1. (25 POINTS) Suppose that \( P(A) = 0.15 \), \( P(B) = 0.2 \) and \( P(C) = 0.35 \). Moreover, events \( A \) and \( B \) are mutually exclusive. Events \( A \) and \( C \) are independent and the events \( B \) and \( C \) are also independent.

(a) (5 POINTS) Since \( A \) and \( B \) are mutually exclusive, what event is \( (A \cap B)^c \)?

\[
A \cap B = \emptyset \Rightarrow (A \cap B)^c = \Omega \quad \text{the outcome space}
\]

(b) (10 POINTS) Determine the probability that at least one of the events \( A, B, C \) occurs.

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\]

\[
A \cap B = \emptyset \Rightarrow P(A \cap B) = 0 \quad A \cap B \cap C = \emptyset \cap C = \emptyset \Rightarrow P(A \cap B \cap C) = 0
\]

\[
A \text{ and } C \text{ independent} \Rightarrow P(A \cap C) = P(A)P(C)
\]

\[
B \text{ and } C \text{ independent} \Rightarrow P(B \cap C) = P(B)P(C)
\]

\[
\therefore P(A \cup B \cup C) = 0.15 + 0.2 + 0.35 - 0.15 \times 0.35 - 0.2 \times 0.35
\]

\[
= 0.5775
\]

(c) (10 POINTS) Determine \( P(B^c | A \cup B) \).

\[
P(B^c | A \cup B) = 1 - P(B | A \cup B)
\]

\[
P(B | A \cup B) = \frac{P(B \cap (A \cup B))}{P(A \cup B)} = \frac{P(B)}{P(A \cup B)} \quad \text{since } B \subseteq A \cup B
\]

\[
P(A \cup B) = P(A) + P(B) \quad \text{since } A \text{ and } B \text{ are mutually exclusive}
\]

\[
= 0.35
\]

\[
\therefore P(B^c | A \cup B) = 1 - \frac{0.2}{0.35} = \frac{3}{7} \approx 0.42857
\]
2. (30 POINTS) A bin contains three components from supplier A, four from supplier B, and five from supplier C. Five of the components are randomly selected without replacement for testing.

(a) (15 POINTS) What is the probability that the sample contains one from supplier A, two from supplier B and two from supplier C?

\[ \Omega = \text{ set of all unordered samples containing 5 components } \]

\[ \# \Omega = \binom{12}{5} = \frac{(12)!}{5!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792 \]

\[ E = \text{ event sample contains 1 from A, 2 from B and 2 from C} \]

\[ \# E = \binom{3}{1} \binom{4}{2} \binom{5}{2} = 3 \times \frac{4 \times 3}{2} \times \frac{5 \times 4}{2} = 180 \]

\[ \therefore P(E) = \frac{\# E}{\# \Omega} = \frac{180}{792} \approx 0.2273 \]

(b) (15 POINTS) If the first component selected is from supplier A, what is the probability that the second component selected is from supplier A, the third component selected is from supplier C and the fourth component selected is from supplier B?

\[ A_1 = \text{ event i th component selected is from supplier A} \]

\[ B_i = \text{ second component selected is from supplier B} \]

\[ C_i = \text{ third component selected is from supplier C} \]

\[ P(A_2 \cap C_1 \cap B_4 \mid A_1) = P(A_2 \mid A_1) \cdot P(C_1 \mid A_1 \cap A_2) \cdot P(B_4 \mid A_1 \cap A_2 \cap C_1) \]

\[ P(A_2 \mid A_1) = \frac{2}{11} \]

\[ P(C_1 \mid A_1 \cap A_2) = \frac{5}{10} \]

\[ P(B_4 \mid A_1 \cap A_2 \cap C_1) = \frac{4}{9} \]

\[ \therefore P(A_2 \cap C_1 \cap B_4 \mid A_1) = \frac{4}{99} = 0.040400 \]

continues...
3. (30 POINTS) There are three boxes, each with two drawers. Box A has a gold coin in each
drawer and Box B has a silver coin in each drawer. Box C has a silver coin in one drawer and
a gold coin in the other. One box is chosen, then a drawer is chosen at random from the box.
Box A is chosen with probability 0.4, Box B with probability 0.25 and Box C with probability
0.35, respectively.

(a) (15 POINTS) What is the probability that the chosen drawer has a gold coin?

\[ G_i = \text{event that the chosen drawer has a gold coin} \]
\[ B_i = \text{event that Box } i \text{ is selected, } i = A, B, C \]

\[ P(G \mid B_A) = 1 \quad P(G \mid B_B) = 0 \quad P(G \mid B_C) = 0.5 \]
\[ P(B_A) = 0.4 \quad P(B_B) = 0.25 \quad P(B_C) = 0.35 \]

\[ \begin{align*}
0.4 & \quad \begin{array}{c} 1 \quad 0 \\ 0.5 \quad 0.5 \end{array} \\
0.25 & \quad \begin{array}{c} 0 \quad 0 \\ 0 \quad 0.5 \end{array} \\
0.35 & \quad \begin{array}{c} 0.5 \quad 0.5 \\ 0.5 \quad 0 \end{array}
\end{align*} \]

\[ B_A, B_B, B_C \text{ partition } \Omega \]

\[ P(G) = P(G \mid B_A) P(B_A) + P(G \mid B_B) P(B_B) + P(G \mid B_C) P(B_C) \]
\[ = 0.575 \]

(b) (15 POINTS) Given that the chosen drawer has a gold coin, what is the probability
that the other drawer of the box has a silver coin?

\[ S = \text{event the other drawer of the box selected has a silver coin} \]

\[ P(S \mid G_i) = \frac{P(S \cap G_i)}{P(G_i)} \quad S \cap G_i \iff B_c \cap G_i \quad \text{and hence} \quad S \cap G_i = B_c \cap G_i \]

\[ \therefore P(S \mid G_i) = \frac{P(B_c \cap G_i)}{P(G_i)} = \frac{P(G \mid B_c) P(B_c)}{P(G_i)} = \frac{1}{3} \times 0.3043 \]

continues...
4. (15 points) A shop makes customized ice-cream cakes. A particular individual has come up with 5 different designs and gives an ice-cream birthday cake each year to their significant other who loves ice-cream. To ensure that the design is a surprise, the individual chooses one of the 5 designs at random each year. What is the probability that over 6 years, each design will have been used at least once?

We are sampling at random with replacement 6 times from 5 designs

\[ \Omega = \text{set of all ordered samples containing 6 designs} \]

\[ \# \Omega = 5^6 \text{ and these are equally likely} \]

\[ A = \text{event that each design is used at least once over 6 years.} \]

Hence, one design is used twice and the others are used once.

Number of times design used: 2 1 1 1 1

Design: 1 2 3 4 5

\[ \binom{5}{1} \text{ ways to choose the design that will be used twice and for} \]

each of these different kinds of "each design is used at least once over 6 years"

there are \[ \binom{6}{2} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1} = 15 \]

ordered outcomes giving rise to it

Note that \[ \binom{5}{1} = \binom{4}{1} \text{ number of ways of partitioning 5 digits of} \]

\[ \# A = \binom{5}{1} \binom{6}{2} = 5 \times 6 \times 5 \times 4 \times 3 = 1800 \]

\[ P(A) = \frac{\# A}{\# \Omega} = \frac{1800}{15625} = 0.1152 \]

The end