

Portfolio Choice and Life Insurance

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18 September 2005

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Abstract

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We study a class of *portfolio choice* problems that combine life insurance and labor income, within the optimal control framework pioneered by Merton (1969, 1971). Our model differs from previous research by (i) focusing attention on the correlation between human capital and financial capital, and (ii) modeling the utility of the family as opposed to separating consumption and bequest.

From a technical point of view we show how the underlying Hamilton-Jacobi-Bellman (HJB) equation can be simplified using a similarity reduction technique, which then allows for the implementation of an efficient numerical solution. And, for reasonable financial economic parameter values, a closed-form approximation is derived which greatly simplifies the numerical calculations.

A variety of example illustrating our numerical algorithm are also provided. Our main qualitative result is that households whose primary breadwinner's wages are negatively correlated with financial market returns, should optimally purchase more life insurance and can afford to take more risky positions with their financial portfolio. In addition, we find that the optimal face value of life insurance is remarkably insensitive to the family's risk aversion.

KEYWORDS: Actuarial Finance, Personal Risk Management, Portfolio Theory, Hamilton-Jacobi-Bellman Equation, Similarity Reduction, Finite Difference Method.

JEL Codes: D91, G11

1 Introduction and Motivation

There is a glaring disconnect between the way most financial advisers sell life insurance versus how they sell or promote investment products such as mutual funds, stocks and bonds. Aside from the regulatory environment and the different licenses required – namely securities as opposed to insurance – these two *financial decisions* are presented as if there were a separation theorem that justified their relative invariance. Moreover, while the concepts of risk tolerance, risk aversion and utility are ubiquitous in the lingo of the securities industry, the same lexicon rarely enters the dialogue in the life insurance arena. This is quite odd, since historically the economics of insurance was the breeding ground for much of the development in utility theory.

This observation is more than just anecdotal. The insurance literature’s starting point for the optimal quantity of life insurance is the original work by Dr. Solomon Huebner (U. of Penn.) in the early 1900s, based on the concept of a Human Life Value (HLV). This idea is also at the heart of much forensic economics and litigation, where the courts must determine the value of a lost life for the purposes of compensation. See Zietz (2003) and Todd (2004) for a *state of the art* review of the literature regarding life insurance. Most financial planning and investment textbooks advocate something called a *needs analysis* where the family determines how much would be required to meet long-term expenses and other financial goals if the primary source of income were lost. Other authors focus on the present value of lost wages. Regardless of the precise mechanics, rarely is this decision presented as a portfolio choice problem akin to the investment in stocks and bonds. Our main point is that, we believe it should. After all, life insurance is a hedge against the loss of human capital and portfolio hedging decisions should be made jointly, not independently.

Thus, our objective in this paper is to jointly analyze the decision of how much life insurance a *family unit* should have to protect against the loss of it’s *breadwinner* as well as how the family should allocate it’s financial resources between risk-free and risky assets, vis a vis the dynamics of labor income and human capital. We view this problem within the paradigm of portfolio choice and use the tools of financial economics pioneered by Merton (1969, 1971) to arrive at optimal controls for investment, consumption and life insurance – but where the family unit is placed front and center.

A number of recent papers have extended the set of decisions included in the portfolio choice problem to highlight the interaction and the risks faced by the household, broadly defined. Thus, for example, Goetzmann (1993), Yao and Zhang (2005) as well as Cocco (2005) focus on the role of the housing portfolio; Campbell and Cocco (2003) focus on optimal mortgage choices; Sundaresan and Zapatero (1997) examine the role of defined

benefit pensions, while Dybvig and Liu (2004) model the impact of flexible retirement dates; Jagannathan and Kocherlakota (1996) and Viceira (2001) stress the impact of aging; Faig and Shum (2002) are motivated by the demand for illiquid assets; Koo (1998) as well as Davis and Willen (2000) model the role of labor income; Dammon, Spatt and Zhang (2001) focus their attention on capital gains and income taxes; Heaton and Lucas (2000) focus on the role of entrepreneurial risk. Others go back to basics and extend portfolio choice models to include more sophisticated (and realistic) processes for investment returns, such as Chacko and Viceira (2005) or time-varying and mean-reverting risk premiums, such as Kim and Omberg (1996).

And, while the above list is clearly not exhaustive, the unifying theme of the burgeoning portfolio choice literature is that a singular *personal financial* feature is highlighted and carefully modeled in order to tease-out the relevant financial economic insights related to that one feature. Some of these papers go-on to test or calibrate their models against real world data, while others end their contribution with a normative model. All of them use the Merton (1969, 1971) framework as a starting point.

Our objective is similar. We start with a traditional diffusion model of asset dynamics and utility of consumption, but we are focused in our modeling of labor income and life insurance purchases. A central feature of our model is the correlation between innovations to the labor income process and financial returns. We are specifically interested in the interaction between the utility-maximizing demand for life insurance *vis* the optimal consumption pattern and asset allocation when there is a non-zero correlation between these two critical state variables. We deviate somewhat from previous models by focusing on the *family unit* that derives utility from consumption and the risk to this family of losing the wage/income source. The emphasis on the family as a unit, which survives independently of the life status of the breadwinner, alleviates the need for specifying a separate utility of bequest.

That is not to say that life insurance has never been analyzed in a financial context. Early work by economists such as Yaari (1965), Hakanson (1969), Fischer (1973), Campbell (1980), Lewis (1989), Hurd (1989) and Babbel and Ohtsuka (1989) helped the literature develop a number of insights into the demand for life insurance. These papers illustrated how the nature of mortality uncertainty, the utility and strength of bequest, risk aversion, increasing household wealth and inter-temporal rates of substitution impact the decision to purchase life insurance and the type of life insurance to buy. But these models are not couched in the current language of portfolio choice not do they give us much insight into the role of labor income dynamics and human capital over the life-cycle. The same is true of the work by Richards (1975), which extended Merton (1969, 1971) to life insurance, but with little numerical insight or analytic solutions given the complexity of the problem. Along the

same lines, Buser and Smith (1983) adopt a portfolio approach to life insurance, but from a strictly one-period framework. Nevertheless, we will return to our contributions above-and-beyond previously known results later on in the analysis, when we present the specifics of our model. Likewise, our work differs from the recent paper by Chen, Ibbotson, Milevsky and Zhu (2005) where the analysis is done in discrete time and via simulations. One of the main differences between the current work and most previous research is that we avoid specifying a utility of bequest yet are still able to talk intelligently about the demand for life insurance. Nevertheless, our results confirm many of the earlier insights.

From a technical point of view, this two state variable class of problems – wage income and asset prices – normally leads to a highly nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equation. Yet, one of contributions of our paper is that despite this nonlinearity, analytical solutions can be obtained in special cases when wage income and asset prices are perfectly (positively or negatively) correlated. When this correlation is between -1 and +1, we obtain approximate solutions. We first use the method of *similarity reduction* to reduce the dimension of the HJB equation. For some special cases we show that this leads to a set of ordinary differential equations, which can be solved explicitly. For the general case, we show that using *similarity reduction* can either help to find approximate solutions or simplify the problem significantly before applying numerical methods. As a result, the computational cost will be reduced, compared to that of a full numerical approach applied to the original HJB.

Technicalities aside, our main qualitative results are as follows. First, we find the optimal amount of (face value) life insurance a family should have, depends quite strongly on both volatility of the wage/income process as well as it's correlation to investment returns. Higher volatility and higher correlations lead to lower, optimal, life insurance levels. *Ceteris paribus*, a tenured university professor needs more life insurance than a Wall Street investment banker, even if they both expect to earn the same wages over time. Furthermore, as one would expect, the risk profile of the wage/income process has a strong impact on the optimal mix between risky and risk-free assets for the family. This result should come as no surprise. The original work by Bodie, Merton and Samuelson (1991) all the way to the recent work by Cocco, Gomez and Maenhout (2005) have stressed the financial characteristics of human capital and how it serves as a substitute for financial capital. Our model easily reproduces this result and emphasizes the role of life insurance, which is a hedge for human capital. Finally, our model confirms the intuition shared by many financial planners that the optimal amount of life insurance a family should hold is not sensitive to the risk aversion or risk tolerance of the family. In our constant relative risk aversion (CRRA = γ) model, the optimal demand for life insurance is insensitive to γ , even though the family's financial portfolio is obviously

quite sensitive to this parameter. As such, our model opens up the possibility of testing whether the actual holding of life insurance is, in fact, correlated with other proxies for a family’s risk aversion. Our model provides justification for finding no such relationship.

The rest of the paper is organized as follows. The setup of the problem is described in Section 2 followed by the setup of the HJB equation in Section 3. Solution methodologies are discussed in Section 4. In Section 4.1, we focus on special cases when the wage and risky asset are perfectly correlated. The general case is discussed in Section 4.2. We further simplify the HJB equation using the similarity reduction technique, which is followed by the presentation of the numerical method for solving the reduced HJB equations. Issues of finding the proper boundary conditions are also discussed. In Section 4.3, we present a closed form approximate solution for practically relevant parameter values. In Section 5 typical results and the financial implications of the results are given. We finish the paper with a discussion of future research directions in Section 6. The derivations of the HJB equation and the closed form solutions are given in the appendices.

2 Model of the Family Life-cycle

2.1 The Input Variables

The variable t denotes the current time and we will work with three dates of interest. The first date is the time horizon T_H of the family, which is assumed exogenous and deterministic on the order of 50 to 100 years. The second date is the time of retirement $T_R < T_H$, which is when the wage/income process (job) terminates and the breadwinner enters his or her retirement years. The third date of interest is the death of the breadwinner, and end of the wage/income process, which takes place at a random stopping time denoted by τ . The wage/income process jumps to zero at the minimum of T_R and τ . We do not allow flexible retirement as in Sundaresan (1997).

We assume the family purchases short-term insurance on the life of the breadwinner which is renegotiated and guaranteed renewable on an ongoing basis at a pre-determined schedule which is driven by an *instantaneous force of mortality (IFM) curve* denoted by λ_{y+t} , where y is the age of the primary breadwinner at inception of the model. We then let I_t denote the insurance premium (in dollars) payable per unit time; a variable which is under the direct control of the family. One can think of I_t as a budget for insurance which will then induce a certain face value or death benefit I_t/λ_{y+t} . For example, if the IFM curve at age $y = 35$ is $\lambda_{35} = 0.001$, and the family spends $I = 50$ dollars on life insurance, this entitles the family to a death benefit of $50/0.001 = \$50,000$.

We let M_t denote the market value of the family's assets which includes the value of all (risky) stocks and (risk-free) bonds on a mark-to-market basis. We assume that M_0 denotes the initial marketable wealth at time $t = 0$. Heuristically, the breadwinner works and converts labor and time into wages and income. A portion of this income is consumed and the remainder is saved in a diversified portfolio consisting of risky stocks and safe bonds.

The variable α_t denotes the fractional allocation of the family's marketable wealth M_t to the risky asset at time t . Thus, if $\alpha_t = 0$ the family allocates all its marketable wealth – but no more – to risk-free bonds and if $\alpha_t = 1$ the family allocates all its marketable wealth – but no more – to risky stocks. The model also allows for $\alpha_t > 1$ which would imply leverage. Thus, for example, $\alpha_t = 2$ implies that 200% of wealth is invested in the risky stock. This is financed by borrowing 100% of wealth at some (constant) rate of interest denoted by r . Recall that α_t is under direct control of the family and is one of the *three* choice variables in our model. We do not address liquidity constraints and the difficulty faced by younger families when borrowing. Note that many of our numerical results involve large leverage.

We let c_t denote the instantaneous consumption rate of the family (in real terms) per unit time. In general, our model is specified in real (after inflation) terms and all parameters and choice variables will reflect this. Note that the consumption rate is our third and final choice (a.k.a. control) variable and c_t is chosen to maximize the family's *utility* of consumption.

Even though a more realistic state dependent instantaneous utility function can be used, under our PDE-based methodology, we adopted a simpler CRRA utility function in this study. The precise functional form is

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma} \quad (1)$$

for some positive constant $\gamma > 0$, which is labeled the coefficient of relative risk aversion.

Let X_t denote the market value of the risky asset (stock index, market portfolio) at time t . This stochastic process will be modeled as a geometric Brownian motion so that:

$$dX_t = \mu_m X_t dt + \sigma_m X_t dB_t^m, \quad (2)$$

where μ_m denotes the drift and σ_m denotes the diffusion coefficient of the process. This, of course, implies that $\ln(X_t/X_0)$ is normally distributed with a mean value of $(\mu_m - 0.5\sigma_m^2)t$ and a standard deviation of $\sigma\sqrt{t}$. Stated differently, the geometric mean return (a.k.a. growth rate of the risky asset) is $\mu_m - 0.5\sigma_m^2$ per annum. Typical values of the parameters μ_m fall in the range of (5%, 15%) and typical values of σ_m fall in the range of (5%, 50%). The risk-free rate r , which is also the rate at which the family can borrow money, is on the order of magnitude of (1%, 3%), which must be lower than μ_m for economic equilibrium purposes.

We now let W_t denote the real (after-inflation) wage/income rate of the family's breadwinner per unit time. This stochastic process is expected to increase in real terms over time and might be correlated with the investment performance of the 'risky asset'. We assume the wage process satisfies the following geometric Brownian motion (GBM), but with specification:

$$dW_t = \begin{cases} \mu_w W_t dt + \sigma_w W_t dB_t^W & t < \tau \\ 0 & t \geq \tau \end{cases}, \quad (3)$$

where τ is the random time of death, μ_w denotes the drift and σ_w denotes the diffusion coefficient of the process, and B_t^W denotes the Brownian motion driving the wages process W_t . Similar to the risky asset, we assume that the real wage at any future time $t+s$ is lognormally distributed with parameters (μ_w, σ_w) . We could just as easily specify a mean-reverting process. Note that the Brownian motion B_t^m driving the risky asset is instantaneously correlated with the B_t^W driving the wage process via the relationship $d\langle B_t^m, B_t^W \rangle = \rho \sigma_m \sigma_w dt$. Later we will talk about this correlation variable ρ , which is the primary focus of our numerical case study and results.

As mentioned earlier, the instantaneous force of mortality (or hazard rate) is denoted by λ_{y+t} , where y is the initial age. The quantity $(\lambda_{y+t})dt$ can be thought of as the rate of death within a small time interval dt at time t . The conditional probability of survival, from age y to age $y+t$, under the law of mortality defined by λ_{y+t} can be computed via:

$$({}_t p_y) = e^{-\int_0^t (\lambda_{y+s}) ds}, \quad (4)$$

where the notation on the left-hand side of the equation is standard in the actuarial literature. For example, if the future lifetime random variable is exponentially distributed, we have that $\Pr[\tau \geq s] = e^{-\lambda s}$, and the hazard rate is constant at all ages, at a value of $\lambda_{y+t} = \lambda$. In the general case, the function λ_{y+t} is expected to increase with time (age). It is important to stress that in our model the family unit knows exactly how much they will have to pay for insurance – regardless of how much and when they want to purchase it – from the current time t , to the time of retirement T_R . Thus, in addition to precluding whole-life and other more complicated forms of insurance, we do not allow for stochastic mortality rates or anti-selection effects which might complicate the insurance purchase problem. Thus, once again, if the family purchase (invests) I_t dollars in life insurance at time t , they will be entitled to a death benefit of I_t/λ_{y+t} if the breadwinner dies at time t . For now, we ignore loading and commissions which can easily be handled by working with a loaded hazard rate $\widehat{\lambda}_{y+t}$ instead of a biological mortality rate λ_{y+t} .

2.2 The Financial Wealth Dynamics

Based on the construction of the wage/income process W_t and the evolution of the risky asset price X_t , the family budget constraint for the marketable wealth process M_t will satisfy the following stochastic differential equation:

$$\begin{aligned} dM_t &= W_t dt - c_t dt - I_t dt + \alpha_t M_t (\mu_m dt + \sigma_m dB_t^m) + (1 - \alpha_t) r M_t dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t + W_t - c_t - I_t] dt + \sigma_m \alpha_t M_t dB_t^m \end{aligned} \quad (5)$$

for $t < \min[\tau, T_R]$, and

$$\begin{aligned} dM_t &= -c_t dt + \alpha_t M_t (\mu_m dt + \sigma_m dB_t^m) + (1 - \alpha_t) M_t r dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t - c_t] dt + \sigma_m \alpha_t M_t dB_t^m \end{aligned} \quad (6)$$

for $t > \min[\tau, T_R]$. The intuition for the various pieces in equations (5) and (6) is as follows. Firstly, we add wage income – when there is some – via $W_t dt$. Secondly, we subtract discretionary consumption $c_t dt$ and insurance premiums $I_t dt$. Finally, we add instantaneous investment returns from the allocation to the risky asset, $\alpha_t M_t dX_t/X_t$ as well allocations to the risk-free asset $(1 - \alpha_t) M_t r dt$. Finally, we have substituted dX_t/X_t from equation (2) to eliminate any reference to X_t from here on.

At the instant of death $t = \tau$, we must carefully add the death benefit by distinguishing between the value of marketable wealth one instant prior to the “arrival” of death and one instant after this arrival. More precisely:

$$M_{\tau+} = M_{\tau-} + \frac{I_\tau}{\lambda_{y+\tau}} \quad (7)$$

where $I_\tau/\lambda_{y+\tau}$ is the death benefit or the face value of the insurance policy at time τ and $\lambda_{y+\tau}$ is the instantaneous force of mortality, or hazard rate.

Finally, the family-unit or household objective function is defined by:

$$\max_{\{\alpha_s, I_s, c_s\}} E_t \left[\int_t^{T_H} e^{-\delta s} u(c_s) ds \middle| \mathcal{F}_t \right], \quad (8)$$

where δ is the subjective discount rate and $u(c)$ denotes the instantaneous utility of consumption at time t . In words, the family is searching for an asset allocation strategy α_s , an insurance buying strategy I_s , and consumption strategy c_s that maximizes the discounted value of utility of consumption between time t (now) and the terminal horizon of the family unit.

We stress once again that we have “eliminated” the need for a utility of bequest by treating the family as one unit that is dependent on the breadwinner for their source of

wages. Thus, in contrast to the classical insurance models of Yaari (1965), Fischer (1973) or even Campbell (1980), the primary breadwinner is not forced to decide how much “they love their family”, versus how much they “love themselves”. Rather, the family decides to allocate consumption across the entire horizon of the family, while protecting the wage/income flow using insurance. This assumption obviously alleviates the need to measure the strength of bequest, but implicitly assumes the family derives the exact same level of utility whether the primary breadwinner is alive, or not. Of course all psychic value of life is ignored as well. Interestingly, we find that even if we change $u(c)$ to HARA utility with “Required Lower Bounds” on consumption, results can be obtained.

3 The Hamilton-Jacobi-Bellman Equation

With the model foundations and formulation behind us, we proceed in the usual fashion for solving these problems. First, let

$$J(M_t, W_t, t) = \max_{\{\alpha_s, I_s, c_s\}} E \left[\int_t^{T_H} e^{-\delta s} u(c_s) ds | \mathcal{F}_t \right], \quad (9)$$

denote the indirect utility function. Now, assume we can find the optimal control that satisfies equation (9), then J satisfies the following Hamilton-Jacobi-Bellman (HJB) equation (we refer the readers to Appendix A for the detailed derivation):

$$\begin{aligned} \lambda_{y+t} J &= J_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t J_M) \\ &+ \max_{I_t} \left(\Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} - I_t J_M \right) \\ &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t J_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 J_{MM} + \alpha_t \rho \sigma_W \sigma_m W_t M_t J_{WM} \right] \\ &+ r M_t J_M + W_t J_W + \mu_W W_t J_W + \frac{1}{2} \sigma_W^2 W_t^2 J_{WW} \end{aligned} \quad (10)$$

where the symbol Φ satisfies the following HJB equation, but without wage income and life insurance,

$$\begin{aligned} 0 &= \Phi_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t \Phi_M) \\ &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t \Phi_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 \Phi_{MM} \right] + r M_t \Phi_M. \end{aligned} \quad (11)$$

For constant relative risk aversion utility (CRRA), the analytic form of $\Phi(M, t; T)$ itself can be obtained as follows

$$\Phi(M, t; T) = h(t; T) \frac{M^{1-\gamma}}{1-\gamma}, \quad (12)$$

where $h(t; T)$ satisfies the following ordinary differential equation

$$\frac{1}{1-\gamma} \frac{h'}{h} + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}} + r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} = 0. \quad (13)$$

Hereafter the prime symbol is used to denote the derivative with respect to time. The analytical solution of equation (13) with zero bequest at time T can be obtained as:

$$h(t; T) = e^{-\delta t} \left(\frac{e^{\xi(T-t)} - 1}{\xi} \right)^\gamma, \quad (14)$$

where

$$\xi = -\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma} \left(r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} \right).$$

After applying the first order conditions, the HJB equation for J can be rewritten as

$$\begin{aligned} \lambda_{y+t} J = & J_t + \frac{\gamma}{1-\gamma} c_t^* J_M + \frac{\lambda_{y+t}\gamma}{1-\gamma} (M + \lambda_{y+t}^{-1} I_t^*) J_M + \lambda M J_M \\ & + \alpha_t^* (\mu_m - r) M J_M + \frac{1}{2} (\alpha_t^*)^2 \sigma_m^2 M^2 J_{MM} + \alpha_t^* \rho \sigma_W \sigma_m W M J_{WM} \\ & + r M J_M + W J_M + \mu_W W J_W + \frac{1}{2} \sigma_W^2 W^2 J_{WW}. \end{aligned} \quad (15)$$

The terminal condition is

$$J(M, W, T_R) = \Phi(M, T_R; T_H). \quad (16)$$

Finally, the optimal control c_t^* (consumption), α_t^* (allocation to risky equity) and I_t^* (amount spent on life insurance purchases) is given by:

$$c_t^* = (e^{\delta t} J_M)^{-\frac{1}{\gamma}}, \quad (17)$$

$$\alpha_t^* = -\frac{(\mu_m - r) J_M + \rho \sigma_m \sigma_w W J_{WM}}{\sigma_m^2 M J_{MM}}, \quad (18)$$

$$I_t^* = \left[\left(\frac{J_M}{h(t; T_H)} \right)^{-\frac{1}{\gamma}} - M \right] \lambda_{y+t}. \quad (19)$$

4 Solution Methodology

In this section we will consider a special case $\rho = \pm 1$, where a closed form analytical solution can be found. This is the case where human capital (i.e. the stochastic wage process) is either 100% positively or 100% negatively correlated with financial market returns.

4.1 Special Case: $\rho = \pm 1$

In this case, as shown in Appendix B, the solution of the HJB (15) equation takes the form

$$J = \frac{h(t)(M + k(t)W)^{1-\gamma}}{1-\gamma} \quad (20)$$

where h is given by (14) and k satisfies

$$\frac{k'}{k} + \frac{1}{k} + \eta = 0, \quad (21)$$

with $\eta = \mu_w - r - \lambda_{y+t} - \beta(\mu_m - r)$ and $\beta = \rho\sigma_w/\sigma_m$. Using the terminal condition $k(T_R) = 0$, k can be solved as

$$k = e^{-\int_{T_R}^t \eta(s) ds} \int_{T_R}^t e^{\int_{T_R}^s \eta(n) dn} ds. \quad (22)$$

When $\lambda_{y+t} = \lambda$, i.e. a constant mortality rate, we have

$$k = \frac{1}{\eta} (e^{\eta(T_R-t)} - 1). \quad (23)$$

The optimal control functions are

$$\frac{I_t^*}{\lambda_{y+t}M} = kx, \quad (24)$$

$$\alpha_t^* = \frac{\mu_m - r}{\gamma\sigma_m^2} (1 + kx) - \beta kx, \quad (25)$$

$$\frac{c_t^*}{M} = e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}} (1 + kx) \quad (26)$$

where $\beta = \rho\sigma_w/\sigma_m$ and $x = W/M$.

Remark. Since $\beta = \pm\sigma_w/\sigma_m$ for $\rho = \pm 1$, η is affected by the value of ρ , but not by γ . This relationship helps to explain, for example, why risk aversion does not impact the face value of life insurance.

4.2 General Case: $-1 < \rho < 1$

For the general case of correlated wage and risky asset, the closed form solution cannot be obtained. However, we can use the similarity variable $x = W/M$ without the restriction of a simple analytical form.

4.2.1 Similarity reduction

We now show that the two dimensional HJB equation can be reduced to a one dimensional one via the similarity reduction technique. We start by letting

$$J = \frac{M^{1-\gamma}}{1-\gamma} F(x, t).$$

This leads to a new partial differential equation,

$$F_t + c_0(1 - \gamma)F + c_1xF_x + c_2x^2F_{xx} = 0, \quad (27)$$

where

$$\begin{aligned} c_0 &= \frac{c_t^*}{M} \frac{\gamma}{1-\gamma} + \frac{I_t^*}{M} \frac{\gamma}{1-\gamma} + \alpha_t^*(\mu_m - r) + x + r - \frac{1}{2}\gamma(\alpha_t^*\sigma_m)^2, \\ c_1 &= \mu_w - \frac{c_t^*}{M} \frac{\gamma}{1-\gamma} - \frac{I_t^*}{M} \frac{\gamma}{1-\gamma} - \frac{\lambda_{y+t}}{1-\gamma} \\ &\quad - x - r - \alpha_t^*(\mu_m - r) - \alpha_t^*\rho\sigma_m\sigma_w\gamma + (\alpha_t^*\sigma_m)^2\gamma, \\ c_2 &= \frac{1}{2}(\alpha_t^*\sigma_m - \rho\sigma_w)^2 + \frac{1}{2}\sigma_w^2(1 - \rho^2), \\ \frac{c_t^*}{M} &= \exp\left(-\frac{\delta t}{\gamma}\right) \left(F - \frac{x F_x}{1-\gamma}\right)^{-\frac{1}{\gamma}}, \\ \frac{I_t^*}{M} &= \left[h^{-\frac{1}{\gamma}} \left(F - \frac{x F_x}{1-\gamma}\right)^{-\frac{1}{\gamma}} - 1 \right] \lambda_{y+t}, \\ \alpha_t^* &= \frac{(\mu_m - r)[(1-\gamma)F - x F_x] - \rho\sigma_m\sigma_w[\gamma x F_x + x^2 F_{xx}]}{\sigma_m^2[\gamma(1-\gamma)F - 2\gamma x F_x - x^2 F_{xx}]}. \end{aligned} \quad (28)$$

The terminal condition is now:

$$F(x, T_R) = h(T_R, T_H). \quad (29)$$

Note that the coefficients c_0 , c_1 , c_2 and the terminal conditions are functions of x only. Therefore, the HJB equation permits the similarity solution, which is a function of x and t .

4.2.2 Numerical Method

We use the method of lines to solve (27), i.e., we approximate the spatial derivatives by finite differences which results in a system of ordinary differential equations with respect to time on each grid point x_j

$$\dot{F}_j + c_0 F_j + \frac{c_1 + |c_1|}{2\delta x} (F_{j+1} - F_j) + \frac{c_1 - |c_1|}{2\delta x} (F_j - F_{j-1}) + \frac{c_2}{\delta x^2} (F_{j+1} + F_{j-1} - 2F_j) = 0 \quad (30)$$

where δx is the grid size, c_0 , c_1 and c_2 are evaluated on the grid points, based on the formula in (28). In addition, we truncate the semi-infinite domain ($x > 0$) into a finite one ($0 < x < X$). Thus we need an extra boundary condition at $x = X$. Note that the upwind method is used for the first order derivative term in x .

The system of ordinary differential equations is then solved by an appropriate numerical integration technique. For example, we can use a semi-implicit method where the coefficients c_0 , c_1 and c_2 are evaluated at the previous time level and the spatial derivatives are treated implicitly.

4.2.3 Boundary conditions

We observe that $x = 0$ corresponds to $W = 0$, i.e., no wage income. Therefore, the solution is of a Merton type (with the addition of insurance), which means that F is not a function of x . In other words, F satisfies the following ordinary differential equation:

$$F_t + c_0(1 - \gamma)F = 0. \quad (31)$$

At $x = X$, the situation is more complicated. Motivated by the solution for the special case of $\rho = \pm 1$

$$F \sim (1 + kx)^{1-\gamma},$$

we postulate that

$$F^{\frac{1}{1-\gamma}} \sim 1 + kx.$$

In other words, it behaves as a linear function of x , thus $\left(F^{\frac{1}{1-\gamma}}\right)_{xx} \sim 0$. We can approximate this by

$$F_{N+1}^{\frac{1}{1-\gamma}} = 2F_N^{\frac{1}{1-\gamma}} - F_{N-1}^{\frac{1}{1-\gamma}} \quad (32)$$

where $x_N = X$ is where we truncate the domain.

4.3 Approximate Solution

Motivated by the solution for the special case $\rho = \pm 1$, we now seek an approximate solution for the general case $-1 < \rho < 1$. To proceed formally, we define a new parameter

$$\epsilon = (1 - \rho^2) \left(\frac{\gamma \sigma_m \sigma_w}{\mu_m - r} \right)^2$$

and consider the case when $\epsilon \ll 1$. Note that the special cases considered earlier correspond to $\epsilon = 0$.

Based on the procedure outlined in Appendix B, we can write our solution in the form $J = J_0 + O(\epsilon)$. For J_0 we have the following explicit form

$$J_0 = \frac{M^{1-\gamma}}{1-\gamma} h(1 + kx)^{1-\gamma}, \quad (33)$$

where h and k are given earlier by (14) and (22). For a constant mortality rate, they can be solved explicitly as

$$h(t) = e^{-\delta t} \left[\frac{e^{\xi(T_H-t)} - 1}{\xi} \right]^\gamma, \quad (34)$$

$$k(t) = \frac{1}{\eta} (e^{\eta(T_R-t)} - 1) \quad (35)$$

with ξ and η are defined earlier. Recall that a constant mortality rate $\lambda_{y+t} = \lambda$, implies that future lifetime is exponentially distributed.

5 Numerical Results and Discussion

With the model, derivation and approximations behind us, we now move-on to present some case studies for a realistic set of economic parameters. And so, the parameter values we have chosen for our results are as follows: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$, $M_0 = 1, 200$, $W_0 = 50$, $T_R = 10, 20, 30$, $T_H = 40, 50, 60$, $\delta = 2\%$, $y = 35, 45, 55$.

The financial justification for these numbers are as follows. We assume that the family can invest their financial wealth in a broadly diversified index-fund that is expected to earn an inflation-adjusted arithmetic mean of $\mu_m = 7\%$ per annum, with a volatility of $\sigma_m = 20\%$. These number are consistent with other asset allocation research in the literature, see for example Campbell and Viciera (2002) or Chen, Ibbotson, Milevsky and Zhu (2005). The risk free rate of $r = 2\%$ after inflation is also consistent with current conditions in the inflation-adjusted bond market. We further assume that wages will grow in real (after inflation) terms by approximately $\mu_w = 1\%$ per annum, with a standard deviation of $\sigma_w = 5\%$. This is consistent with wage inflation being higher than price inflation, but also allows for shocks to wages, such as unemployment or business cycle effects. Finally, we assume the family's discount rate – or subjective rate of time preference – is $\delta = r = 2\%$, which is consistent with various macro economic models.

In terms of other embedded parameters, we assume that insurance prices (and mortality rates) are driven by a Gompertz law of mortality, which at age 45 starts at: $\lambda_{45+t} = \exp\{(45+t-86.3)/9.5\}/9.5$. These numbers are consistent with survival rates implicit in pension-based mortality tables and the analytic law of mortality has been used in a variety of actuarial and insurance papers, for example Frees, Carriere and Valdez (1996).

Either way, our first order of business is to confirm that when the parameter values are in the above-mentioned range, that our analytic approximation for $\rho \neq \pm 1$ is valid and accurate. Indeed, we are fortunate that this is the case and Figure #1 provides a visual indication of this fact by comparing the optimal insurance purchase I_0^*/M_0 , as a function of the ratio between wages W_0 and marketable wealth M_0 , when the correlation parameter is $\rho = 0.5$. Notice the close fit between the analytic (numerical) result and the analytic approximation result. For completeness, comparison is also given for $\rho = 1$ as well as for other control variables c_0^*/M_0 and α_0^* .

Figure #1 Placed Here

Indeed, based on the above-mentioned parameter values, we can compute the value of $\epsilon = 0.35(1-\rho^2)$, which is small compared to unity. This explains the close agreement between the numerical and analytical solutions. We note that for the non-constant mortality rate

λ_{y+t} , numerical methods have to be used to compute the function k . However, we find that compared to the full numerical approach, the computational cost for computing k is negligible.

As a side note, the numerical solution we display was obtained using 100 grid points on a spatial interval $x = [0, 0.25]$ and a time step size of 10^{-5} on the interval of $t = [0, 20]$. Our numerical experiments reveal that the computation becomes unstable at a larger time step due to oscillations in the optimal value of α_t^* , which is the holding of risky stocks. This severe stability constraint may be caused by the semi-explicit nature of our algorithm since we compute the coefficients c_t^* , α_t^* and I_t^* explicitly. An alternative approach would be to use a fully implicit method. However, it is not clear how much improvement can be made since an iterative procedure needs to be used at each time step. Despite the fact that small time step sizes have to be used, our numerical method is still competitive since we have reduced the dimension of the HJB, compared to the method in Purcal (2003) where a two-dimensional HJB was solved directly.

Having established the accuracy of the approximate solution for the particular set of parameter values listed earlier, we now turn our attention to the financial implications of the solution. In particular, we have computed a number of cases which demonstrate the impact of risk aversion and wage correlation on the optimal asset allocation and demand for term life insurance.

Table #1 Placed Here

More specifically, Table #1 displays the optimal allocation to risky (market) stocks and the optimal face amount (death benefit) for life insurance, assuming the primary breadwinner has $T_R = 20$ years to retirement and the family horizon is $T_H = 50$ years. This individual is $y = 45$ years-old, earns $W_0 = 50$ thousand dollars per year in real (after inflation) terms and currently has $M_0 = 200$ thousand dollars saved. These numbers are fairly realistic when we consider the fact that a 45 year-old should already have some amount of savings set aside for retirement purposes.

Notice that to be consistent with how insurance is discussed by industry practitioners, we have chosen to display the actual face value of life insurance I_t^*/λ_{y+t} , instead of the amount spent on life insurance I_t^* or the ratio of wealth insurance I_t^*/M_t .

The main qualitative insights from our model are as follows. First, there is a clear link between the optimal amount of life insurance and allocation to stocks as a function of the correlation between (the shock to) wages and the investment return on stocks. For example, a 45-year old who earns \$50 thousand per annum and who already has \$200 in accumulated savings will optimally purchase between \$762 - \$965 thousand of life insurance depending on

the correlation between wages (i.e. human capital) and markets (financial capital). If this individual is working in a job/career whose wages profile is counter-cyclical to the financial market ($\rho = -1$), then the optimal amount of insurance is \$964 thousand. On the other hand, if the wage profile is perfectly correlated with financial markets, the optimal amount of life insurance is a lower \$762 thousand. The economic intuition for the impact of wage-market correlations on the optimal demand for insurance – which is also confirmed in the recent paper by Chen, Ibbotson, Milevsky and Zhu (2005) – is that when human capital is highly correlated with financial capital, the *utility-adjusted value* of human capital is much lower and hence the family requires less life insurance. A similar story applies to the impact of this correlation on the optimal allocation to risky stocks. When the wage process is highly correlated with the returns from the risky asset, the allocation to financial risky asset is reduced to counteract the existing (and implicit) allocation to risky assets within the wage process. Stated differently (and quite obviously), if your job is very risky in the sense that your wages depend on the performance of the stock market, then you should not be investing too much of your financial wealth in the same stock market. A number of models developed in the early cited papers confirm this. Notice, however, that the optimal amount of face value life insurance, I_t^*/λ_{y+t} , does not depend on the risk aversion of the family. For example, at zero wage to market correlation ($\rho = 0$), the optimal face value is approximately \$855 thousand, which is more than 16 times the breadwinner's annual wage. Notice that this number is somewhat higher than the often-heard rule of thumb that people should have 5-8 times their annual income in life insurance. Once again, this number does not depend on the level of risk aversion γ , of the family unit or hazard rate.

Some economists might find it puzzling that our model does not predict higher levels of life insurance for families that are more risk averse. Technically this can be sourced to the CRRA utility of consumption that is identical whether the primary breadwinner is alive or dead. However, we believe this result is fairly consistent with practitioner intuition when it comes to life insurance. Namely, a family should be protected with a minimal level of insurance regardless of how risk tolerant the family (or breadwinner) consider themselves. Obviously, risk tolerance and risk aversion has a very strong impact on the demand for the risky asset α_t . Families that are more risk tolerant will allocate more financial and marketable wealth to risky equities, in highly leveraged amounts, as evidenced by our numerical values of $\alpha_t \gg 1$. It is not clear at this point how liquidity and borrowing constraints would impact these results.

Table #2 provides similar results, but with the key difference that the individual is now $y = 55$ years-old and therefore $T_R = 10$ years from retirement and $T_H = 40$ years from the terminal horizon of the family unit. We assume all other parameters are exactly the same

and simply focus on the impact of “aging” on the optimal face value (i.e. death benefit) of life insurance and the optimal asset allocation.

Table #2 Placed Here

A casual review of the Table #2 reveals substantially lower values for both the optimal asset allocation proportions α_t , as well as the face amount of life insurance I_t^*/λ_{y+t} . Intuitively, the individual is closer to retirement, with only 10 more remaining years of work and wages. In this case, the optimal amount of life insurance lies between \$430 and \$480 thousand, depending on the wage-market correlation ρ . This is only 8 times the annual wage of $W_0 = \$50$ thousand. Notice how the optimal amount of life insurance coverage closely tracks the expected discounted value of future wages. The breadwinner has 10 more years of labor income and buys 8 times annual income of life insurance. In Table #1 the breadwinner had 20 more years of labor income and he/she purchased 16 times annual income. In addition, notice that once the family unit is 10 years away from retirement the amount of financial wealth invested in the risky asset is greatly reduced. In fact, when the wage-market correlation $\rho > 0$, the family eliminates leverage and stops borrowing to invest, as evidenced by $\alpha_t < 1$ values.

Finally, Table #3 takes us back to age 35, which is $T_R = 30$ years prior to retirement and $T_H = 60$ prior to the end of the family’s planning horizon. One other change in Table #3 is that we have reduced the initial (current) wealth $M_0 = 1$ in contrast to the $M_0 = 4W_0 = 100$ of the previous two tables. This particular parameter assumption is motivated by the low levels of wealth one would expect to observe amongst (young) 35 year-old individuals.

Table #3 Placed Here

In this third and final example, the amount of life insurance purchased by the family falls in the millions of dollars, ranging from \$1.03 million to \$1.45 million depending on the correlation variable ρ . With 30 years to retirement the family purchases almost 20 times annual wages in life insurance. Once again, these numbers are consistent with a human capital perspective that protects the family against the loss of the income source.

Another important insight from Table #3 is the (abnormally) high amounts allocated to the risky investment asset α_t . Notice that regardless of the level of risk aversion, whether it is a high γ or a low γ , the family places thousands of a percent in risky stocks. This is financed by borrowing thousands of a percent, a.k.a. shorting the risky free bond in our model. Clearly, these very high number are unrealistic (in practice) and at the very least are unfeasible given the various institutional restrictions on borrowing with very little collateral. However, we do not believe this (odd) result represents a flaw or problem with our model,

since this tends to plague most portfolio choice and asset allocation models in which risk aversion is low, borrowing is unlimited and horizons are long. We refer the interested reader to Campbell and Viciera (2002) for an in-depth discussion of the impact of time horizon, liquidity constraints on the demand for risky assets, since this falls outside the main scope of our analysis.

6 Conclusion and Highlights

This paper focused on a subset of *portfolio choice* problems, where the emphasis was placed on the demand for life insurance as a function of labor income. Our primary research question was to investigate the interaction between the demand for life insurance, the demand for risky assets and the optimal level of consumption as function of one's occupation. This life insurance problem has been investigated by a number of classical papers in the literature, including more recent papers by Chen, Ibbotson, Milevsky and Zhu (2005) and Purcal (2003). Our underlying model differs from those in a number of methodological and conceptual ways, which we have explained in the body of the paper.

From a technical point of view we show that the underlying Hamilton-Jacobi-Bellman (HJB) equation can be simplified using the similarity reduction technique, which then allows for the implementation of an efficient numerical method. Furthermore, for realistic financial economic parameter values, a closed-form approximation is derived which greatly simplifies the numerical calculations. Numerical tests confirm the robustness and accuracy of the approximate solution compared to the full numerical solutions.

A variety of financial case studies illustrating our numerical algorithm are also provided within the paper. Our main qualitative and practical result is that households whose primary breadwinner's wages are negatively correlated with financial market returns, should optimally purchase more life insurance and can afford to take more risky positions with their financial portfolio. However, we also find that the optimal face value of life insurance is remarkably insensitive to the family's risk aversion.

From a practical perspective our model validates a number of rules-of-thumb used by financial planners when making portfolio and investment recommendations for their wealth management clients. First, we find that younger investors – i.e. people who are farther away from the date at which their employment wage process will hit zero – are more able to tolerate financial risk and will therefore invest more in the risky asset compared to the risk-free bond. In fact, we find a very high (optimal) leverage ratio that can range in the hundreds of a percent, for families whose risk aversion is low. Second, we find that households should insure against the loss of their primary breadwinner's wages quite independently of how risk

tolerant or risk averse the family defines itself. This is consistent with the Human Life Value (HLV) concept originally introduced by Dr. Solomon Huebner in the early 1900s. According to this approach to financial planning and insurance, there is some universal multiple of wages the a family should insurance against losing, regardless of whether this particular family “likes to gamble”.

On a slightly less intuitive manner, our model lends credence to the idea of classifying human capital as either a stock or a bond. Namely, if the breadwinner’s human capital is risky and highly correlated to the returns on the equity market, then the family should invest less in the same risky asset. Likewise, a family whose human capital is uncorrelated with equity markets can afford to take more financial risk with their portfolio. Stated simply. If you are stock, you should own more bonds. If you are a bond, you should own more stock. This type of thinking has only recently started to resonate with financial advisers, but has started to appear in related articles in the literature.

Finally, we find that the demand for life insurance should depend on the riskiness of the human capital wages process and its correlation with financial returns. High anticipated levels of wage volatility and/or high levels of correlation induce a reduction in the demand for life insurance.

Future research by the authors will focus on a variety of extensions to the basic model presented in this paper, that would further enriched the literatures understanding of portfolio choice problems *vis a vis* the demand for life insurance. High on our list is extending our objective function from a CRRA utility to a HARA utility function, where the family unit imposes a baseline constraint, or fixed level of consumption that must be maintained under all circumstances. This baseline or minimal income might depend on whether the primary breadwinner is alive, which might implicitly induce a type of bequest motive. Another extension involves relaxing the deterministic force of mortality assumption, where there is uncertainty regarding the future costs or price of insurance. This might lead to yet another justification for the coexistence and optimality of various forms of life insurance, such as term, whole-life and universal policies – as originally explored by Babbel and Ohtsuka (1989). Finally, we will extend the model to multiple insurance products and risky assets – each with their own correlation to possibly more than one risky wage process – where the portfolio choice problem will involve and element of intra-occupational hedging strategies.

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A Derivation of the HJB Equation

We derive the HJB equation by using the dynamic programming principle as following. Here we assume in the next time period h , the survival probability of the individual is $({}_h p_{y+t})$, the death probability is $({}_h q_{y+t})$, $({}_h p_{y+t}) = 1 - {}_h q_{y+t}$ and y is the current age of the individual.

Let h be an arbitrarily small time increment, we have

$$J(M_t, W_t, t) \geq ({}_h p_{y+t}) E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + J(M_{t+h}, W_{t+h}, t+h) \right] \\ + ({}_h q_{y+t}) E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \Phi(M_{t+h} + I_{t+h} \lambda_{y+t+h}^{-1}, t+h) \right], \quad (\text{A-1})$$

where $\Phi(\cdot, \cdot)$ is the optimal objective function when there is no wage income and no need for paying insurance premium. By Ito's formula, we have

$$dJ(M_t, W_t, t) = (J_t + A_t^J) dt + \sigma_m M_t J_M dB_t^m + \sigma_W W_t J_W dB_t^W, \quad (\text{A-2})$$

where

$$A_t^J = (\alpha_t(\mu_m - r)M_t + rM_t + W_t - c_t - I_t) J_M \\ + \mu_W W_t J_W + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 J_{MM} + \frac{1}{2} \sigma_W^2 W_t^2 J_{WW} + \alpha_t \rho \sigma_W \sigma_m M_t W_t J_{WM} \quad (\text{A-3})$$

Integration equation (A-2), we obtain

$$J(M_{t+h}, W_{t+h}, t+h) = J(M_t, W_t, t) \\ + \int_t^{t+h} (J_s + A_s^J) ds + \int_t^{t+h} \sigma_m M_s J_M dB_s^m + \int_t^{t+h} \sigma_W W_s J_W dB_s^W. \quad (\text{A-4})$$

Similarly, for Φ we have

$$d\Phi(M_t, t) = (\Phi_t + B_t^\Phi) dt + \sigma_m \alpha_t M_t \Phi_M dB_t^m, \quad (\text{A-5})$$

where

$$B_t^\Phi = ((\mu_m - r)M_t \alpha_t + rM_t - c_t) J_M + \frac{1}{2} \sigma_m^2 \alpha_t^2 M_t^2 J_{MM}. \quad (\text{A-6})$$

Integrating equation (A-5) yields

$$\Phi(M_{t+h}, t+h) = \Phi(M_t, t) + \int_t^{t+h} (\Phi_s + B_s^\Phi) ds + \int_t^{t+h} \sigma_m \alpha_s M_s \Phi_M dB_s^m. \quad (\text{A-7})$$

Combining (A-1), (A-7) and (A-4), we obtain

$$J(M_t, W_t, t)(hq_{y+t}) \geq (hp_{y+t})E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \int_t^{t+h} (J_s + A_s^J) ds \right] \\ + (hq_{y+t})E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \Phi(M_t, t) + \int_t^{t+h} (\Phi_s + B_s^\Phi) ds \right]. \quad (\text{A-8})$$

Dividing by h and rearranging equation (A-8), as $h \rightarrow 0$, we obtain:

$$J(M_t, W_t, t)\lambda_{y+t} \geq J_t + A_t^J + \Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} + e^{-\delta t} u(c_t). \quad (\text{A-9})$$

The equality holds when we take the optimal control:

$$J(M_t, W_t, t)\lambda_{y+t} = \max_{\alpha_t, I_t, c_t} \left(J_t + A_t^J + \Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} + e^{-\delta t} u(c_t) \right). \quad (\text{A-10})$$

Using equations (A-3) and (A-10), after rearrangement, we obtain

$$\lambda_{y+t}J = J_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t J_M) \\ + \max_{I_t} \left(\Phi \left(M + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} - I_t J_M \right) \\ + \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M J_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M^2 J_{MM} + \alpha_t \rho \sigma_W \sigma_m W M J_{WM} \right] \\ + r M J_M + W J_M + \mu_W W J_W + \frac{1}{2} \sigma_W^2 W^2 J_{WW}. \quad (\text{A-11})$$

Here Φ satisfies the following HJB (without wage and insurance)

$$0 = \Phi_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t \Phi_M) \\ + \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M \Phi_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M^2 \Phi_{MM} \right] + r M \Phi_M. \quad (\text{A-12})$$

In order to simplify the presentation, we have dropped the subscript t from M_t and W_t .

B Derivation of the Closed Form Solution

B.1 $\rho = \pm 1$

Motivated by the closed form solution for constant wage Merton (1971), we seek the solution of the HJB (15) in the form

$$J = \frac{h(t)(M + k(t)W)^{1-\gamma}}{1-\gamma}. \quad (\text{B-1})$$

Note that when $W = 0$ (no wage income) this solution reduces to the Merton formula (12). Simple calculation and algebraic manipulation lead to the following expressions

$$\frac{I_t^*}{\lambda_{y+t}M} = kx, \quad (\text{B-2})$$

$$\alpha_t^* = \frac{\mu_m - r}{\gamma\sigma_m^2}(1 + kx) - \beta kx, \quad (\text{B-3})$$

$$\frac{c_t^*}{M} = e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}}(1 + kx) \quad (\text{B-4})$$

where $\beta = \rho\sigma_w/\sigma_m$ and $x = W/M$. From (15), we obtain

$$\begin{aligned} & -\frac{\lambda_{y+t}}{1-\gamma}(1+kx)^2 + \frac{h'}{h}\frac{(1+kx)^2}{1-\gamma} + \frac{k'}{k}(1+kx)kx \\ & + \left[\frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(1+kx) + \frac{\lambda_{y+t}}{1-\gamma}(1+\gamma kx) \right. \\ & \quad \left. + r + x + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2}(1+kx) - \frac{\beta(\mu_m - r)}{2}kx \right] (1+kx) \\ & - \frac{\mu_m - r}{2}\beta(1+kx)kx + \mu_w(1+kx)kx + \frac{\gamma}{2}\sigma_w^2(kx)^2(\rho^2 - 1) = 0. \end{aligned} \quad (\text{B-5})$$

When $\rho = \pm 1$, the last term in the equation above drops out, and when $1 + kx \neq 0$, we have

$$\begin{aligned} & \frac{h'}{h}\frac{1+kx}{1-\gamma} + \frac{k'}{k}kx + \frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(1+kx) - \lambda_{y+t}kx \\ & + r + \frac{1}{k}kx + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2}(1+kx) - (\mu_m - r)\beta kx + \mu_w kx = 0. \end{aligned} \quad (\text{B-6})$$

This can be viewed as a linear algebraic equation for x and for the solution to exist, we must have the coefficients vanish for all x , which leads to the following two first order ordinary differential equations for h and k

$$\frac{1}{1-\gamma}\frac{h'}{h} + \frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}} + r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} = 0, \quad (\text{B-7})$$

$$\frac{k'}{k} + \frac{1}{k} + \mu_w - r - \lambda_{y+t} - \beta(\mu_m - r) = 0. \quad (\text{B-8})$$

Note that (B-7) is the same as (13), and h is simply the Merton solution given by (14). Furthermore, our solution reduces to (12) at $t = T_R$, which yields the terminal conditions for k ,

$$k(T_R) = 0. \quad (\text{B-9})$$

Remark. When (B-5) is viewed as a quadratic equation of x , the solution (B-1) exists when this quadratic equation is of a special form (with a common factor $1 + kx$) since we only

have two degrees of freedom (h and k). Another special case when the quadratic equation has a common factor $1 + kx$ is when $\sigma_w = 0$, in which case we can also eliminate a common factor $1 + kx$. Merton's solution of constant wage (Merton (1971)) belongs to this case when μ_w is also zero.

B.2 $-1 < \rho < 1$

We assume that there exists an asymptotic expansion of the value function in the form $J = J_0 + \epsilon J_1$ where

$$J_0 = \frac{M^{1-\gamma}}{1-\gamma} h(t)(1 + k(t)x)^{1-\gamma},$$

$$J_1 = \frac{M^{1-\gamma}}{1-\gamma} F_1(x, t).$$

In the following we show that h and k are the same functions obtained in Section 4.1, valid for $-1 \leq \rho \leq 1$ as long as ϵ is small.

We proceed as before by substituting J into the HJB equation (15) and collect the terms of the same order of ϵ . At the zeroth order, we have

$$\begin{aligned} & -\frac{\lambda_{y+t}}{1-\gamma}(1+kx)^2 + \frac{h'(1+kx)^2}{h} \frac{1}{1-\gamma} + \frac{k'}{k}(1+kx)kx \\ & + \left[\frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}}(1+kx) + \frac{\lambda_{y+t}}{1-\gamma}(1+\gamma kx) \right. \\ & \quad \left. + r + x + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2}(1+kx) - \frac{\beta(\mu_m - r)}{2}kx \right] (1+kx) \\ & - \frac{\mu_m - r}{2}\beta(1+kx)kx + \mu_w(1+kx)kx = 0, \end{aligned} \tag{B-10}$$

which is (B-6) multiplied by $1 + kx$. At the first order of ϵ , we have

$$\frac{\partial F_1}{\partial t} + \hat{c}_0(1-\gamma)F_1 + \hat{c}_1x \frac{\partial F_1}{\partial x} + \hat{c}_2x^2 \frac{\partial^2 F_1}{\partial x^2} = G \tag{B-11}$$

where \hat{c}_0 , \hat{c}_1 and \hat{c}_2 are similarly defined as c_0 , c_1 and c_2 in Section 4.2 with F replaced by $h(1+kx)^{1-\gamma}$. G is a function of F_1 as well as h and k .

For practically relevant parameter values, as shown by the numerical examples, the zeroth order approximation gives sufficiently accurate results. In principle, if we are interested in a more accurate approximation, we could always use the numerical method outlined earlier.

Table #1: Life Insurance (in Thousands) and Allocation to Stocks						
20 Years Away From Retirement, At Age 45						
		Correlation Between Wages and Stock Market				
Risk Aversion		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	6.06	5.18	4.40	3.69	3.06
Low	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1
$\gamma = 3$	α_t^*	3.63	2.88	2.20	1.59	1.05
Medium	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1
$\gamma = 4.5$	α_t^*	2.82	2.11	1.47	0.894	0.384
High	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1
Parameter Assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$						
$M_0 = 200, W_0 = 50, T_R = 20, T_H = 50, \delta = 2\%, y = 45, m = 86.3, b = 9.5$						

Table #2: Life Insurance (in Thousands) and Allocation to Stocks						
10 Years Away From Retirement, at Age 55						
		Correlation Between Wages and Stock Market				
Risk Aversion		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	3.46	3.09	2.73	2.40	2.09
= Low	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6
$\gamma = 3$	α_t^*	2.03	1.69	1.37	1.06	0.775
= Medium	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6
$\gamma = 4.5$	α_t^*	1.56	1.22	0.911	0.616	0.337
= High	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6
Parameter Assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$						
$M_0 = 200$, $W_0 = 50$, $T_R = 10$, $T_H = 40$, $\delta = 2\%$, $y = 55$, $m = 86.3$, $b = 9.5$						

Table #3: Life Insurance (in Thousands) and Allocation to Stocks?						
30 Years Away From Retirement, at Age 35						
		Correlation Between Wages and Stock Market				
Risk Aversion		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	1570	1270	1010	789	599
= Low	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030
$\gamma = 3$	α_t^*	966	717	505	325	171
= Medium	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030
$\gamma = 4.5$	α_t	765	533	337	170	28.8
= High	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030
Parameter Assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$						
$M_0 = 1, W_0 = 50, T_R = 30, T_H = 60, \delta = 2\%, y = 35, m = 86.3, b = 9.5$						

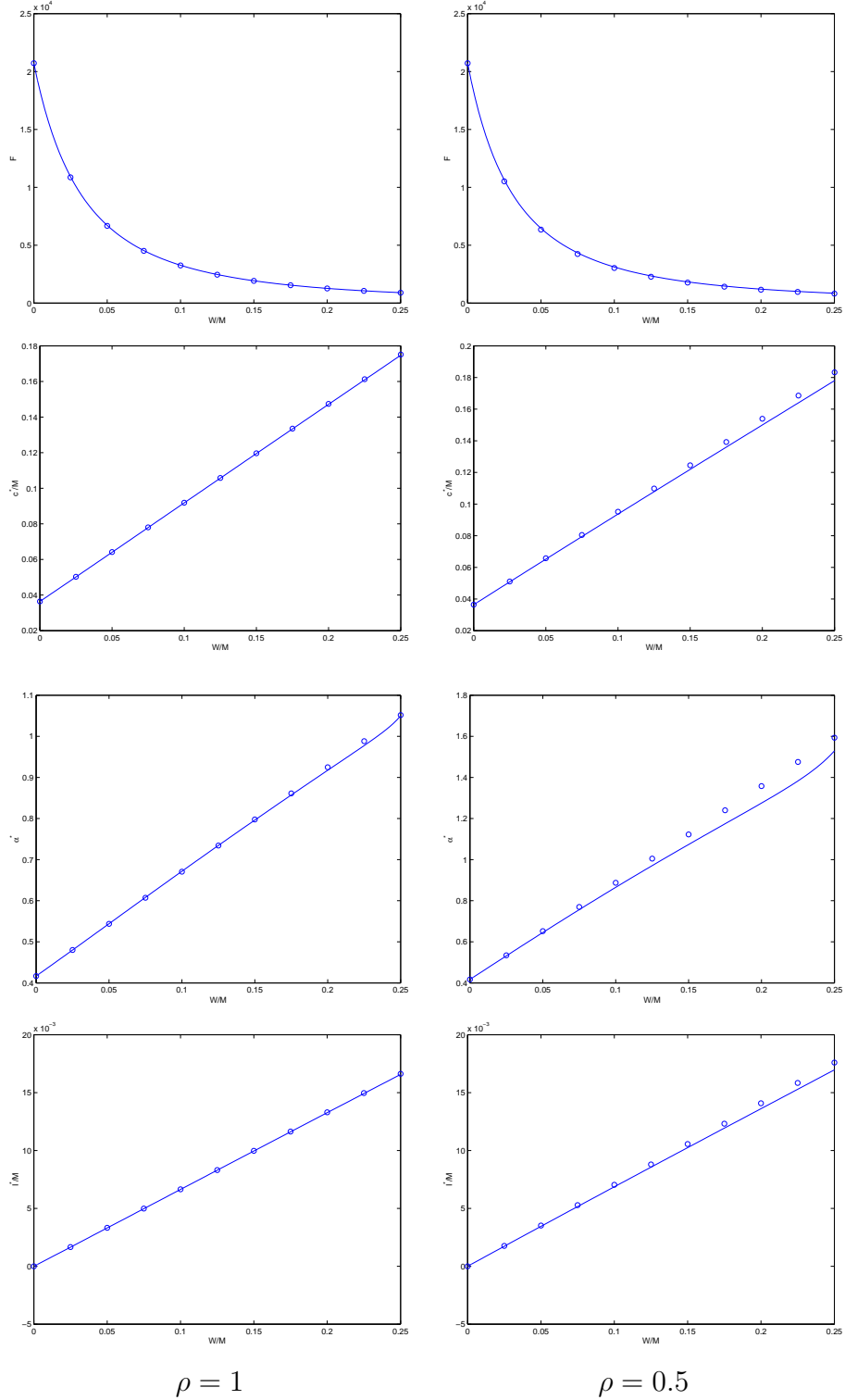


Figure 1: Comparison between numerical (lines) and analytical solutions (symbols). From the top to bottom are the scaled value function F , the optimal consumption ratio, the optimal ratio between the risky and risk-less assets and optimal insurance premium ratio (with respect to the wealth), all as functions of the wage and wealth ratio.