

Dynamic Geometry Programs and the Practice of Geometry

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Dynamic Geometry Programs such as Cabri¹ and Geometer's Sketchpad² have appeared over the last decade as a critical new tool in geometry. Originally designed for teaching geometry in secondary schools, these programs include the classical 'ruler and compass' constructions as well as isometries. These constructions have an accuracy far surpassing any person with a physical compass. Their characteristic dynamic behaviour comes from dragging or animating the initial choices while remaining objects are constrained by their constructed properties. One construction generates a range of examples and possibilities, including loci. This changes the entire quality and impact of the experience.

Within teaching, the use has expanded: down to K-8 teaching and up to universities, primarily for pre-service teachers. These programs also generate Java Applets on a web page for the internet. The features are expanding to encourage visual investigation of other topics, such as algebra, calculus, modeling and science. There are a number of articles³ and resources^{4,5} that describe effective uses for these tools.

These programs were quickly taken up by researchers in discrete geometry as valuable tools of our own research and our communication. To come full circle, the dynamic geometry program Cinderella⁶ was originally designed for researchers, with Euclidean, spherical and hyperbolic constructions, and was immediately used for teaching geometry and for geometry on the internet. This overlap of learning tools and geometry research tools is very strong and should be made visible in the classroom. The student is not going through a phase with an educational toy but is learning a key modern tool of the trade for geometers.

How geometers use the programs can suggest way in which students as 'cognitive apprentice' can practice geometry. If such an apprenticeship is the goal, then we face issues that are hidden if the discussion is based on current curriculum and preconceptions. The preparation of teachers in the use of living geometry tools is key, if they begin as outsiders to the practices of modern geometry.

I am a discrete geometer⁷. For almost thirty years, I have worked on fundamental and applied areas involving constraints in geometry: Which bar and joint frameworks are rigid in 3-space? When is a plane drawing the correct picture of a 3-D polyhedron? What portions of a protein are flexible? Many current research problems examine how geometry can be done on computers and what algorithms will be efficient. Some of the new geometry problems are at the heart of what future versions of dynamic geometry programs will be. How to explore configurations, respecting specific sets of constraints, is an issue everywhere from parametric CAD through robotics and medical

imaging to modeling large molecules. Some of these problems are extensions of classical geometry problems: Which measurements confirm that two configurations of four points (or n points) are congruent, as SSS, ASA and SAS confirm the congruence of configurations of three points? In the past forty years, geometry has come alive in ways that are very different from the content of our current curriculum⁸.

I am also a geometry teacher - teaching geometry courses to pre-service and in-service teachers of mathematics. Over the past 6 years, I have increasingly used Geometer's Sketchpad, combined with physical objects for the plane, the sphere and 3-D geometry. I use Geometer's Sketchpad and Cinderella in my own research, as well as in communicating my explorations and results to undergraduate and graduate students who are apprentice geometers. Ontario has purchased Geometer's Sketchpad for all schools and is requiring its use in all Grade 9 classes. Teachers and students have a strong incentive to learn the program and we regularly reflect about how to use dynamic geometry programs effectively - for learning and for teaching.

Reflecting on these experiences, I have a couple of central points:

- (i) Learning to use these programs like a geometer includes learning to 'see' differently and therefore think differently.
- (ii) As experts shift their communication and practices, and students experiment into similar practices, teachers need to experience these current practices of geometry, including the use of dynamic diagrams so that they can model for and mentor their students.

How do geometers use dynamic geometry? In the words of the cognitive scientist, they are one of the 'things that make us smart'⁹. Our sometimes crude and imprecise mental processes for visualizing configurations, animating and transforming possibilities and investigating consequences are extended into a high precision tool for constructing and exploring geometric diagrams. One can explore the consequences of prescribed constraints or transformations and clarify a number of features in a rich form that is rich in information. The programs expand the role of precise visual and diagrammatic reasoning in all stages of our work: posing questions, making conjectures, creating counterexamples, seeking and recognize connections. The change is dramatic and we leave a session with the program with more refined information, connections and images. These refined images then run in our heads as we work with the problems and the solutions, both mentally and with words and pictures on paper, guided by these dynamic visual processes. These diagrams become part of our internal vocabulary and our ongoing mental processes. We also see unexpected events: an extra coincidence of lines or points in the construction; a transformation; or a mental association with the pattern in some previous geometric study.

With dynamic geometry tools, the previously invisible internal life of geometers becomes visible to the public and to our students. We can now show others what we

already ‘see’ privately. These tools were accepted by geometers so quickly because of their similarity to our existing mental world - a world which few outsiders were aware of. It is important to repeat that, as a practicing geometer, I did use dynamic visuals in my mind’s eye, long before I saw such a program. These programs have refined my use of visuals. My use of visuals is now accessible to non-experts in a way my mental images never were, no matter how hard I waved my hands over still diagrams! As I turn these mental images into a critical part of my communication, I am expanding the internal and external use of images in my own private work. I experience an accelerating shift in my use of visuals in all areas my work¹⁰.

As I teach geometry classes these days, there is a regular routine. I leave an encounter with students with new images in my mind’s eye which I recreate within a dynamic geometry program so that the processes we analyzed in class become more transparent and ‘visible’ to the students. I struggle with features such as colour, buttons, and animations to pull their viewing along the tracks of how I ‘see’ the connections. As with most tools in science and in intellectual life, these dynamic sketches create new answers, new methods or reasoning, and new questions. We change what we ‘see’ and work to help others ‘see’ as much or more.

I have put the word ‘see’ in quotation marks to emphasize that this activity of seeing is not some self-evident, innate process, but something created and learned¹¹. The ‘seeing’ of an expert is rich with conventions. It is filled with practices that start with particular groups of details for attention, followed by the essential perceptual skill of ignoring prior details and focusing on new groups. This focusing, then ignoring and refocusing, is something which I simulate with hide/show buttons in some programs. There is clear expertise in the use of these dynamic diagrams - both mental and computer drawn - expertise that must be learned and which can benefit from guidance and intervention.

Student activities with dynamic geometry can be modeled on the types of explorations and communication developed by practicing geometers. In saying this, I recognize the many stages of growth and learning which must be followed in such a visual and geometric cognitive apprenticeship. It takes more than raw exposure to the programs. It requires mentors and teachers who ‘see’ and question in an expert mode.

Experts have a different way of ‘seeing’ a given problem and a given diagram. In many ways, this difference is analogous to the ways experts and apprentices ‘see’ an algebra equation. An expert reads information in the appearance that predicts possible next steps towards the desired goal, or recalls similar patterns and suggest strategies and changing forms. A big part of this visual process is what cognitive scientists call ‘chunking’: a mental grouping of pieces which ignores many details, for this step, and records a critical feature: e.g. Where is the outermost parenthesis or operation in an algebra equation? Many students do not group the algebraic expression in the same

way, do not focus on the same features and are not prompted to execute the same steps. One learns to 'see'.

The same thing is true of expert use of geometric diagrams: scanning for possible pairs of triangles with three common measures; searching for a symmetry of large pieces of the diagram which will capture the relationships in a single sweep. Within dynamic geometry programs, one can add animation of the initial data as a scanning technique that lets our eyes detect a coincident which remains constant, or note angles or lengths that remain equal, or in constant ratio, while other features are in flux. Added 'measurement' features, combined with calculators, can be used to confirm such equalities or ratios, once we suspect they are present. In the spirit of Felix Klein, geometry is the study of invariant properties and invariance under dragging or animation is a major source of relevant information. Again it takes practice and skill to focus on the critical parts of a larger diagram and to select the significance features while discounting other equally obvious but insignificant features of the same diagram.

Students using dynamic geometry programs exhibit different forms of behaviour reflecting different levels of expertise in their thinking about the sketch. An example would be general dragging which is unfocused and undirected, as opposed to dragging which is directed at probing a specific relationship, such as a hidden isometry between figures. How does a student apprentice in such skills? Guided explorations, pre-made sketches with animations, observing an expert, knowledge of what fixed features can occur in a visual form, are all part of this apprenticeship. Discovery during individual and group explorations is a step. However it is a mistake to assume that, having the sketch open, a learner will see the same things as the teacher or an expert. I now know that my students do not 'see' what I 'see', until there is this exploration, dialog, refocusing of attention, and added background for interpretation of the visual experience. This is a mistake I made for several years and am daily struggling to overcome in my own teaching and mentoring. I must regularly pause and ask the innocent but critical questions: What do they see? How can they also see what I see? This is a core message I extract: the critical role of correct 'seeing' in the practice of geometry and the vital need for novices and teachers to learn to use these visual images as carefully as they learn to use algebra or any other important part of mathematics.

'Observing an expert' as a strategy to learn how a geometer uses dynamic geometry programs raises an important question: How does the student get access to such expertise? Their normal contact is with their mathematics teacher. In a typical North American education, the teacher will never have been in the classroom with a practicing geometer nor with someone who is a practiced user of dynamic geometry programs. More generally, the teacher has never apprenticed in visual literacy or in the current uses of diagrams in the practice of mathematics.

Teachers who are embedded in the classical curriculum and classroom practices may bring very different attitudes to the pedagogy of dynamic geometry and towards

the use of these tools for students. Where I see these tools as essential, others find them distractions from the 'true spirit of geometry'. Where I struggle with central questions about how students 'see the diagrams' and how to integrate dynamic diagrams into their practice, others treat such diagrams as obvious, optional, or both, within some larger agenda of logic. To introduce these programs as valuable tools for the current practice of geometry, geometers will have to play a larger role in these reflections and in the preparation of teachers.

For emphasis, I distinguish two roles for geometry courses: teaching geometric concepts and processes; and teaching abstract reasoning and proofs. Many geometry courses are not centered on teaching geometry but on teaching of 'logic and proof', as a highly structured, non-visual task. Teachers who have emerged from a course in axiomatics, illustrated with examples from the history of geometry, will have little experience in how current geometers use visual forms and dynamic geometry. They may not know the kinds of questions, answers and methods that are the fabric of current geometry. To paraphrase one of my mentors, Gian-Carlo Rota: "Axiomatics is to geometry as medicine is to food". Medicine may be necessary to clear up certain difficulties, but it is not the stuff of our daily life in mathematics. Too much of our geometry curriculum is closer to a roll call at the medicine cabinet than a banquet at a well-set table.

I conclude that mentors of students must themselves experience geometry as a live subject, filled with amazing questions, revealing experiences and a rich visual content which includes dynamic images of the type displayed in dynamic geometry programs. Transmitting such experiences encourages a full range of engagement in geometry by students with many different approaches to learning and many different abilities. Proof will be one portion of the larger range of ways of reasoning, all of which benefit from the use of dynamic geometry programs.

How do 'experts' encounter proof with these programs? One obvious role is to generate counterexamples. The great accuracy means that observing that three lines do not meet in a point is solid evidence of a counterexample to any conjecture that they do meet. This ease of generating counterexamples encourages the wide but disciplined exploration of conjectured relationships. A second critical connection to proof is the comparison of two constructions (sets of constraints) which appear to generate the same visible pattern. One immediately asks why. Why do we get a visual parallelogram by taking a triangle and doing a half-turn about the midpoint of one edge? Do all parallelograms have this half-turn symmetry? These questions about constructions beg for verification and proof in a way that makes sense, even to students.

Suitably manipulated, some plane sketches are seen by students as 3-D diagrams. In a way that never happened with single still diagrams, students embed the plane properties and results, appropriately, into a continuum of 3-D examples. This points out both a new possibility and a critical gap in terms of current software. Although the

primary experience of young children is with 3-D geometry, our teaching of geometry focuses for too long on 2-D geometry. The existing software feeds this trend, because it lives on a 2-D screen and receives input by dragging on a 2-D pad. The full explosion of dynamic geometry awaits suitable 3-D input devices for 3-D constructions and 3-D animations. These 3-D images already flow across my mind, daily. When such new programs arrive, we will have another qualitative change in the experience, communication, and pedagogy of geometry. This too will offer wonderful new opportunities for the teaching and learning of geometry.

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