Visualization in Mathematics: Claims and Questions towards a Research Program

Walter Whiteley, Director of Applied Mathematics, Graduate Programs in Mathematics, in Computer Science and in Education, York University, Toronto Canada. June 2004

Context:

I am a research mathematician, working in discrete applied geometry. My own practice of mathematics is deeply visual: the problems I pose; the methods I use; the ways I find solutions; the way I communicate my results. The visual is central to mathematics as I experience it. It is not central to mathematics as many teachers present it nor as students witness it. This contrast is striking.

I also work with future and in-service teachers of mathematics: elementary, secondary and post-secondary. They are surprised to learn that modern abstract and applied mathematics can be intensely visual, combining a very high level of reasoning with a solid grounding in the senses. They wonder how good visual work in elementary school connects to their own experiences with algebraic and formula centric (but visually meager) presentations of mathematics in courses. They wonder how any of these approaches connect to students' future work that might use mathematics. They suspect that visual and hands-on work is not the 'real math' but is a crutch or bridge to be left behind as one matures. They are surprised at distinct and varied forms of visual reasoning within mathematics. They are surprised that what their students see is not what they see. We cannot just show a visual and say 'behold'. Learning to effectively use visuals takes as much teaching and time as algebraic and symbolic reasoning [7,31]. The challenge is that visual representation and thinking skills can be as important to students' futures as the symbolic and language based reasoning.

From these experiences as a working mathematician, from my work with students of mathematics, from a decade of reading, speaking, and teaching on these topics, and from the challenges which teachers present to me, I offer a series of claims about visualization in mathematics and mathematics education, and some possible research questions for future work on visualization in mathematics and mathematics education.

I am aware, as Sfard [27] suggests, that mathematicians and mathematics educators may not be talking about the same subject. However, I see some strong supportive connections to be made, as we rework and rethink the role of visuals, manipulatives, and related technology in both fields.

Claim 1: Visuals are widely used, in diverse ways, by practicing mathematicians.

Comments: The evidence is widespread, from direct observations, through collected anecdotes, to studies of cognition [2,3,12,16,17,19,29]. There has been less commentary on the diversity of visual forms, the different conventions and even cognitive processes evoked by these different forms. The diversity is wider than the diversity of languages and symbolic systems and this diversity can confound generalizations about 'visuals' as a general class of representations and forms of reasoning.

It is an illusion that mathematical reasoning is done in the brain with language. Standard presentations of mathematics foster this illusion, but this formal public appearance does not represent the problem solving, the thinking, the reasoning of many mathematicians. A ballet performance does not embody the way this performer walks around their home, or the way they practice. Analogously, what you observe in a mathematics paper or lecture does not embody what a mathematician does while solving the problem, or when talking with a colleague. It does not match the cognitive processes of the mathematician, the teacher, or the learner of mathematics.

What students commonly see in a mathematics classroom is also an illusion. Do our future teachers actually know how mathematics is done? Are they comfortable with the real variety that will support the mathematical work of their students? Do students who do mathematics visually, the way many professional mathematicians do it, get the reinforcement to continue in mathematics, or does this performed illusion convince them they do not belong?

Questions for research:

- Which visuals are used, by experts, when solving mathematical problems? How are they used?
- Do teachers know how mathematics is done?
- Do teachers have the ability to do mathematics this way?
- Which visual practives for mathematics can be consistently presented and supported from the earliest grades through post-secondary education?

Claim 2: Visual reasoning in solving problems is central to numerous other fields: engineering, computer science, chemistry, biology, applied statistics.

Comments: When civil engineers, with long experience in private industry were asked what mathematics they used, the answer was: "back of the envelope calculations and geometric reasoning". When I describe a geometric result to computer scientists or biochemists, I use physical models, drawn sequences of visually presented examples for algorithms, and now variables dynamic images. The words, sometimes even the symbols are confusing, but some images are shared and sharable. Scientific visualization, using new technologies, highlight this reliance on visual communication and visual reasoning in a wide spectrum of science and mathematical fields [10,13,25].

Cognitive studies suggests that scientists read graphs differently (in context) than mathematicians [24]. Scientists spend significant time 'agreeing' on what they see, and what they want others to see. A book on diagrams for teaching science notes: "We see things not as they are, but as we are".

Questions:

- Are the visual practices outside of mathematics consistent with the visuals practices of mathematicians?
- Do teachers of science and other disciplines know and value the use of visuals for mathematics?
- How do the visual practices of experts compare with the visual representations and processes supported by our teaching materials and pedagogies?

Claim 3: Children se visual processes for early work in mathematics

Comments: From age 3 days, children process some number comparisons in visual areas. Early experiences include visually (and kinesthetically) based reasoning with concepts that will become mathematized.

Questions:

- How is this connected through the later mathematization and pedagogy on to effective practices in the mature learner?

Claim 4: Visual reasoning is not restricted to geometry or spatially represented mathematics.

Comments: As an example, combinatorics is very rich in visual patterns and associated reasoning [22]. Even the algebra, and symbolic logic, rely on visual form and appearance to evoke appropriate steps and comparisons [2]. All fields of mathematics contain processes and properties that afford visual patterns and visually structured reasoning [3].

I spend a lot of time showing my students what I see, what I focus on and what I ignore, what I image next. Within our group, we develop shared support for reasoning, and for communication. However, even with others in the same areas of mathematics, it is a struggle to evolve common conventions and methods.

Mathematicians have not developed clear, consistent ways of working with visuals, as we have with algebra and other symbolic forms. While the larger community has the discipline to agree on shared definitions and algebraic forms, we continually develop new diagrammatic representations, in undisciplined ways. This mixes sustainable visuals with good cognitive fit with local eccentricities. This gap between individual or local practices and shared conventions is an obstacle to effective sharing and learning.

Questions:

- Are there community practices within mathematics and within mathematics education that are obstacles to effective use of visuals and diagrams?
- Can mathematics educators extract shared visual practices and representations from mathematical practice?

Claim 5: We create what we see. Visual reasoning or 'seeing to think' is learned. It can also be taught and it is important to teach it.

Comments: As cognitive science reports, we learn to see [13,21,23,24,26]. We learn what to notice and what to ignore, and how to interpret ambiguous cues. We work with images in the brain, as wholes and as parts, with symmetry, and with transformations at many levels [15]. In mathematics, what the expert sees and does with an image is not what the novice sees, even with the same diagrams. What the teacher sees is not what the students see. What one student sees is not what their neighbor sees. All of these differences impact our classroom work with diagrams and visuals.

Since we create what we see, we can change what we see. Consciousness of alternative ways to see, and of the value of seeing differently, is one step. It takes ongoing guidance (cognitive apprenticeship), practice and evolving imaging (and imagination) to 'learn to see like a mathematician' [7,31]. A well-known book on learning to draw says: "I will change how you see and the rest will be easy" [9]. Something analogous can be true for the learning of mathematics.

Questions

- What examples, exercises, and activities develop the awareness, and support the changes in what we see and how we think in mathematics and as users of mathematics?
- How does a teacher recognize the mis-seeing and misinterpretations and support change?

Claim 6: Visual and diagrammatic reasoning is cognitively distinct from verbal reasoning.

Comments: Brain imaging, neuroscience, and anecdotal evidence confirm this distinction, in the brain and in functional problem solving. Imaging suggests connections of mathematical reasoning with brain areas for eye-hand coordination, and an association of visual and kinesthetic reasoning. For example, we do proportional reasoning in this area of the brain, appearing to use a logarithmic number line associated with eye-hand coordination [1,5,16,17]. Studies of the brain during problem solving show distinct paths and forms for visual reasoning and verbal reasoning. Symbolic reasoning appears to be a distinct amalgam of these, with parallel paths dependent on parallel representations. To quote C.S. Pierce: "Diagrammatic reasoning is the only really fertile reasoning."

Questions

- Can one be effective in mathematics with only one of these forms of reasoning?
- How do we develop balanced skills in students, and the meta-cognitive skill to select when to use appropriate diagrammatic representations and visual reasoning?
- How do we ensure teachers have these balanced skills, so they can support the variety of students who can be successful within mathematics?

Claim 7: Visual reasoning is connected to kinesthetic and emotional reasoning.

Comments: These ways of grasping concepts are indispensable to 'making sense' in mathematics. We are less aware of these aspects than of the visual forms, and such forms of thinking are even farther from re-presentation in words than visual presentations. However, brain imaging and modeling suggests significant problem solving in mathematics works with areas of the brain associated with eye-hand coordination.

Working with 'abstract mathematics' is still 'sensible' (and not none-sense as some philosophers and many students might suspect). Significant levels of our reasoning are guided by associational reasoning connected to the visual and kinesthetic senses. Such non-verbal selections and connections may be captured in our discussions as 'intuition'. However, under that umbrella for deep, essential processes that are not articulated in words, they are still learned, learnable and teachable.

All forms of effective problem solving and decision making connect to the emotions – and the emotional sense of fit [4]. This cannot be ignored in a well-grounded mathematics education.

Questions for research:

- How strong is the evidence for continued kinesthetic reasoning as mathematical practices mature?
- Does development of visual and kinesthetic representation and reasoning also develop what is otherwise called 'intuition'?
- How is aesthetics connected to kinesthetic and visual reasoning?

Claim 8: Children begin school with relevant visual abilities, including 3-D. In North America, this declines through school.

Comments: Young children learn to see and to plan in a 3-D world. They also learn to problem solving at multiple levels, from topology (what can I reach?) through transformations (what would it look like from another point of view – or is this another view of the same cat?), to straight paths, lengths and other measurements. Children recognize pattern, including symmetry, in 3-D objects and in the plane. They work with 'number' long before they have language, and proportional reasoning which may be embedded in visual kinesthetic brain regions. Children move, and plan motions, within space. Children image and imagine in rich ways that can also be connected to patterns of creative adults.

Typically, the education do not value and develop these skills in ways to intimately connect to mathematical concepts and problem solving. We do not explicitly build on these foundations, though by chance some students and teachers ground their mathematics in these ways. At ages 12-15, children go through a burst of brain development and trimming of connections. What is the impact of this on the development of visual reasoning.

Curriculum suggests that 2-D is easier than 3-D, although it is cognitively less natural for many modes of reasoning, and 3-D skills are the needed goal for later work. The domination of analytic over synthetic reasoning encourages the pattern that 2-D is the starting point, and the disconnection between early childhood reasoning, and latter problem solving both of which engage 3-D reasoning.

Questions for research:

- What 3-D abilities do children bring to mathematics, and how do we sustain these for essential later work?
- Does technology assist with these connections, or does it pull us away from 3-D?
- Is transformational thinking more natural than ‘geometric constructions’ for elementary students and elementary teachers?
- Is some portion of visual thinking ‘use it or loose it’? At what ages?

Claim 9: Visually based pedagogy opens mathematics to students who are otherwise excluded.

Comments: Studies suggests that students (and adults) with autism and dyslexia may rely more on visual reasoning than on verbal reasoning [11,26,30]. Studies also confirm that this can form a very solid basis for effective work with mathematics [10,26,30]. This effective use of visuals is sustained beyond childhood into mature work in mathematics, computer science, and allied fields. If we fail to support visual methods in mathematics, we exclude an important group of students.

Technology, such as dynamic geometry programs, offers much wider options for effective visual representations and visually based explorations. This technology, combined with scientific visualization of data and structures also support high level work in many of the sciences. However, studies in diagrammatic reasoning indicate that effective use of flat screen technology for 3-D reasoning needs to build on top of hands-on 3-D work.

Questions for research:

- What proportion of our students would engage mathematics more effectively and more enthusiastically through visual processes.
- Which uses of technology supports this, and which does not?
- Are there curriculum / pedagogy which build appropriate skills and connections for all students.

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whiteley@mathstat.yorku.ca http://www.math.yorku.ca/~whiteley/