(N')$ consists of 1, 2 or 3 cosets in $\mathcal{C}_3/\mathcal{N}$. The proposition follows from the fact that $[3_3^0 : \mathcal{N}] = k$. ☐

We illustrate the proposition by the following examples.

The truncations of the regular maps of Schläfli type $\{4, 4\}$ and $\{6, 3\}$ are 3-orbit maps. The faces of the former are 4-gons and 8-gons, whereas the latter has 3-gons and 12-gons as faces.

The truncation of the regular map $\{3, 6\}_{(t,0)}$ is the regular map $\{6, 3\}_{(t,t)}$, whereas the truncation of the regular map $\{3, 6\}_{(t,t)}$ is the regular map $\{6, 3\}_{(3t,0)}$.

The map given in dotted lines in the figure below is a 3-orbit map on an orientable surface of genus 2. As we can see, this map can be obtained by truncating the regular map $\{3, 8| \cdot, \cdot, 2\}$ (see Figure 8 a) or by truncating the 2-orbit map given in Figure 8 b (belonging to class 2_{01}).

The core of 3_3^0 is the index two subgroup $K = \langle s_0, s_{101}, s_{21012} \rangle$ of $\mathcal{C}_3$. The group K plays an important role for the truncation operation, as is shown by the following two results.

Proposition 4.6. *Let $M = (\mathcal{C}/N, \mathcal{C})$ be a k-orbit map such that $\mathrm{Tr}(M)$ is $\frac{3k}{2}$-orbit or k-orbit. Then M must necessarily be 2_{01}-compatible.*

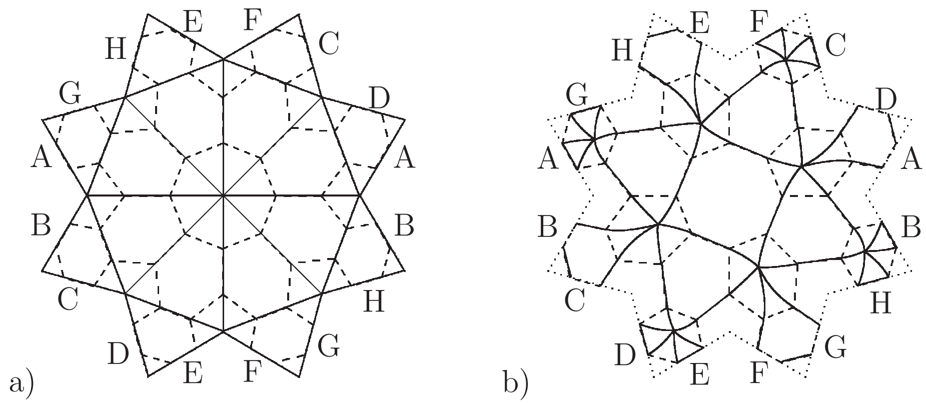


Figure 8: A map obtained by truncation of a regular (a) or of a 2-orbit map (b).

Proof. It is not hard to see that $[\mathcal{C}_3 : K] = 6$, and that $\mathcal{C}_3 = 3^0_3 \cup 3^3_0 s_1 \cup 3^3_0 s_{121}$. Let $H = f_3^{-1}(\varphi(N))$, $H' = f_3(H) = \varphi(N) \leqslant 3^0_3$, $A = H' \cap K$ and $B = H' \cap K s_2$. Note that $3^0 s_1 \cup 3^0_3 s_{121} = K s_1 \cup K s_{21} \cup K s_{121} \cup K s_{12}$ and conjugations by elements of $\mathcal{C}_3$ induce permutations of cosets $\mathcal{C}_3/K$. In particular, no element of any coset $K s_1$, $K s_{21}$, $K s_{121}$ or $K s_{12}$ fixes the coset $K s_2$ (by conjugation). Therefore, if $B \neq \emptyset$ no element in $3^0 s_1 \cup 3^0_3 s_{121}$ normalizes H'. As $K = \varphi(2_{01})$ and $2_{01} \lhd \mathcal{C}$, it follows that if N is not completely contained in 2_{01} then $\mathrm{Tr}(M)$ is a $3k$-orbit map. □

Proposition 4.5 implies that the truncation of a regular map can be either a 1-orbit or a 3-orbit map. The following proposition provides necessary and sufficient conditions to distinguish between the two cases.

Proposition 4.7. *Let M be a regular map and $G = \langle \alpha_0, \alpha_1, \alpha_2\alpha_1\alpha_2 \rangle$ a subgroup of* $\mathrm{Aut}(M)$. *The truncation of M is regular if and only if* $[\mathrm{Aut}(M) : G] = 2$ *and there exists an automorphism $\tau \in$* $\mathrm{Aut}(G)$ *interchanging α_0 and α_1 and fixing $\alpha_2\alpha_1\alpha_2$.*

Proof. Let $M = (\mathcal{C}/N, \mathcal{C})$. Then $\mathrm{Tr}(M) = (\mathcal{C}/f_3^{-1}(\varphi(N)), \mathcal{C})$ and $\mathrm{Tr}(M)$ is regular if and only if $N' = \varphi(N) \lhd \mathcal{C}_3$.

Assume $N' \lhd \mathcal{C}_3$. Note that $[\mathcal{C} : f^{-1}(K)] = [\mathcal{C}_3 : K] = 6$, $[3^0_3 : K] = 2$. and $N' \leqslant K$. Conjugation of K by s_1 induces an automorphism γ of K interchanging s_0 and s_{101} and fixing s_{21012}. Let $H = \varphi^{-1}(K)$. Then $\varphi|_H : H \to K$ is a group isomorphism. Note that φ^{-1} maps s_0, s_{101}, s_{21012} respectively to r_0, r_1 and r_{212}, the generators of $H \leqslant \mathcal{C} = \langle r_0, r_1, r_2 \rangle$. The automorphism $\gamma \in \mathrm{Aut}(K)$ induces the automorphism $\delta \in \mathrm{Aut}(H)$ which interchanges r_0 and r_1 while fixes r_{212}. Also, δ fixes N and therefore induces the required automorphism τ of $H/N \cong G \leqslant \mathrm{Aut}(M)$. Note that $[\mathcal{C} : H] = [3^0_3 : K] = 2$.

Assume now the existence of τ and that $[\mathrm{Aut}(M) : G] = 2$. Since $G \lhd \mathrm{Aut}(M)$, the conjugation of G with α_2 induces the automorphism $\mu \in \mathrm{Aut}(G)$ that fixes α_0 and interchanges α_1 and $\alpha_2\alpha_1\alpha_2$. Note that $U = \langle \tau, \mu \rangle \leqslant \mathrm{Aut}(G)$ is a group of order 6 isomorphic to the symmetric group S_3. It is easy to see that $G \rtimes U$ is generated by

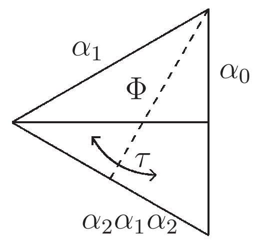


Figure 9: Geometric interpretation of the condition in Proposition 4.7 for the truncation of a regular map to be regular.

elements $S_0 = (\alpha_0, id)$, $S_1 = (id, \tau)$ and $S_2 = (id, \mu)$. Note that $S_1S_0S_1 = (\alpha_1, id)$ and $S_2S_1S_0S_1S_2 = (\alpha_{212}, id)$. The mapping $q : \mathcal{C}_3 \mapsto G \rtimes U$, defined on generators by $s_i \mapsto S_i$, $i = 0, 1, 2$, extends to an epimorphism since the generators S_i satisfy all the relations imposed by $\mathcal{C}_3$. This implies that $q(K) = (G, id) \lhd G \rtimes \langle \mu, \tau \rangle$. Since q is an epimorphism and $[G \rtimes \langle \mu, \tau \rangle : (G, id)] = [\mathcal{C}_3 : K] = 6$, it follows that $q^{-1}(G) = K$ and $\ker q \leqslant K$. In addition, $G \rtimes \langle \mu \rangle = \langle (\alpha_0, id), (\alpha_1, id), (id, \mu) \rangle \cong \mathrm{Aut}(M)$, where the isomorphism $\psi : G \rtimes \langle \mu \rangle \to \mathrm{Aut}(M)$ is defined by $(\alpha_0, id) \mapsto \alpha_0$, $(\alpha_1, id) \mapsto \alpha_1$ and $(id, \mu) \mapsto \alpha_2$. Therefore $(\psi \circ q)|_K$ is a surjection that takes the generators s_0, s_{101}, s_{21012} of K to α_0, α_1 and α_{212}, respectively. Note that $\ker((\psi \circ q)|_K) = \ker q$. Since M is regular, $\theta : \mathcal{C} \to \mathrm{Aut}(M)$, defined by $\theta : w \mapsto \alpha_w$ induces the isomorphism $\mathcal{C}/\ker\theta = \mathcal{C}/N \cong \mathrm{Aut}(M)$, as $\ker\theta = N$. Therefore, $(\theta \circ \varphi^{-1})|_K : K \to G$ is an epimorphism with kernel $\varphi(N)$, which maps the generators s_0, s_{101}, s_{21012} of K exactly the same way as $(\psi \circ q)|_K$. Hence $(\theta \circ \varphi^{-1})|_K = (\psi \circ q)|_K$ implying that $N' = \varphi(N) = \ker q \lhd \mathcal{C}_3$, $f_3^{-1}(\varphi(N)) \lhd \mathcal{C}$ implying that $\mathrm{Tr}(\mathcal{C}/N, \mathcal{C})$ is regular. □

Geometrically, the truncation $\mathrm{Tr}(M)$ of a regular map M is regular, if and only if for any flag Φ of M there exists a reflection which interchanges flags 1 and 2 obtained from Φ as in Figure 6. Such a reflection interchanges α_0 and α_1 while fixes $\alpha_2\alpha_1\alpha_2$ as Figure 9 shows.

Observe that the obvious necessary condition for the truncation of a regular map to be regular is that the map has Schläfli type $\{p, 2p\}$.

We conclude the section with several results on truncation of 2-orbit and 3-orbit maps. The proofs are mostly straight forward using local arrangements of flags. For example, to prove Proposition 4.9 we use the following argument.

Since maps in class 3^2 cannot have vertices of degree 3, they cannot be the truncation of any map. If M is a map in class 3^{02} such that is the truncation of a map M', then there is a set V of faces of M that correspond to vertices of M'. In other words, for any flag Ψ in a face in V we can assemble the flags Ψ, Ψ^2 and $\Psi^{2,1}$ into a new flag Φ_Ψ. In Figure 6, the flags Ψ, Ψ^2 and $\Psi^{2,1}$ correspond to the flags 3, 2 and 1 respectively. The number of orbits of the map M' is determined by the different types of flags obtained in the process described above. Since M is face transitive, and every face contains flags on the three orbits, we get three different flags as Figure 10 show (see also Figure 12). Hence M' is a 3-orbit map.

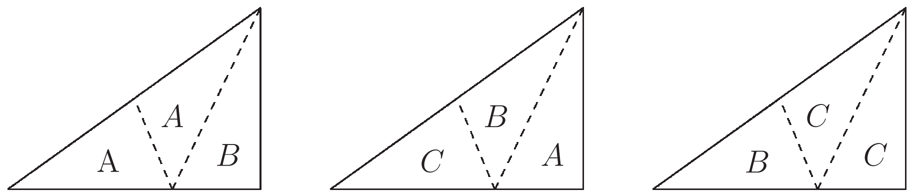


Figure 10: A map in class 3^{02} is a truncation of another map

Note that for a map M in class 3^0 we can define the set V containing one of two different types of faces, however only one choice of them will produce a 2-orbit map M' by assembling triplets of flags in M. The assembled flags are the ones shown in Figure 11

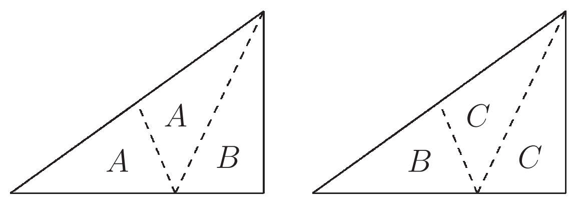


Figure 11: A map in class 3^0 is a truncation of another map

The local configurations shown in Figure 12 imply the adjacencies between these two flags. We conclude now that the map M' is in class 2_{01}.

The proofs of Proposition 4.10 is very similar to that of Proposition 4.9, whereas the proof of Proposition 4.8 is more tedious since we have to involve the seven classes of 2-orbit maps.

Proposition 4.8. *If the truncation* $\mathrm{Tr}(M)$ *of a* 2*-orbit map* M *is again a* 2*-orbit map, then one of the following must happen:*

a) M *and* $\mathrm{Tr}(M)$ *are in class* 2*, or*

b) M *is in class* 2_{01} *and* $\mathrm{Tr}(M)$ *in class* 2_0*, or*

c) M *is in class* 2_2 *and* $\mathrm{Tr}(M)$ *is in class* 2_{12}*.*

Proposition 4.9. *If the truncation* $\mathrm{Tr}(M)$ *of a* 2*-orbit map is a* 3*-orbit map, then* M *is in class* 2_{01} *and* $\mathrm{Tr}(M)$ *in class* 3^0*.*

Proposition 4.10. *If the truncation* $\mathrm{Tr}(M)$ *of a* 3*-orbit map is again a* 3*-orbit map, then* M *and* $\mathrm{Tr}(M)$ *are in class* 3^{02}*.*

The following result follows directly from the geometric definitions of truncation and 3^0-compatibility.

Proposition 4.11. *The truncation* $\mathrm{Tr}(M)$ *of any map* M *is* 3^0*-compatible.*

5 Three-orbit maps on surfaces of small genus

In this section we analyze 3-orbit maps on orientable surfaces of genera 0 and 1 as well as those on non-orientable surfaces of genera 1 and 2. We also provide "Hurwitz-like" upper bounds for orders of automorphism groups of 3-orbit maps on compact closed surfaces of other genera.

Algebraic considerations discussed in Section 3 imply that there are only three classes of 3-orbit maps, namely 3^0, 3^2 and 3^{02} (see Table 1).

Alternatively, for a geometric argument in enumerating classes of k-orbit maps we use the following observation. If B and C are two flag-orbits, $\Phi \in B$ a flag such that $\Phi^i \in C$, then for any $\Psi \in B$, $\Psi^i \in C$. This implies that the four flags containing a single edge belong to either one, two or four orbits.

Consequently, the local configurations of orbits around the base flag must be one of the three configurations shown in Figure 12.

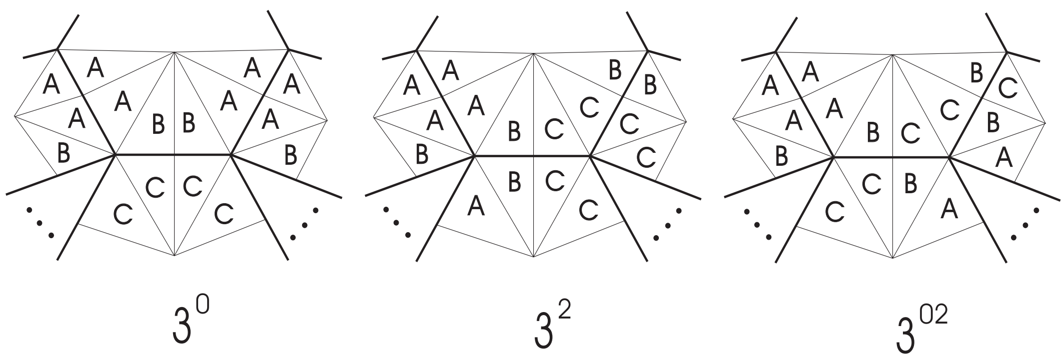


Figure 12: Local flag distributions

All three classes have two edge-orbits. One third of edges of the map are in the edge-orbit which contains edges with all flags from the same flag-orbit. The remaining two thirds of the edges belong to the edge-orbit with flags from the other two flag-orbits.

The maps in class 3^0 are vertex-transitive with two face-orbits. One face-orbit contains faces of even co-degree consisting of flags from two different flag-orbits. The remaining faces contain flags from the third flag-orbit. The bigger edge-orbit contains the edges between faces of different types while the smaller edge-orbit consists of the edges between two faces, each containing flags on two different orbits. The class 3^2 contains exactly all duals of the maps in class 3^0. The maps in class 3^{02} are vertex- and face-transitive.

The degrees of the vertices of all maps in classes 3^0 and 3^{02}, and the co-degrees of the faces of all maps in classes 3^2 and 3^{02} are divisible by 3.

The following proposition establishes a connection between 3-orbit maps and the truncation operation (compare with Theorem 4.3).

Proposition 5.1. *A map with vertices of degree 3 is* 3^0*-admissible if and only if it is the truncation of a regular map.*

Proof. Recall that a map $M = (\mathcal{C}/N, \mathcal{C})$ is 3^0-admissible whenever $\mathrm{Norm}_{\mathcal{C}}(N) \geqslant 3^0 \geqslant N$. This implies that if M has vertices of degree 3 then it is the truncation of the map $(\mathcal{C}/\varphi^{-1}(f_3(N)), \mathcal{C})$. Since

$$\mathrm{Norm}_{\mathcal{C}}(\varphi^{-1}(f_3(N))) = \varphi^{-1}(\mathrm{Norm}_{3^0_3}(f_3(N))) = \varphi^{-1}(f_3(\mathrm{Norm}_{3^0}(N))) = \mathcal{C},$$

the map $(\mathcal{C}/\varphi^{-1}(f_3(N)), \mathcal{C})$ is regular. It follows directly from the definitions that the truncation of a regular map is 3^0-admissible. ☐

The statement of Propostion 5.1 has also a nice geometrical interpretation. For any flag Φ in a face which consists of flags in the same orbit, consider the triple of flags Φ, Φ^2 and Φ^{21} (orbits C, B and A, respectively, in Figure 12). Since the three flags in each triple are in distinct orbits, the automorphism group of the map acts regularly on the triples. Each such triple can be assembled into a single flag like in Figure 6, in which Φ, Φ^2 and Φ^{21} correspond to 3,2,1, respectively. Clearly the map with so assembled flags is regular.

Consider a vertex transitive map on a surface S with vertices of degree d. From Euler formula it is easy to see that $d < 6$ when S is either the sphere or the projective plane, and $d \leqslant 6$ when S is either the torus or the Klein bottle.

Therefore, any map on the sphere or on the projective plane in classes 3^0 or 3^{02} has vertices of degree 3. Since a dual of a map in class 3^{02} is also in class 3^{02}, it follows that each such map has Schläfli type $\{3, 3\}$. But the only map of Schläfli type $\{3, 3\}$ is the regular tetrahedron. Proposition 5.1 now implies the following theorem.

Theorem 5.2. *Any 3-orbit map on the sphere (projective plane) is either the truncation of a regular map on the sphere (projective plane), or its dual.*

For example, the truncation of the regular polyhedron of type $\{2, q\}$ ($q \geq 3$) is a prism with two q-gonal faces. It is a 3-orbit map if and only if $q \neq 4$. The dual of this prism is the bipyramid over a q-gon.

In order to classify the 3-orbit maps on the torus we require the following definition. The vertices of the Euclidean tessellation $\{3, 6\}$ are represented by the integer lattice generated by the vectors $\overline{u} = (1, 0)$ and $\overline{v} = (1/2, \sqrt{3}/2)$. Each integer linear combination $a\overline{u} + b\overline{v}$ defines the translation $t_{(a,b)}$, which is an automorphism. The toroidal map $\{3, 6\}_{\{(a,b),(c,d)\}}$, $ad - bc \neq 0$, is the quotient of the Euclidean tessellation $\{3, 6\}$ by the subgroup $\langle t_{(a,b)}, t_{(c,d)}\rangle$. Similarly, we define the toroidal map $\{4,4\}_{\{(a,b),(c,d)\}}$ with vertices of the Euclidean tesselation $\{4,4\}$ in points generated by the integer lattice with the basis $\overline{u} = (1,0)$ and $\overline{v} = (0,1)$.

Let u be a unit vector, $\alpha, \beta > 0$ real numbers and $k \in \{0, \frac{1}{2}\}$. We denote by $\mathcal{L}(u,\alpha,\beta,k)$ the integer lattice $\mathcal{L}(u,\alpha,\beta,k) = \{a\alpha u + b(\beta v + k\alpha u) \mid a, b \in \mathbb{Z}\}$. In what follows we shall make use of the following lemma.

Lemma 5.3. *Let Λ be the integer lattice generated by two independent vectors in the plane. Denote by ρ the reflection in the line ℓ going through the origin $\mathcal{O} = (0,0)$ in direction of the unit vector s.*

1) Λ is invariant under ρ if and only if $\Lambda = \mathcal{L}(s,\alpha,\beta,k)$ for some choice of α, β and k. In particular α is the distance from $\mathcal{O}$ to a closest point in $(\Lambda \cap \ell) \setminus \{\mathcal{O}\}$, while β is the distance from ℓ to a closest point in $\Lambda \setminus \ell$.

2) Λ is invariant under ρ if and only if Λ is invariant under the reflection in the line through $\mathcal{O}$ perpendicular to ℓ.

The proof of part (1) is given in [14] and part (2) is a direct consequence of part (1).

Note that, for maps of type $\{3,6\}$, if we assume that the line ℓ is parallel to the x axis then the lattices $\Lambda = \mathcal{L}(s,\alpha,\beta,0)$ and $\Lambda = \mathcal{L}(s,\alpha,\beta,\frac{1}{2})$ can be reinterpreted as the $\mathbb{Z}$-span of αe_1 and $\sqrt{3}\beta e_2$, and αe_1 and $\frac{\sqrt{3}}{2}\beta e_2 + \frac{1}{2}\alpha e_1$ respectively, with $\{e_1, e_2\}$ the standard basis of $\mathbb{R}^2$.

Consider a map M in class 3^0 on the torus. If the degree of a vertex is not 3, it must be 6. Euler formula implies that M is a triangulation and therefore M is of Schläfli type $\{3,6\}$. Since all maps of Schläfli type $\{3,6\}$ are quotients of the Euclidean tessellation of the plane by a rank 2 abelian subgroup of translations, the automorphism group $\mathrm{Aut}(M)$ contains half-turns about the midpoint of every edge [20]. From Figure 12 we conclude that 3-orbit maps of Schläfli type $\{3,6\}$ on the torus can only be in class 3^{02}. Therefore, all the toroidal maps in class 3^0 are truncations of regular maps on the torus of Schläfli type $\{4,4\}$ or $\{6,3\}$, and the ones in class 3^2 are the duals of the maps in class 3^0.

Equivelar maps on the torus must have Schläfli type $\{4,4\}$, $\{3,6\}$ or $\{6,3\}$. Clearly, maps in class 3^{02} cannot have type $\{4,4\}$. Since the maps of type $\{6,3\}$ are duals of maps of type $\{3,6\}$, in classifying 3-orbit maps in class 3^{02} on the torus we need only to consider one of them, for example type $\{3,6\}$.

Theorem 5.4. *Any 3-orbit map on the torus is one of the following:*

1. *The truncation of a regular map on the torus of type $\{4,4\}$ or $\{6,3\}$ or its dual.*
2. *The toroidal map $\{3,6\}_{\{(a,0),(-b,2b)\}}$ for $a, b > 0$ or its dual.*
3. *The toroidal map $\{3,6\}_{\{(a,0),(b,a-2b)\}}$, where $a > 2b$, $a > 0$, $b \neq 0$ and $a \neq 3b$, or its dual.*

Proof. By the above discussion it only remains to show that the 3-orbit maps in class 3^{02} of type $\{3,6\}$ are precisely those described in (2) and (3).

Let $M = \{3,6\}_{\{(a,b),(c,d)\}}$ be a map in class 3^{02}. According to [20], Proposition 5.2, or by [18], an automorphism α_w of the plane tessellation $\{3,6\}$, $w \in \mathcal{C}$, induces an automorphism of M if and only if α_w normalizes the group of translations $H = \langle t_{(a,b)}, t_{(c,d)} \rangle$. From Table 1 we have that M is in class 3^{02} if and only if α_{s_0} and α_{s_2} induce automorphisms of M while $\alpha_{s_1 s_2}$ does not. In general (see [14] for details) a conjugation by α_w, $w \in \mathcal{C}$, induces a linear transformation $\mathcal{A}_w$ of the integer lattice and α_w normalizes H if and only if $\mathcal{A}_w$ preserves the lattice. Assuming that the edge of the base flag of the map is parallel to the x-axis and the vertex of the base flag is in origin, $\mathcal{A}_{s_2}$ and $\mathcal{A}_{s_0}$ correspond to the reflections over the x-axis and y-axis, respectively, while $\mathcal{A}_{s_1 s_2}$ corresponds to the rotation by 60^o angle around the origin.

Lemma 5.3 now implies that the integer lattice Λ induced by H is either $\mathcal{L}(e_1, a, \sqrt{3}b, 0)$ or $\mathcal{L}(e_1, a, \frac{\sqrt{3}}{2}b, \frac{1}{2})$ for integers $a, b > 0$. We choose the base $\{(a,0),(0,b\sqrt{3})\}$ for $\mathcal{L}(e_1, a, \sqrt{3}b, 0)$ and the base $\{(a,0),(a/2, b\sqrt{3}/2)\}$ for $\mathcal{L}(e_1, a, \frac{\sqrt{3}}{2}b, \frac{1}{2})$, and express these bases in terms of a linear combination of $\overline{u}$ and $\overline{v}$. The maps obtained are $\{3,6\}_{\{(a,0),(-b,2b)\}}$ and $\{3,6\}_{\{(a,0),((a-b)/2,b)\}}$ respectively. Making $c = (a-b)/2$ the latter is transformed into $\{3,6\}_{\{(a,0),(c,a-2c)\}}$ with $b > 0$ transforming into $a > 2c$. Lemma 5.3 also shows that the lattice Λ is $\mathcal{A}_{s_0}$ invariant.

It remains to determine which of the above lattices are not $\mathcal{A}_{s_1 s_2}$ invariant. It is easy to see that the rotation $\mathcal{A}_{s_1 s_2}$ never fixes the lattice $\mathcal{L}(e_1, \alpha, \sqrt{3}\beta, 0)$ and fixes the lattice $\mathcal{L}(e_1, \alpha, \beta, \frac{1}{2})$ only for $\beta \in \{\alpha, \alpha/3\}$. The case $\beta = \alpha$ corresponds to the regular map $\{3,6\}_{(a,0)}$ and occurs when $c = 0$, whereas the case $\beta = \alpha/3$ corresponds to the regular map $\{3,6\}_{(a/3,a/3)}$ and occurs when $a = 3c$ (see [6] for the notation).

□

Note that all the maps described in Theorem 5.4 are related to regular maps by the operation truncation since the petrial of the dual of the maps of type $\{3,6\}$ in class 3^{02} are trivalent maps in class 3^0 (see Table 1). The map of type $\{3,6\}_{\{(a,0),(-b,2b)\}}$ is the dual of the petrial of the regular map of type $\{2m, 2a\}$ with group $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ factored by the extra relations $(\alpha_0(\alpha_2\alpha_1)^2)^2 = id$ and $(\alpha_2\alpha_1)^{2b}(\alpha_1\alpha_0)^{2b} = id$, where $m = [a,b]$ is the least common multiple of a and b. The map of type $\{3,6\}_{\{(a,0),(b,a-2b)\}}$ is the dual of the petrial of the regular map of type $\{n(a-2b), 2a\}$ factored by the relations $(\alpha_0(\alpha_2\alpha_1)^2)^2 = id$ and $(\alpha_2\alpha_1)^{2b}(\alpha_1\alpha_0)^{a-2b} = id$, where n is $[a,b]/b$.

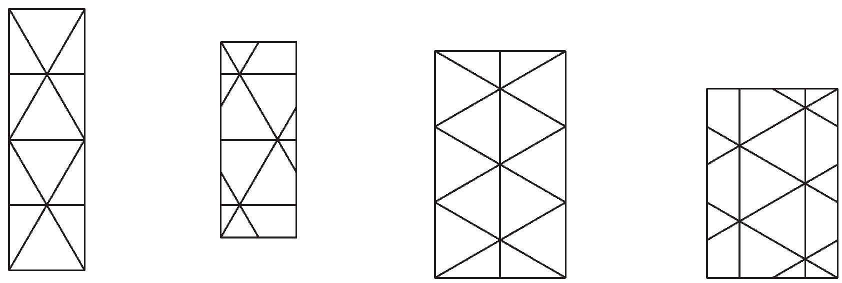
Proposition 5.1 implies that any map in class 3^0 with vertices of degree 3 is the truncation of a regular map. Since there are no regular maps on the Klein bottle (see [5]), it follows that any 3-orbit map in class 3^0 on the Klein bottle has vertices of degree 6. Using Euler's formula we conclude that any such map is a triangulation of the Klein bottle, and hence, equivelar.

The following theorem follows from the complete description of all equivelar maps on the Klein bottle given in [24].

Theorem 5.5. *The only 3-orbit maps on the Klein bottle are the maps* $\{3,6\}_{|m,1|}$ *and* $\{3,6\}_{\backslash m,1\backslash}$, $m \geq 1$, *and their duals, all of them in class* 3^{02}.

Corollary 5.6. *The only 3-orbit polyhedra in the Klein bottle are the maps* $\{3,6\}_{\backslash m,1\backslash}$, *and the maps* $\{6,3\}_{\backslash m,1\backslash}$, *for* $m \geq 3$ *(see Figure 13).*

Proof. The maps $\{3,6\}_{|m,1|}$, $\{6,3\}_{|m,1|}$, $\{3,6\}_{\backslash m,1\backslash}$ and $\{6,3\}_{\backslash m,1\backslash}$, for $m = 1, 2$ fail to satisfy the diamond condition. □



$\{3,6\}_{|4,1|}$ $\{3,6\}_{|3,1|}$ $\{3,6\}_{\backslash 6,1\backslash}$ $\{3,6\}_{\backslash 5,1\backslash}$

Figure 13: 3-orbit maps on the Klein bottle

Now we give Hurwitz-type bounds for 3-orbit maps.

Proposition 5.7. *Let* M *be a 3-orbit map with minimal vertex degree and minimal face co-degree at least 3 on a compact closed surface* S *of genus* $g \geq k$, *where* $k = 2$ *if* S *is orientable, and* $k = 3$ *if* S *is non-orientable. If* M *is not the truncation of a regular map or the dual of the truncation of a regular map, then*

$$|\mathrm{Aut}(M)| \leq \begin{cases} 12(2g-2) & \text{if } S \text{ is orientable,} \\ 12(g-2) & \text{if } S \text{ is not orientable.} \end{cases}$$

Proof. Let v, e and f be the number of vertices, edges and faces of M, respectively. Then by the Euler characteristic of S, $\chi = v - e + f = 2 - mg$, where $m = 2$ if S is orientable and $m = 1$ if S is non-orientable. Since the number of flags of M is $4e$, we have that $|\mathrm{Aut}(M)| = 4e/3$.

For M in class 3^0, we let $3s$ be the degree of a vertex, $2p$ the co-degree of the faces in one face-orbit with f_1 faces, and q the co-degree of the faces in the other face-orbit with f_2 faces. Then $v = 2e/3s$ and $f = f_1 + f_2 = 2e/3p + 2e/3q$. It can be easily verified that the following equality holds.

$$|\mathrm{Aut}(M)| = \frac{4(2-mg)}{2(1/s + 1/p + 1/q) - 3}.$$

For genus $g \geq k$ the numerator is negative, so the denominator has to be negative as well. Since the numerator does not depend on s, p or q, the upper bound for $|\mathrm{Aut}(M)|$ is achieved when $x = 2(1/s + 1/p + 1/q) - 3$ takes its largest value subject to $x < 0$. Since $s = 1$ indicates that M is a truncation of a regular map we restrict ourselves to the case $s \geq 2$. Taking $s = p = 2$ and $q = 3$ we have that $x = -1/3$ and the proposition follows.

If M is in class 3^2 then its dual is of type 3^0 and is a map on the same surface with the same automorphism group, hence the proposition also holds.

If M is in class 3^{02}, let $3p$ be the degree of a vertex and $3q$ be the co-degree of a face. Then $v = 2e/3p$, $f = 2e/3q$ and

$$|\mathrm{Aut}(M)| = \frac{4(2 - mg)}{2(1/p + 1/q) - 3} \ .$$

If $p, q \geq 2$ then $x = 2(1/p + 1/q) - 3 \leq -1$, and if $p = 1$ (say) then the maximum value of x less than 0 is $-1/3$ when $q = 3$. Hence the proposition holds. □

6 Two-orbit maps on surfaces of small genus

By a theorem of McMullen [16] any equivelar map on the sphere is regular. Hence, to find all 2-orbit maps on a sphere it suffices to consider only the class 2_{01}, since the only remaining class which contains non-equivelar polytopes is 2_{12}, but this class contains exactly all the dual maps of the maps on class 2_{01}.

The maps on class 2_{01} are vertex transitive with vertices of even degree, implying that every vertex has degree 4, since degree 2 forces equivelar maps to be regular. Using Theorem 4.3 and the fact that the canonical double cover on the sphere of any equivelar map in the projective plane is again equivelar we conclude the following theorem.

Theorem 6.1. *Every 2-orbit map on the sphere (projective plane) is either the medial of a regular map on the sphere (projective plane) or the dual of the medial of a regular map on the sphere (projective plane).*

As a consequence of Theorem 6.1 we have that the only convex 2-orbit polyhedra are the cuboctahedron, the icosidodecahedron and their duals, the rhombic dodecahedron and the rhombic triacontahedron.

To classify 2-orbit maps on the torus we shall make use of the following lemma whose proof follows directly from Lemma 5.3 (2).

Lemma 6.2. *Let M be an equivelar map on the torus. Then $\rho_0 \in \mathrm{Aut}(M)$ if and only if $\rho_2 \in \mathrm{Aut}(M)$.*

Theorem 6.3. *Any 2-orbit map on the torus is either the medial of a regular map of type $\{3, 6\}$, its dual, or belongs to one of the following families:*

1. $\{4,4\}_{(a,b)}$ *in class* 2*,*

2. $\{4,4\}_{\{(a,a),(b,-b)\}}$ *in class* 2_1*,*

3. $\{4,4\}_{\{(a,a),(b,a-b)\}}$ *in class* 2_1*,*

4. $\{4,4\}_{\{(a,0),(0,b)\}}$ *in class* 2_{02}*,*

5. $\{4,4\}_{\{(2a,0),(a,b)\}}$ *in class* 2_{02}*,*

where in every case $a, b > 0$ *and* $a \neq b$*.*

Proof. Lemma 6.2 implies that there are no maps on the torus in classes 2_0 and 2_2.

Any map M on class 2_{01} on torus must have vertices of even degree less or equal than 6. Theorems 4.3 and 4.2 imply that if the degree of every vertex of M is 4 then M is the medial of a regular map M must be of type $\{3,6\}$. But Lemma 6.2 implies that there are no such maps in class 2_{01}. Any map in class 2_{12} can be obtained as the dual of a map in class 2_{01}, implying that any map on the torus in class 2_{12} must be the dual of the medial of a regular map of type $\{3,6\}$.

Chiral maps (class 2) on torus have been classified by Coxeter.

In the remaining classes 2_1 and 2_{12}, every map has type $\{2p, 2q\}$. Hence to be a map on the torus they must be of type $\{4,4\}$. An argument similar to the one used for type $\{3,6\}$ in the proof Theorem 5.4 can now be used to complete the proof. □

From the above theorem one can derive an alternative proof for the classification of edge transitive 2-orbit maps on the torus to the one given by Širáň, Tucker and Watkins (see [20]).

Since there are no regular maps on the Klein bottle, Theorem 4.3 implies that there cannot be any map in class 2_{01} with vertices of degree 4 on this surface. Hence, any map on the Klein bottle in class 2_{01} must have vertices of degree 6, and by the Euler's formula, must have Schläfli type $\{3,6\}$. This implies that every 2-orbit map on the Klein bottle is equivelar of type $\{4,4\}$, $\{3,6\}$ or $\{6,3\}$.

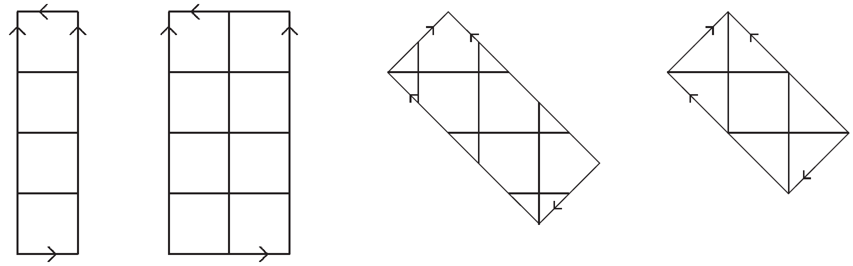
From the description of the equivelar maps on the Klein bottle in [24] we conclude the following theorem.

Theorem 6.4. *The only* 2*-orbit maps on the Klein bottle are the maps* $\{4,4\}_{|m,1|}$ *and* $\{4,4\}_{|m,2|}$ *for* $m \geq 1$ *in class* 2_{02}*, and* $\{4,4\}_{\backslash m,1\backslash}$ *for* $m \geq 1$ *in class* 2_1 *(see Figure 14).*

Corollary 6.5. *The only* 3*-orbit polyhedra on the Klein bottle are the maps* $\{4,4\}_{|m,2|}$ *for* $m \geq 2$*.*

Proof. The maps $\{4,4\}_{|m,1|}$ and $\{4,4\}_{|1,2|}$ do not satisfy the diamond condition, while the maps $\{4,4\}_{\backslash m,1\backslash}$ fail to satisfy strong flag connectivity. □

We finish giving Hurwitz-like bounds for 2-orbit maps. The proof of the next proposition is similar to that of Proposition 5.7.



$\{4,4\}_{|4,1|}$ $\{4,4\}_{|4,2|}$ $\{4,4\}_{\backslash 5,1\backslash}$ $\{4,4\}_{\backslash 4,1\backslash}$

Figure 14: 2-orbit maps on the Klein bottle

Proposition 6.6. *Let M be a 2-orbit map with minimal vertex degree and minimal face co-degree at least 3 on a compact closed surface S of genus $g \geq k$, where $k = 2$ if S is orientable and $k = 3$ if S is non-orientable. If M is not the medial of a regular map, then $|\mathrm{Aut}(M)| \leq n$ with the values of n given in table 3.*

Class of M	S orientable	S non-orientable
2	$84(g-1)$	—
2_0, 2_2	$48(g-1)$	$24(g-2)$
2_1, 2_{02}	$24(g-1)$	$12(g-2)$
2_{01}, 2_{12}	$168(g-1)$	$84(g-2)$

Table 3: Hurwitz bounds for 2-orbit maps

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