Ruined Moments in Your Life: How Good Are the Approximations?

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Abstract

In this paper we employ numerical PDE techniques to compute the *probability of lifetime ruin* which is the probability that a fixed retirement consumption strategy will lead to financial insolvency under stochastic investment returns and lifetime distribution. Using equity market parameters derived from US-based financial data we conclude that a 65-year-old retiree requires 30 times their desired annual (real) consumption to generate a 95% probability of sustainability – which is equivalent to a 5% probability of lifetime ruin – if the funds are invested in a well-diversified equity portfolio. The 30-to-1 margin of safety can be contrasted with the relevant annuity factor for an inflation-linked income which would obviously generate a *zero* probability of lifetime ruin.

Our paper then goes on to compare our numerical PDE values with various moment matching and other approximations that have been proposed in the literature to compute the lifetime probability of ruin. Our results indicate that the Reciprocal Gamma approximation provides the most accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum, which is consistent with capital market history. At higher levels of volatility the moment matching approximations do break down and we provide some theoretical reasons for this phenomena.

Our numerical and methodological results should be of interest to both academics and practitioners who are interested in methods of approximating stochastic present values as well as methods for computing sustainable consumption and withdrawal rates towards the end of the human life cycle.

KEYWORDS: Annuity, Income, Retirement, Stochastic Present Value

1 Motivation

A number of recent papers in finance and insurance literature have been interested in the probability a retiring individual will exhaust their wealth under a fixed consumption strategy while still alive. This quantity has been coined the *lifetime ruin* probability and has been investigated by Khorasanee (1996), Milevsky and Robinson (2000), Albrecht and Maurer (2002), Gerrard, Haberman and Vigna (2003) and Young (2003) amongst others. The concept of lifetime ruin is at the core of various commercial software packages that provide retirement advice, and a variant of this problem has also been explored within the context of Asian options where the literature is quite extensive. See Goovaerts, Dhaene and de Schepper (2000) for a discussion of the problem from the point of view of stochastic present value functions.

Motivated by the continued interest in the topic, our paper goes back to first principles and employs analytic techniques to represent the *probability of lifetime ruin* as the solution to a Partial Differential Equation (PDE). We then use a Crank-Nickolson scheme to solve this second-order linear PDE.

With a rapid algorithm at our disposal, we apply our procedure using equity market parameters derived from US-based financial data and we conclude that a 65-year-old retiree requires 30 times their desired annual (real) consumption to generate a 95% probability of sustainability – which is equivalent to a 5% probability of lifetime ruin – if the funds are invested in a well-diversified equity portfolio. We provide similar estimates for different ages and under a collection of differing return and volatility assumptions. The 30-to-1 margin of safety can be contrasted with the relevant annuity factor for an inflation-linked income which would obviously generate a *zero* probability of lifetime ruin. Thus, for those retirees who decide to self-annuitize, the lifetime ruin probability can provide a summary *risk metric*.

Our paper then goes on to compare our numerical PDE values with various moment matching and other approximations that have been proposed in the literature to compute the lifetime probability of ruin. We label this a *horse race* with an eye towards testing the robustness of the so-called moment matching methodology – which is explained in the body of the paper – in contrast to approximations which are based on comonotinicity techniques. Our results indicate that the Reciprocal Gamma (RG) approximation provides the most accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum. This volatility range is consistent with capital market history and renders the RG approximation superior for the purpose of approximating lifetime ruin probability. However, we do find that at higher levels of volatility the RG moment matching approximations break down – while the comonotonicity techniques do not – and we provide some theoretical reasons for this phenomena.

The remainder of this paper is organized as follows. Our general model is presented in section 2. The PDE theory and techniques are presented in section 3. In section 4, we provide a variety of numerical approximation techniques for the lifetime probability of ruin. We start with the so-called Reciprocal Gamma approximation – which is based on the work by Milevsky and Robinson (2000) – we then illustrate the same technique using the LogNormal approximation and finally we implement the method proposed by Goovaerts, Dhaene and de Schepper (2000) to compare the various techniques. A broad range of numerical examples are presented in section 5, and the paper concludes in section 6.

2 The Probability of Lifetime Ruin

Without any loss of generality we can scale the problem by assuming a constant consumption rate, taken to be one (real or nominal) for simplicity, with a wealth process that obeys the following stochastic differential equation (SDE):

$$dW_t = (\mu W_t - 1)\, dt + \sigma W_t\, dB_t, \qquad W_0 = w, \tag{1}$$

where μ, σ are the drift and diffusion coefficients and B_t is the Brownian motion driving the process. Note that the net-wealth process defined by equation (1) has a drift $(\mu W_t - 1)$, that *may* become negative if μW_t becomes small enough relative to 1. This, in turn, implies that the process W_t *may* eventually hit zero, in stark contrast to the classical geometric Brownian motion which is bounded away from zero in finite time.

Theorem #1: The net-wealth process W_t, defined by equation (1), can be solved explicitly to yield:

$$W_t = e^{(\mu-\frac{1}{2}\sigma^2)s+\sigma B_s}\left[w - \int_0^t e^{-(\mu-\frac{1}{2}\sigma^2)s-\sigma B_s} ds\right], \qquad W_0 = w. \tag{2}$$

Proof #1: See the book by Karatzas and Shreve (1992, page 361). The proof requires a basic application of the method of variation of coefficients. The solution can be confirmed by applying Ito's Lemma to equation (2) and thus recovering equation (1).

In this paper we are interested in an efficient numerical procedure that will compute three progressive and distinct *ruin probability* values. The first quantity of interest is defined to be:

$$P_1(w, y, t, T \mid \mu, \sigma) := \Pr[W_T \leq y \mid W_t = w], \tag{3}$$

which is the probability that the net-wealth diffusion process W_T will attain a value less than or equal to y, assuming it starts at a value of $W_t = w$ at time $t \geq 0$.

The second quantity of interest is:

$$P_2(w, y, t, T \mid \mu, \sigma) := \Pr[\inf_{t\leq s\leq T} W_s \leq y \mid W_t = w], \tag{4}$$

which is the probability the process W_t *ever* crosses the level of y during the time $[t, T]$.

Finally, the third quantity of interest – and our main objective – represents the *lifetime ruin probability* which is modelled as follows. Let $\mathbf{T}_x$ denote a future lifetime random variable – obviously independent of W_t – with a distribution that is defined to be Gompertz-Makeham (GM) and is parametrized by three variables,

$$\lambda_{x+t} = \lambda + \frac{1}{b} e^{\left(\frac{x+t-m}{b}\right)}, \tag{5}$$

where x denotes the current age of the individual. By definition of the hazard rate function, we have that:

$$
\begin{aligned}
1 - F_x(t) := \Pr[\mathbf{T}_x \geq t] &= e^{-\int_0^t \lambda_{x+s} ds} \\
&= \exp\left\{-\lambda t + b(\lambda_x - \lambda)(1 - e^{t/b})\right\}, \qquad (6)
\end{aligned}
$$

where $F_x(t)$ is the CDF and $f_x(t)$ is the PDF of the random variable $\mathbf{T}_x$. Roughly speaking, one can think of m as the *mode* of the future lifetime and b as a *scale parameter* of $\mathbf{T}_x$. For example, when $\lambda = 0$ and $m = 80$ and $b = 10$, equation (6) stipulates that the probability a current 65-year-old lives to age 85 is: $\Pr[T_{65} \geq 20] = 0.2404$, but the probability that a current 75-year-old lives to age 85 is: $\Pr[T_{75} \geq 10] = 0.3527$. Naturally, the probability of reaching age 85 increases as the individual grows older.

Note some facts about $\mathbf{T}_x$ which will be used later in the analysis. First,

$$\int_0^\infty (1 - F_x(t))\lambda_{x+t} dt = 1, \qquad (7)$$

and therefore a simple application of the chain rule retrieves the convenient relationship:

$$\lambda_{x+t} = \frac{f_x(t)}{1 - F_x(t)}. \qquad (8)$$

Another important (and well known) fact of any future lifetime random variable is that:

$$E[\mathbf{T}_x] = \int_0^\infty t f_x(t) dt = \int_0^\infty \Pr[\mathbf{T}_x \geq t] dt = \int_0^\infty (1 - F_x(t)) dt \qquad (9)$$

Thus, under the above-mentioned parameters of $\lambda = 0$, $m = 80$ and $b = 10$, the life expectancy (median life) at age 65 is 79.18 (79.13) and at age 75 is 83.25 (82.62).

Our third and final probability of ruin is defined as:

$$P_3(w, y, x \mid \lambda, m, b, \mu, \sigma) := \Pr[\inf_{0 \leq s \leq \mathbf{T}_x} W_s \leq y \mid W_0 = w], \qquad (10)$$

which is the probability the process will ever 'hit' a value of y while the random variable $\mathbf{T}_x$ is still alive. This is the so-called *probability of lifetime ruin.*

Theorem #2. The net-wealth stochastic process W_t defined by equation (2) obeys the following property:

$$P_2(w, 0, t, T \mid \mu, \sigma) = P_1(w, 0, t, T \mid \mu, \sigma), \qquad \forall\, T \geq 0 \qquad (11)$$

In other words, the net-wealth process W_t will not cross $y = 0$ more than once. Once it enters the negative region, it stays there.

Proof #2: Equation (2) contains two parts, an exponential function which is strictly greater than zero, multiplied by a term in square brackets whose sign is indeterminate. Therefore, the process W_t,will be less than or equal to zero (ruin) at some future time T, *if, and only if*, the term in square brackets is less than or equal to zero. In other words,

$$W_T \leq 0 \qquad \Longleftrightarrow \qquad w \leq \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds. \qquad (12)$$

On the other hand, the integral term is monotonically non-decreasing with respect to the upper bound of integration T. This means that once it becomes greater than w, it *stays* greater than w. Consequently, we arrive at our result that the probability W_t crosses zero prior to a deterministic time T is equivalent to the probability that $W_T \leq 0$.

Given the result from Theorem #2 applied to any fixed value of T, we can generalize to a relationship involving the lifetime ruin probability P_3. Namely,

$$\Pr[\inf_{0\leq s\leq \mathbf{T}_x} W_s \leq 0 \mid W_0 = w] = \Pr\left[\int_0^{\mathbf{T}_x} e^{-(\mu-\frac{1}{2}\sigma^2)s-\sigma B_s} ds \geq w\right] \tag{13}$$

Our moment matching (MM) methodology will be based on approximating the integral in equation (13) with a *suitably close* random variable that share the first few moments with the true (density unknown) variable. This random variable can be interpreted as the stochastic present value of lifetime consumption of \$1 per annum. More on this later.

The analytic approach which leads to a PDE representation will be based on the following analysis of the problem. Motivated by the structure of $\mathbf{T}_x$ we define a *future ruin time* random variable $\mathbf{R}_w^y$ which captures the amount of time it takes for the net-wealth process W_t to 'die' – which is to hit the value of y – assuming it starts at an initial value of $W_t = w$. Note that $\mathbf{R}_w^y$ is independent of the future lifetime random variable $\mathbf{T}_x$. Using our previous notation the formal definition of $\mathbf{R}_w^y$ satisfies:

$$\Pr[\mathbf{R}_w^y \leq t] := P_2(w, y, 0, t \mid \mu, \sigma). \tag{14}$$

¿From this perspective it should become clear that the lifetime ruin probability P_3 – which is the focus of our analysis – can be represented as follows:

$$P_3(w, y, x \mid \lambda, m, b, \mu, \sigma) = \Pr[\mathbf{R}_w^y \leq \mathbf{T}_x]. \tag{15}$$

It is the probability that the net-wealth process W_t gets ruined *before* the individual dies. We have now transformed the problem to one of computing the cumulative density function (CDF) of the new random variable $\mathbf{R}_w^y - \mathbf{T}_x$, and evaluating this CDF at zero. And, given the natrual independence between $\mathbf{R}_w^y$ and $\mathbf{T}_x$, this becomes a simple exercise in probability *convolutions*.

Akin to the future lifetime random variable, let $G_w(t) = \Pr[\mathbf{R}_w^y \leq t]$ denote the CDF in question. We then define the probability density function (PDF) of $\mathbf{R}_w^y$ via:

$$g_w(t) = \frac{\partial G_w(t)}{\partial t} = \frac{\partial P_2(w, y, 0, t \mid \mu, \sigma)}{\partial t}. \tag{16}$$

Note that for $g_w(t)$ to be a proper density function – so that it integrates to a value of one – we must add a probability mass of $1 - P_2(w, y, 0, \infty \mid \mu, \sigma)$ at $g_w(\infty)$, which is the probability the wealth process W_t *never* hits a value of y. In this way, we obtain:

$$\int_0^{\infty} g_w(t)dt + (1 - P_2(w, y, 0, \infty \mid \mu, \sigma)) = 0. \tag{17}$$

Theorem #3. If two independent random variables X_1 and X_2 have respective PDFs of $f_1(x)$ and $f_2(x)$, then the PDF $f_3(x)$ of the sum of these two random variables $X_3 = X_1 + X_2$, is given by

$$f_3(y) = \int_{-\infty}^{\infty} f_1(y-z) f_2(z) dz, \tag{18}$$

which leads to:

$$\Pr\left[X_3 \leq 0\right] = \int_{-\infty}^{0} \int_{-\infty}^{\infty} f_1(y-z) f_2(z) dz dy \tag{19}$$

Proof #3: Any probability textbook.

In our case, $f_1(x)$ would denote the PDF of future ruin time random variable $\mathbf{R}_w^y$ and $f_2(x)$ would denote the PDF of the negative value of the future lifetime random variable $-\mathbf{T}_x$. The quantity $\Pr[X_3 \leq 0]$ is precisely the probability of lifetime ruin P_3.

Finally, applying some chain-rule calculus to the right-hand-side of equation (19), we are left with:

$$\Pr[\inf_{0 \leq s \leq \mathbf{T}_x} W_s \leq 0 \mid W_0 = w] = \int_0^{\infty} g_w(t)(1 - F_x(t)) dt. \tag{20}$$

One can heuristically think of the integral as 'adding up' the probability of ruin at t, weighted by the probability the individual will survive to this time. Technically, we should add the value $g_w(\infty)$ to the convolution, but since it is weighted by a zero probability of future lifetime survival, we have omitted this term. Note also that in some literature the symbol $({}_t p_x)$ is used to represent $1 - F_x(t)$, which is the conditional survival probability.

3 P.D.E. Representation and Numerical Methods

The ruin probabilities defined in equation (3) and equation (4) are also known as the transition and exit probabilities. It can be shown that they both satisfy the Kolmogorov backward equation, see for example, Bjork (1998).

$$P_t + (\mu w - 1) P_w + \frac{1}{2}\sigma^2 w^2 P_{ww} = 0 \tag{21}$$

with a terminal condition

$$P(w_T, T) = 1 - H(w_T - y) \tag{22}$$

where $H(w)$ is the Heaviside function and w_T is the wealth at T.

The analytic difference between P_1 and P_2 lies in the relevant boundary condition. For P_2 it is obvious that $P_2 = 1$ at $w \leq y$. On the other hand P_1 is non-zero for all $w > 0$. When $w = 0$, we observe that the process defined by (1) implies $dw_t \leq 0$, thus w will remain negative and the proper boundary condition is $P_1 = 1$ at $w \leq 0$. Finally, when $w \to +\infty$, both P_1 and P_2 approach zero.

The next step is to re-scale the problem and realize that:

$$\Pr[\inf_{t \leq s \leq \mathbf{T}_x} W_s \leq y \mid W_t = w] = \Pr[\inf_{0 \leq s \leq \mathbf{T}_{x+t}} W_s \leq y \mid W_0 = w], \tag{23}$$

and therefore

$$P_3(w, y, x+t \mid \lambda, m, b, \mu, \sigma) = \Pr[\inf_{t \le s \le \mathbf{T}_x} W_s \le y \mid W_t = w] \tag{24}$$

from the original definition of the lifetime ruin probability. On the other hand, we also know that

$$P_3(w, y, x+t \mid \lambda, m, b, \mu, \sigma) = \int_0^\infty P_2(w, y, 0, \tau) f_{x+t}(\tau) d\tau. \tag{25}$$

And, from the definition of $f_x(\tau)$ it can be easily verified that

$$f_{x+t}(\tau) = \frac{f_x(t+\tau)}{1 - F_x(t)}. \tag{26}$$

Thus

$$\begin{aligned} P_3(w, y, x+t \mid \lambda, m, b, \mu, \sigma) &= \frac{1}{1 - F_x(t)} \int_0^\infty P_2(w, y, 0, \tau) f_x(\tau + t) d\tau \\ &= \frac{1}{1 - F_x(t)} \int_t^\infty P_2(w, y, t, \tau) f_x(\tau) d\tau. \end{aligned} \tag{27}$$

Some algebraic manipulations leads us to the following expressions for the partial derivatives:

$$\begin{aligned} \frac{\partial P_3}{\partial t} &= -\frac{f_x(t)}{1 - F_x(t)} P_3 - P_2(w, y, t, t|\mu, \sigma) f_x(t) + \int_t^\infty \frac{\partial}{\partial t} P_2(w, y, 0, \tau|\mu, \sigma) f_x(\tau) d\tau \\ &= \lambda_{x+t} P_3 + \int_t^\infty \frac{\partial}{\partial t} P_2(w, y, 0, \tau|\mu, \sigma) f_x(\tau) d\tau, \\ \frac{\partial P_3}{\partial w} &= \int_t^\infty \frac{\partial}{\partial w} P_2(w, y, 0, \tau|\mu, \sigma) f_x(\tau) d\tau, \\ \frac{\partial^2 P_3}{\partial w^2} &= \int_t^\infty \frac{\partial^2}{\partial w^2} P_2(w, y, 0, \tau|\mu, \sigma) f_x(\tau) d\tau. \end{aligned} \tag{28}$$

Note that we have used the identity $f_x(t) = d(1 - F_x(t))/dt = -(1 - F_x(t))\lambda_{x+t}$. Thus P_3 satisfies the following backward equation:

$$P\lambda_{x+t} = P_t + (\mu w - 1) P_w + \frac{1}{2}\sigma^2 w^2 P_{ww}, \tag{29}$$

with the following terminal condition:

$$P(w_\infty, \infty) = 1 - H(w_\infty - y), \tag{30}$$

where λ_{x+t} is the hazard function which is defined by equation (5). The PDE in equation (29) has also been derived by Young (2003) within the context of controlling the net-wealth diffusion to minimize the probability of lifetime ruin.

3.1 Ruin Probability when $T \to \infty$

When $T \to \infty$ which can be viewed as the perpetuity case, the solution for (21) is independent of t and given by the following ODE:

$$(\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2 \frac{\partial^2 P}{\partial w^2} = 0. \tag{31}$$

Note that we have dropped the subscript on P since the equation is the same for both P_1 and P_2. The solution of this ODE can be obtained by integration with respect to w twice, and result in:

$$P = C \int_{1/w}^{\infty} e^{-av} v^{b-1} dv + D \tag{32}$$

where C and D are two constants, $a = 2/\sigma^2$ and $b = 2\mu/\sigma^2 - 1$.

Applying the boundary conditions for P_1 and P_2 yields

$$P_1 = \Gamma(a/w, b), \quad P_2 = \frac{\Gamma(a/w, b)}{\Gamma(a/y, b)}. \tag{33}$$

where

$$a = \frac{2\mu}{\sigma^2} - 1, \quad b = \frac{2}{\sigma^2}, \tag{34}$$

and

$$\Gamma(a, z) = \int_{z}^{\infty} e^{-t} t^{a-1} dt. \tag{35}$$

This closed-form analytic representation for the ruin probability is not new – indeed, it has been 'discovered' by a variety of authors in the actuarial, finance and insurance literature – and simply serves to confirm our PDE representation.

3.2 Numerical Scheme

In equation (21),the ruin probability $P(w, t)$ satisfies a second order linear partial differential equation. We solve this equation by a θ-method which can be written as follows:

$$\begin{aligned} &\frac{P_j^{(n+1)} - P_j^{(n)}}{\delta t} + (\mu w_j - 1)\left(\theta \frac{P_{j^*}^{(n+1)} - P_{j^*-1}^{(n+1)}}{\delta w} + (1-\theta)\frac{P_{j^*}^{(n)} - P_{j*-1}^{(n)}}{\delta w}\right) \\ &+ \frac{\sigma^2 w_j^2}{2}\left(\theta \frac{P_{j+1}^{(n+1)} + P_{j-1}^{(n+1)} - 2P_j^{(n+1)}}{\delta w^2} + (1-\theta)\frac{P_{j+1}^{(n)} + P_{j-1}^{(n)} - 2P_j^{(n)}}{\delta w^2}\right) = 0, \end{aligned} \tag{36}$$

where $P_j^{(n)}$ is a grid function which approximates $P(w, t)$ on the grid points $(w_j,\ t_n)$. A uniform grid with equal spacing δt and δx is used. The parameter θ can be arbitrarily selected, but when $\theta = 1/2$ it corresponds to a second order Crank-Nickolson scheme. An upwind scheme is used for the first order derivative P_w, where the variable j^* is either j or $j + 1$, depending on the sign of the coefficient.

For any implicit method where $0 < \theta \leq 1$, numerical boundary conditions must be provided on the computational boundaries $j = 0$ and $j = J$. This can be derived as:

$$P_0^n = 1, \qquad j = 0 \qquad \text{and} \qquad P_J^n = 0, \qquad j = J \tag{37}$$

$j = 0$ and $j = J$ correspond to the $w_0 = 0$ and $w_J = W$ which are the boundaries of the truncated computation domain for calculating the probability in equation (3), which is P_1. Likewise, for calculating the probability in equation (4), P_2 and the relevant equation (10) for P_3, we use $j = 0$ and $j = J$ with respect to the $w_0 = y$ and $w_J = W$. There are the boundaries of the truncated computation domain. The terminal condition is:

$$P_j^N = 1 - H(w_j - y). \tag{38}$$

With these boundary conditions and the terminal conditions the discrete equations can be solved by matching from time t_n to t_{n+1}, starting from $n = 0$. At t_{n+1}, the equations for $P_j^{(n+1)}$ can be arranged from equation (36). In this space, we can solve for all the probabilities by iteration. For equation (29), we can apply the same method.

4 Analytic Approximations

4.1 Moment Matching for Deterministic T

Using equation (3), equation (4) and Theorem 2 when the ruin level $y = 0$, we can represent our $P_1 = P_2$ probability as:

$$\begin{aligned} P_1 &= \Pr[W_T \leq 0 \mid W_0 = w] \\ &= P_2 = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 \mid W_0 = w] \\ &= \Pr\left[w \leq \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds\right], \end{aligned} \tag{39}$$

which is equivalent to the probability that the stochastic present value is greater than w. We therefore define the stochastic present value random variable as:

$$\mathbf{Z}_T = \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds, \tag{40}$$

and attempt to approximate the (unknown) distribution of this random variable $\mathbf{Z}_T$ by an approximating density curve. The approximating density will be selected so that it's first two moments are identical to the first two moments of the true random variable $\mathbf{Z}_T$. By constructing the 'approximator' in this way, we hope to create a measure of closeness between the two. While the algebra is somewhat tedious, the first moment of $\mathbf{Z}_T$ is:

$$M_1 = E[\mathbf{Z}_T] = \frac{1}{\mu - \sigma^2 - \rho} - \frac{e^{-(\mu - \sigma^2 - \rho)T}}{\mu - \sigma^2 - \rho}, \tag{41}$$

and second moment is:

$$\begin{aligned} M_2 &= E[\mathbf{Z}_T^2] \\ &= \frac{2}{(\mu - 2\sigma^2 - \rho)(\mu - \sigma^2 - \rho)}(1 - e^{-(\mu-\sigma^2-\rho)T}) \\ &+ \frac{2}{(\mu - 2\sigma^2 - \rho)(2\mu - 3\sigma^2 - 2\rho)}(e^{-(2\mu-3\sigma^2-2\rho)T} - 1) \end{aligned} \tag{42}$$

4.2 Moment Matching for Stochastic T

Following the representation derived in equation (13) we now compute the first two moments of the stochastic present value when the terminal horizon is stochastic. In this case the random variable is defined as:

$$\mathbf{Z}_{\mathbf{T}_x} = \int_0^{\mathbf{T}_x} e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds. \tag{43}$$

We intend to 'moment match' the stochastic present value $\mathbf{Z}_{\mathbf{T}_x}$ to both the Reciprocal Gamma (RG) distribution and the LogNormal (LN) distribution. Our assumption remains that the future lifetime random variable is Gompertz-Makeham distributed and is independent of the Brownian motion driving the investment return process. We start with

$$\mathbf{Z}_t = \int_0^{t} e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s} ds, \tag{44}$$

Using the rules for conditional expectations, we know that:

$$E\left[\mathbf{Z}_{\mathbf{T}_x}\right] = E\left[E\left[\mathbf{Z}_t \mid \mathbf{T}_x = t\right]\right] = E\left[E\left[\mathbf{Z} \mid \mathcal{F}_\infty^B\right]\right], \tag{45}$$

where $\mathcal{F}_\infty^B$ is the sigma field generated by the entire path of the Brownian motion. Using the moment generating function for the normal random variable $E\left[\exp\left\{-\sigma B_s\right\}\right] = \exp\{\frac{1}{2}\sigma^2 s\}$, we obtain that:

$$\begin{aligned} E\left[\mathbf{Z}_{\mathbf{T}_x}\right] &= E\left[E\left[\int_0^t \left(\exp\left\{(\mu - \frac{1}{2}\sigma^2)s + \sigma B_s\right\}\right)^{-1} ds\right] \mid \mathbf{T}_x = t\right] \\ &= E\left[\int_0^t E\left[\exp\left\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\right\}\right] ds \mid \mathbf{T}_x = t\right] \\ &= E\left[\int_0^t \exp\{-(\mu - \frac{1}{2}\sigma^2)s\} E\left[\exp\left\{-\sigma B_s\right\}\right] ds \mid \mathbf{T}_x = t\right] \\ &= E\left[\int_0^t \exp\{-(\mu - \frac{1}{2}\sigma^2)s\} \exp\{\frac{1}{2}\sigma^2 s\} ds \mid \mathbf{T}_x = t\right] \\ &= \int_0^\infty \exp\left\{-\left(\mu - \sigma^2\right) s\right\} {}_s p_x \, ds. \end{aligned} \tag{46}$$

We define the function,

$$A(\xi \mid m, b, x) := \int_0^\infty \exp\left\{-\xi s\right\} {}_s p_x ds, \tag{47}$$

which is a form of present value operator. Indeed, after substituting the Gompertz $({}_sp_x)$ from equation (6) and changing variables, we get:

$$A(\xi \mid m, b, x) = b \exp\left\{\exp\left\{\frac{x-m}{b}\right\} + (x-m)\xi\right\} \Gamma\left(-b\xi\,, \exp\left\{\frac{x-m}{b}\right\}\right), \tag{48}$$

where $\Gamma(u,v) = \int_v^\infty e^{-t}t^{(u-1)}dt$ once again denotes the incomplete Gamma function. By construction, the term $A(\xi \mid m, b, x)$ in equation (47) coincides with the Gompertz price of a life-annuity under a continuously compounded force of interest ξ. Thus, the expectation of the stochastic present value of lifetime consumption is:

$$M_1 = E\left[\mathbf{Z}_{\mathbf{T}_x}\right] = A(\mu - \sigma^2 \mid m, b, x), \tag{49}$$

which is the first (non-central) moment. The same technique, as detailed in equation (46), can be employed to obtain all higher non-central moments of the stochastic variable $\mathbf{Z}_{\mathbf{T}_x}$. The second moment is:

$$M_2 = E\left[\mathbf{Z}^2_{\mathbf{T}_x}\right] = \left(\mu/2 - \sigma^2\right)\left(A(\mu - \sigma^2 \mid m, b, x) - A(2\mu - 3\sigma^2 \mid m, b, x)\right). \tag{50}$$

The RG and LN approximation method requires the first two moments.

4.3 Reciprocal Gamma Approximation.

The first and second moments of the Reciprocal Gamma random variable are: $M_1 = 1/(\beta(\alpha - 1))$ and $M_2 = 1/(\beta^2(\alpha-1)(\alpha-2))$ respectively. We can then invert the first two moments and express the 'fitted' variables α and β in terms of M_1 and M_2. They are:

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \qquad \beta = \frac{M_2 - M_1^2}{M_2 M_1}, \tag{51}$$

where M_1 and M_2 are taken from equations (49) and (50) when we are approximating P_3 and they are taken from equations (3) and (4) when we are approximating P_2.

In either event, the stochastic variable

$$\mathbf{Z}_\eta = \int_0^\eta \exp\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\}ds, \tag{52}$$

can be approximated by the Reciprocal Gamma (RG) density function. When we are examining the fixed horizon we use $\eta = T$ and when we are examining the random lifetime horizon we use $\eta = \mathbf{T}_x$. Thus, the probability of lifetime can be approximated by:

$$\Pr[\inf_{0 \le s \le \mathbf{T}_x} W_s \le 0 \mid W_0 = w] \cong \mathbf{G}(1/w \mid \alpha, \beta) \tag{53}$$

where α and β are defined by equation (51). The justification for the RG approximation derives from the limiting arguments provided by equation (33). We refer the interested reader to Milevsky and Robinson (2000) for a similar and more elaborate discussion of this approximation. The current paper is concerned mainly with robustness issues when compared against the PDE values.

4.4 LogNormal Approximation.

We can approximate the unknown distribution of the random variable $\mathbf{Z}_t$ in equation (40) by the LogNormal (LN) distribution instead of the Reciprocal Gamma distribution. The LogNormal density is ubiquitous in the finance literature and is actually used by many practitioners to approximate stochastic present values. Based on a LogNormal 'approximator' the first two moments of the random variable $\mathbf{Z}_t$ are linked via:

$$M_1 = E[\mathbf{Z}_t] = e^{\alpha + \frac{1}{2}\beta^2}, \tag{54}$$

and

$$M_2 = E[\mathbf{Z}_t^2] = e^{2\alpha + 2\beta^2}, \tag{55}$$

where α and β are the two free parameters (or degrees of freedom) available for the LN distribution. Our numerical examples which we present and compare in the next section will employ the LN approximation exclusively for the P_2 (fixed T) case and thus by equation (3) and (4) we can obtain yet another approximation:

$$\Pr[W_T \leq 0 \mid W_0 = w] = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 \mid W_0 = w] \cong 1 - \Phi\left[\frac{\ln(w) - \alpha}{\beta}\right], \tag{56}$$

where Φ is the cumulative distribution function of a standard normal distribution.

4.5 Comonotonicity Approximation.

A series of recent papers starting with Goovaerts, Dhaene and de Schepper (2000) uses comonotonicity arguments to obtain generally defined upper and lower bounds for the stochastic present value of a series of life contingent payments. According to this approach, the distribution of the random variable $\mathbf{Z}_T$ in equation (40) can be approximated by a new random variable $\tilde{\mathbf{Z}}_T$ which is defined by:

$$\tilde{\mathbf{Z}}_T := \int_0^T e^{-(\mu - \frac{1}{2}\sigma^2)s - \sigma\sqrt{s}\Phi^{-1}(\mathcal{U})} ds, \tag{57}$$

where Φ is the cumulative distribution function of a standard normal distribution and $\mathcal{U}$ is a random variable uniformly distributed on the unit interval [0,1].

The survival function of $\tilde{\mathbf{Z}}$ is:

$$\Pr[\tilde{\mathbf{Z}}_T > w] = \Phi(z_w), \tag{58}$$

were z_w is the root of the equation

$$\sum_{i=1}^{n} e^{-(\mu - \frac{1}{2}\sigma^2)idt - \sigma\sqrt{idt}z_w} \Delta t = w, \tag{59}$$

and where $\Delta t = T/n$. Using this approach, from equations (3) and (4) we obtain:

$$\Pr[W_T \leq 0 \mid W_0 = w] = \Pr[\inf_{0 \leq s \leq T} W_s \leq 0 \mid W_0 = w] \cong \Pr[\tilde{\mathbf{Z}}_T > w]. \tag{60}$$

Note that this particular approximation technique has only been proposed and implemented within the context of a fixed (non-stochastic) time T, and we therefore only present results for P_2. It is an open question whether the same arguments can be used to derive similar lower bounds when the horizon itself is a random variable. We refer the interested reader to the work generated by Goovaerts et. al. (2000) in which the justifications, theoretical origins and limitations of this technique are discussed at length.

5 Numerical Examples and Comparison

We now have the ability to compute some explicit ruin probabilities as well as comparing the performance of various approximation methods. Note once again that with the moment matching method (and comonotonicty techniques) it is only possible to calculate the ruin probabilities P_2 and P_3 when $y = 0$ since the *stochastic present value* representation is only defined when the ruin is set at zero.

Our first table illustrates the difference between the probability of the net-wealth process hitting the level y at any time prior to maturity (P_2) and the probability of the process being under level y at maturity. The table also illustrates our claim in Theorem 2 that both probabilities are identical when $y = 0$. Of course, when $y \neq 0$, the ruin probability of equation (4) is greater than that of equation (3).

[Table 1 goes here]

Table 1 displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$.

[Figure 1 goes here]

Figure 1 displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$.

[Figure 2 goes here]

Figure 2 displays the probability that an individual who is 65 years old with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within his lifetime (P_3), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. The mortality parameters are based on a Gompertz approximation with $m = 87.8$ and $b = 9.5$. Thus, for example, there is effectively a 100% probability that wealth will be drawn-down; and will hit 10 dollars while the individual is still alive.

[Figure 3 goes here]

Figure 3 displays the minimum initial wealth level at various ages, that is needed in order to maintain the lifetime ruin probability at 1%, 5% and 10% respectively. Thus, for example, a 70 year old would require $w = 40$ to sustain a 1 dollar per annum consumption rate for life, with a 99% probability, but would only require $w = 27$ to sustain this with a 95% probability. The capital market and mortality parameters are as in Table 2.

5.1 Ruin Probabilities P_1, P_2 and P_3

We now provide some explicit results for the RG approximations and compare those to the (true) PDE values to obtain some reasonable estimates for actual ruin probabilities.

[Figure 4 goes here]

Figure 4 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing levels of expected investment return. Note that the approximate RG value is always greater than the PDE value; i.e. the approximation overstates the ruin probability – and this gap (bias) is an increasing function of volatility. Note the assumption that initial wealth is $w = 12$ and the terminal horizon is $T = 20$ years.

[Figure 5 goes here]

Figure 5 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing terminal horizons and assuming an expected growth rate of $\mu = 12\%$ and an initial wealth of $w = 12$. Once again the approximate RG value is always greater than the PDE value and this gap (bias) is an increasing function of volatility. But note that for levels of volatility under 30%, the RG approximation produces values that are virtually indistinguishable from the PDE values.

[Figure 6 goes here]

Figure 6 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the ruin probability (P_2) as a function of investment volatility, for differing levels of initial wealth and assuming a $T = 30$ year horizon and $\mu = 15\%$. Note that for levels of volatility under 30%, the RG approximation produces values that are virtually indistinguishable from the PDE values, but at higher levels of volatility that approximation is worse the higher the level of initial wealth.

[Figure 7 goes here]

Figure 7 displays the ruin probability as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$ and a terminal horizon of $T = 30$ years. Note that as the volatility increases beyond 30%, the gap in the estimated versus the precise numerical value increases. At very high levels of volatility, the RG approximation breaks down with the ruin probability being given as 100%, which in fact it is much lower.

[Figure 8 goes here]

Figure 8 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility, for differing levels of expected investment return. Although the discrepancy is an increasing function of volatility, it is a decreasing function of the expected investment return. The calculations assume that initial wealth is $w = 12$ and the individual is 65 years old with mortality specified by the Gompertz distribution with $m = 87.8$ and $b = 9.5$.

[Figure 9 goes here]

Figure 9 displays the discrepancy between the Reciprocal Gamma (RG) approximation and the numerical PDE solution for the lifetime ruin probability (P_3) as a function of investment volatility, for differing levels of initial wealth. We assume the same parameters as in Figure 8.

[Figure 10 goes here]

Figure 10 displays the lifetime ruin probability for an individual aged 65, as a function of volatility using the numerical PDE method and the approximate Reciprocal Gamma method assuming an initial wealth of $w = 12$ an expected investment return of $\mu = 12\%$.

5.2 A Horse Race: LN, LB, RG and PDE

In this section we compare and contrast the various approximations that have been described in the earlier sections and examine how they perform when benchmarked against the (true) PDE solution.

[Figure 11 goes here]

Figure 11 compares the results from a variety of methods for computing the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$. Notice that the LB method understates the ruin probability at low levels of initial wealth, and then overstates at higher levels.

[Figure 12 goes here]

Figure 12 displays the ratio of the various approximations to the precise numerical estimate for the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are the same as Figure 11. Note that the RG approximation results are closest to 1 and hence it is a better approximation compared to either the LN or the LB method for these (historical) levels of volatility.

[Table 2 goes here]

Table 2 compares the probability than an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. The deviation of the three approximation methods from the PDE value is listed in brackets. Note that the market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$, which correspond to long-run historical values for these parameters in real (after-inflation) terms.

[Table 3 goes here]

Table 3 compares the probability than an individual with an initial wealth of $w = 15$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. Notice that the ruin probabilities are uniformly higher the lower the level of initial wealth. The capital market parameters are the same as Table 2.

[Table 4 goes here]

Table 4 compares the probability than an individual with an initial wealth of $w = 10$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. The capital market parameters are the same as Table 2.

[Table 5 goes here]

Finally, Table 5 reports a particular result from the LB approximation assuming different discretization schemes. We start with the case where each period is exactly one year – which is the situation reported by Goovaerts et. al. (2000) – and then show the results for more quarterly, monthly, weekly and daily compounding. Note that as one would expect, as n gets large the probabilities converge.

6 Conclusion.

With today's advanced computing power – and the intellectual simplicity of simulation – it is quite easy to fall-back on Monte Carlo techniques to derive all forms of *lifetime ruin* probabilities. This is especially common amongst practitioners who are interested in quick-and-dirty heuristic approximations. In this paper we have shown how to formulate and then numerically solve the PDE representation of the lifetime ruin probability; a quantity which has been investigated by numerous authors in the finance and insurance literature.

Using reasonable parameter values for lifetime and financial market uncertainty we compared the performance with various moment matching and bounding approximations. Our results indicate that under realistic growth rate assumptions the Reciprocal Gamma approximation provides the most accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum. These parameters are consistent with historical capital market experience. However, at higher levels of volatility the RG approximation breaks down. Further research will apply this approach to more complex models for investment returns and consumption strategies.

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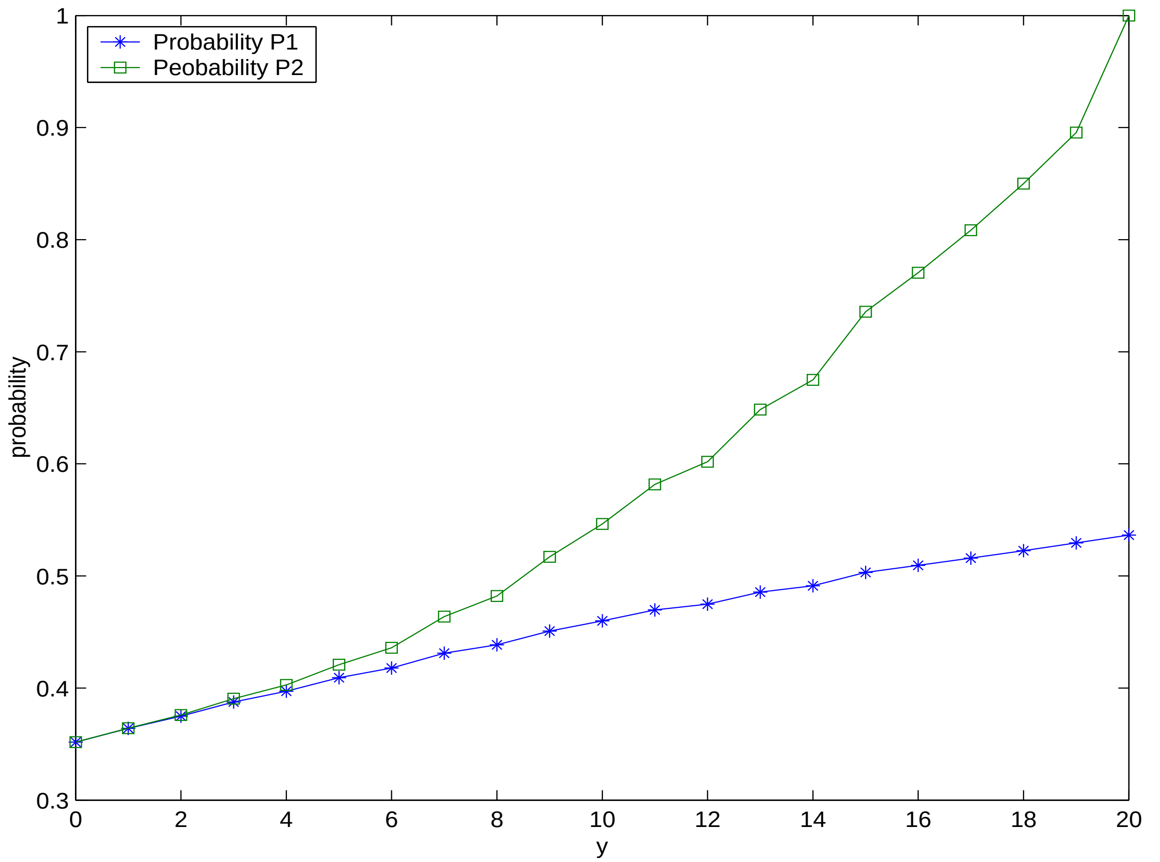


Figure 1: The figure displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$

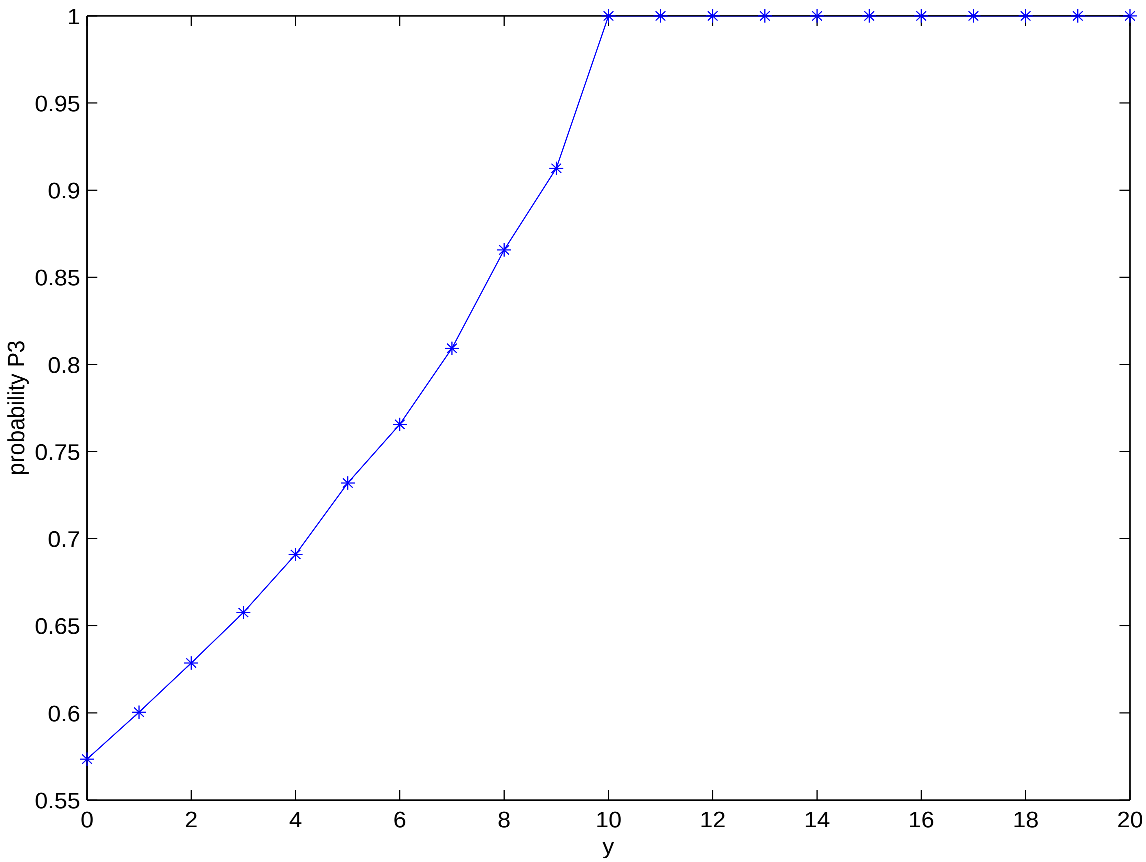
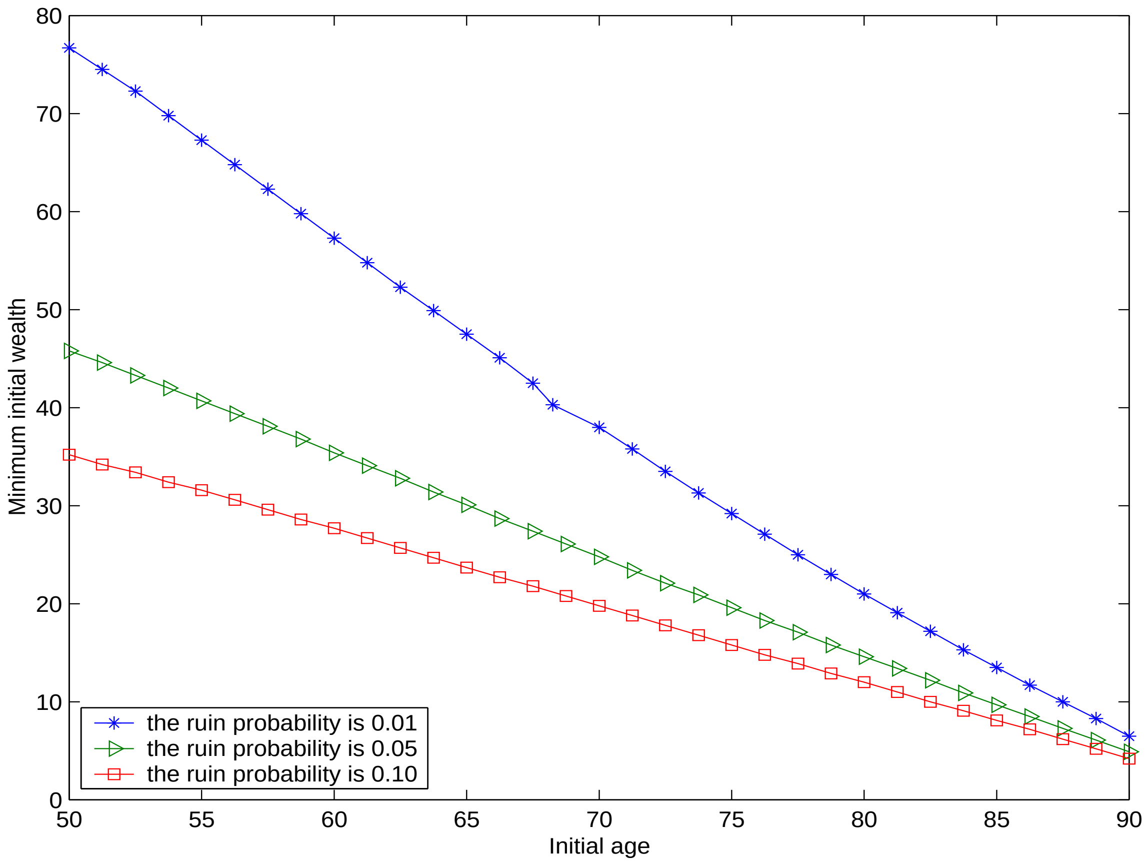


Figure 2: The figure displays the probability that an individual who is 65 years old with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within his lifetime (P_3), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$. The mortality parameters are based on a Gompertz approximation with $m = 87.8$ and $b = 9.5$. Thus, for example, there is effectively a 100% probability that wealth will be drawn-down; and will hit 10 dollars while the individual is still alive.



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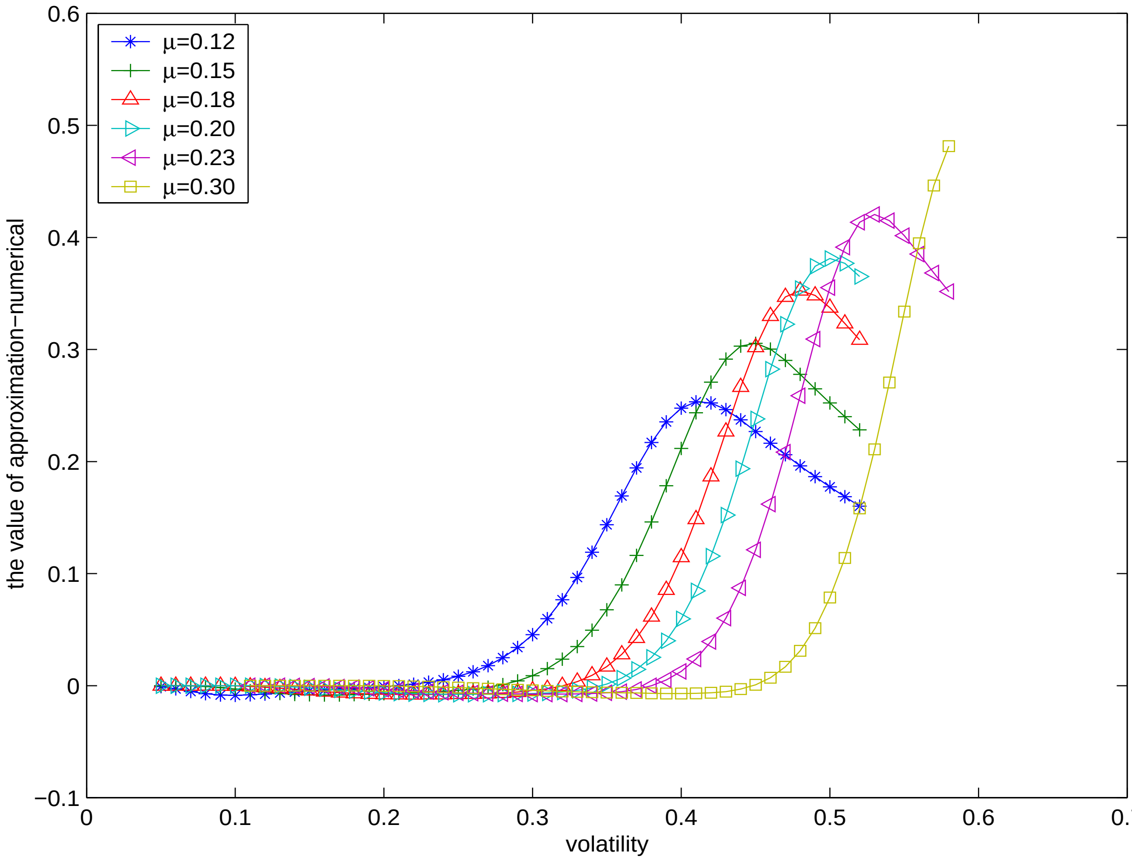

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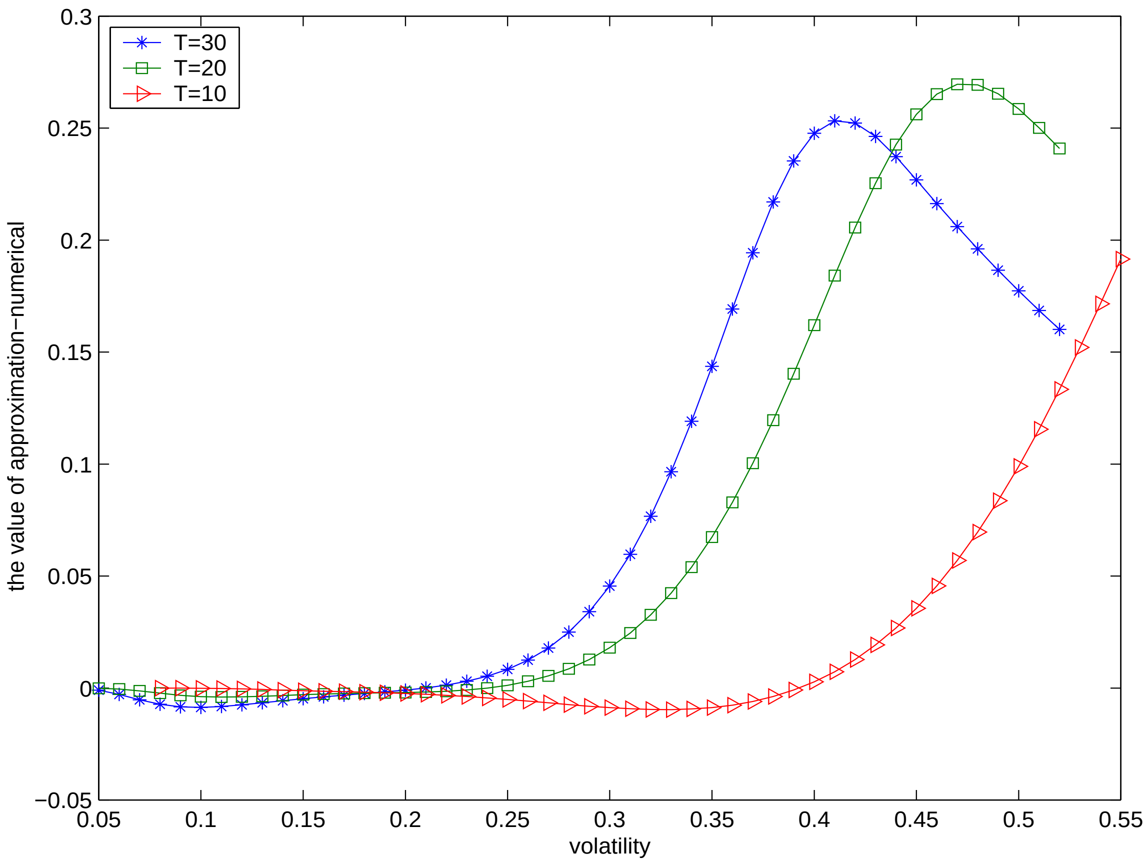

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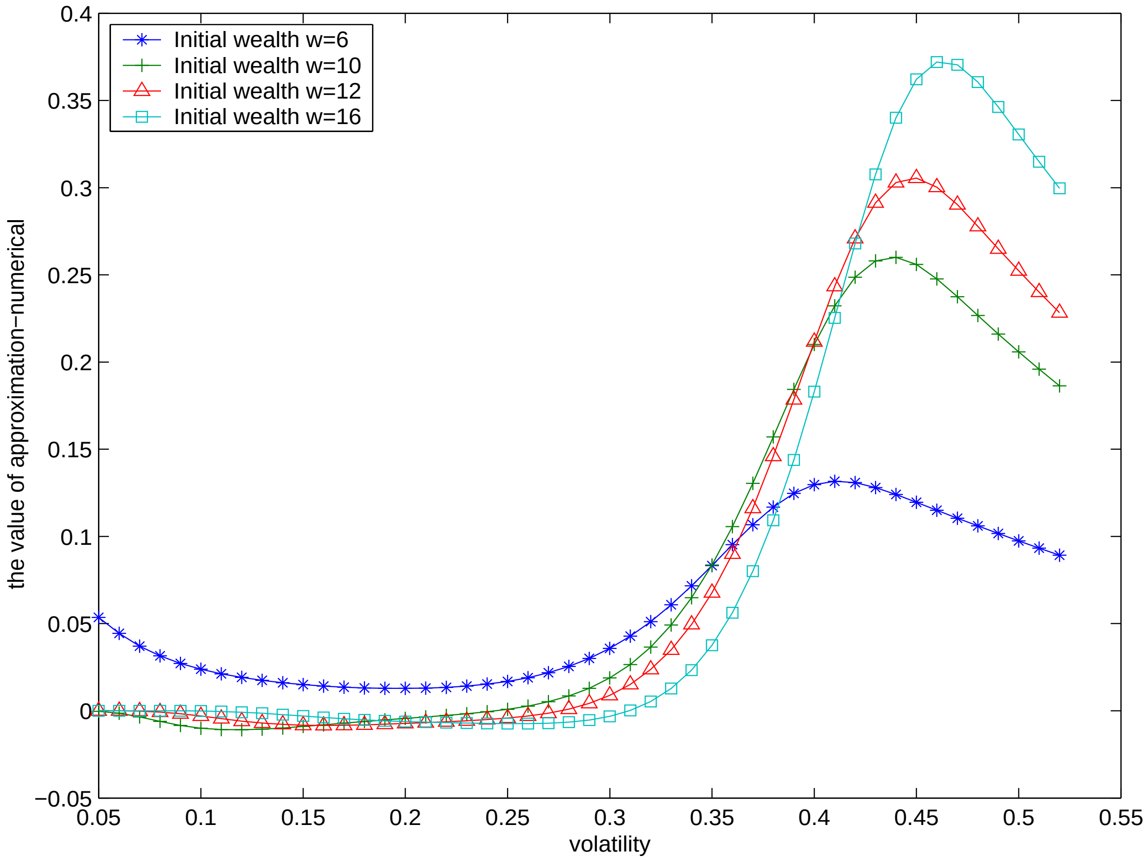

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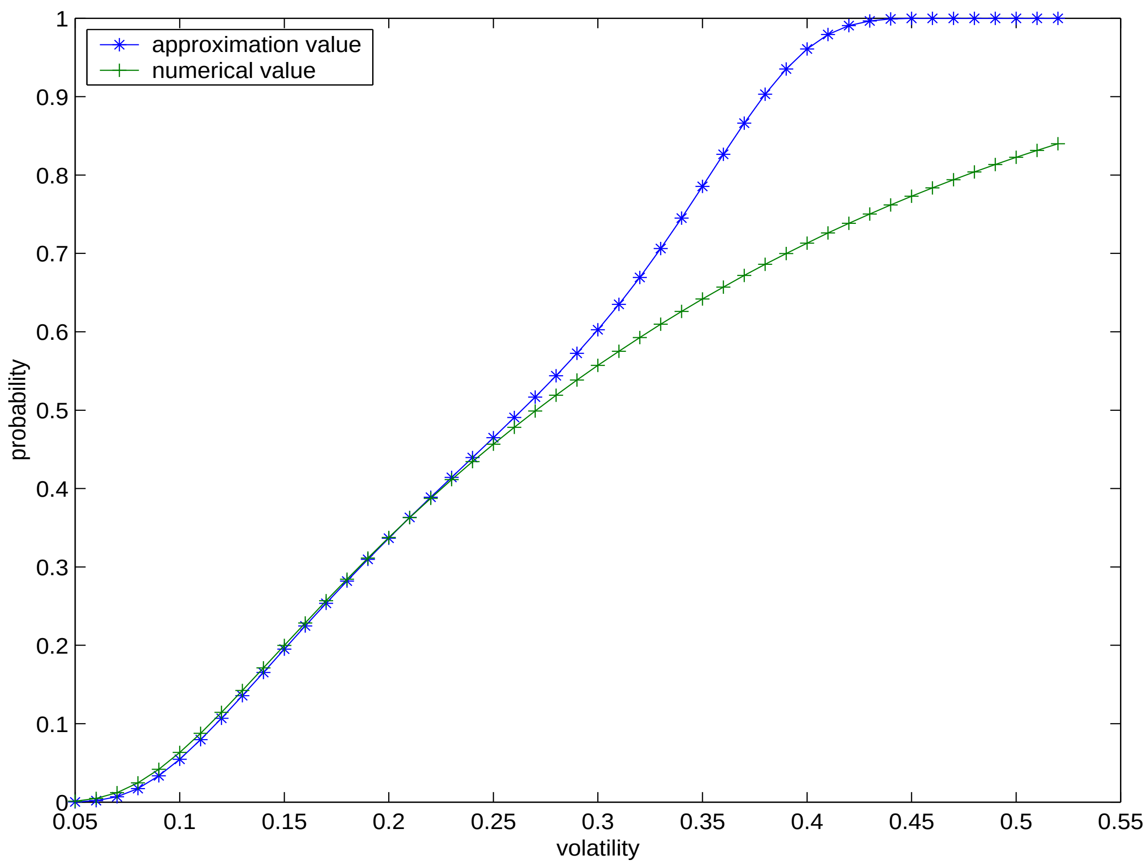

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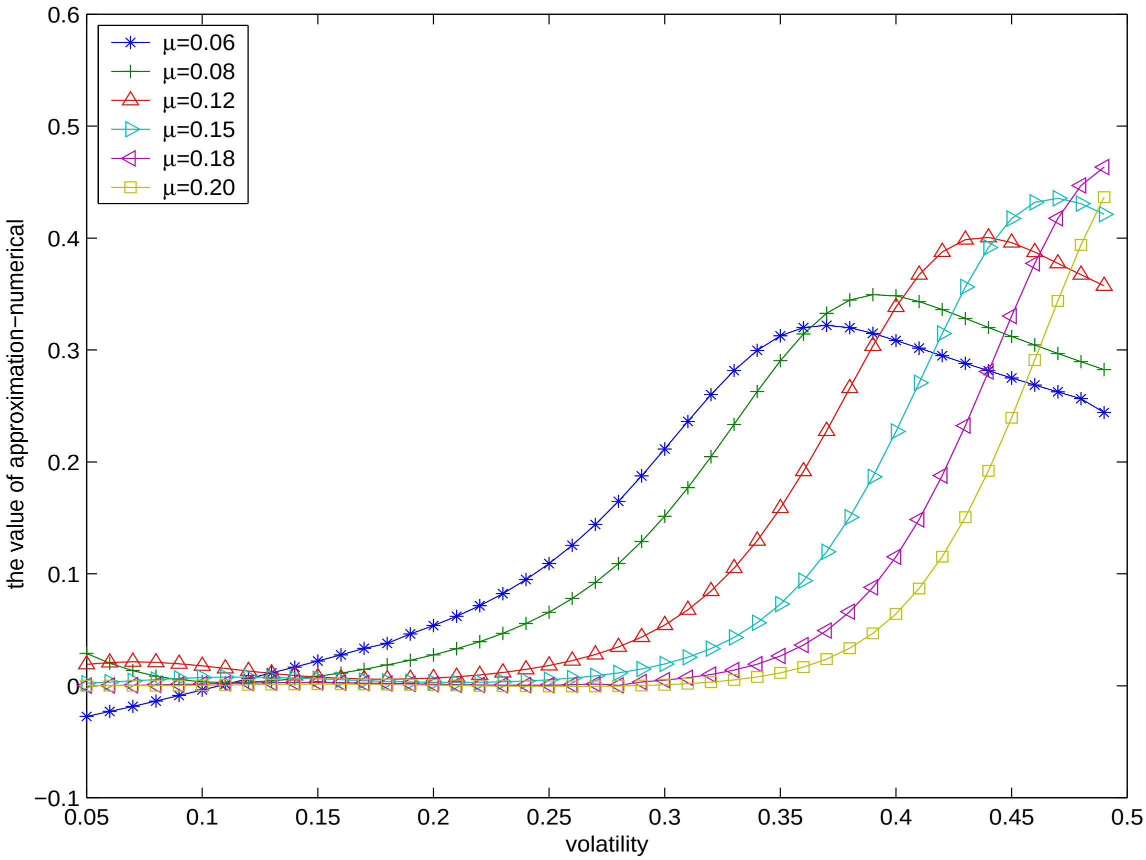

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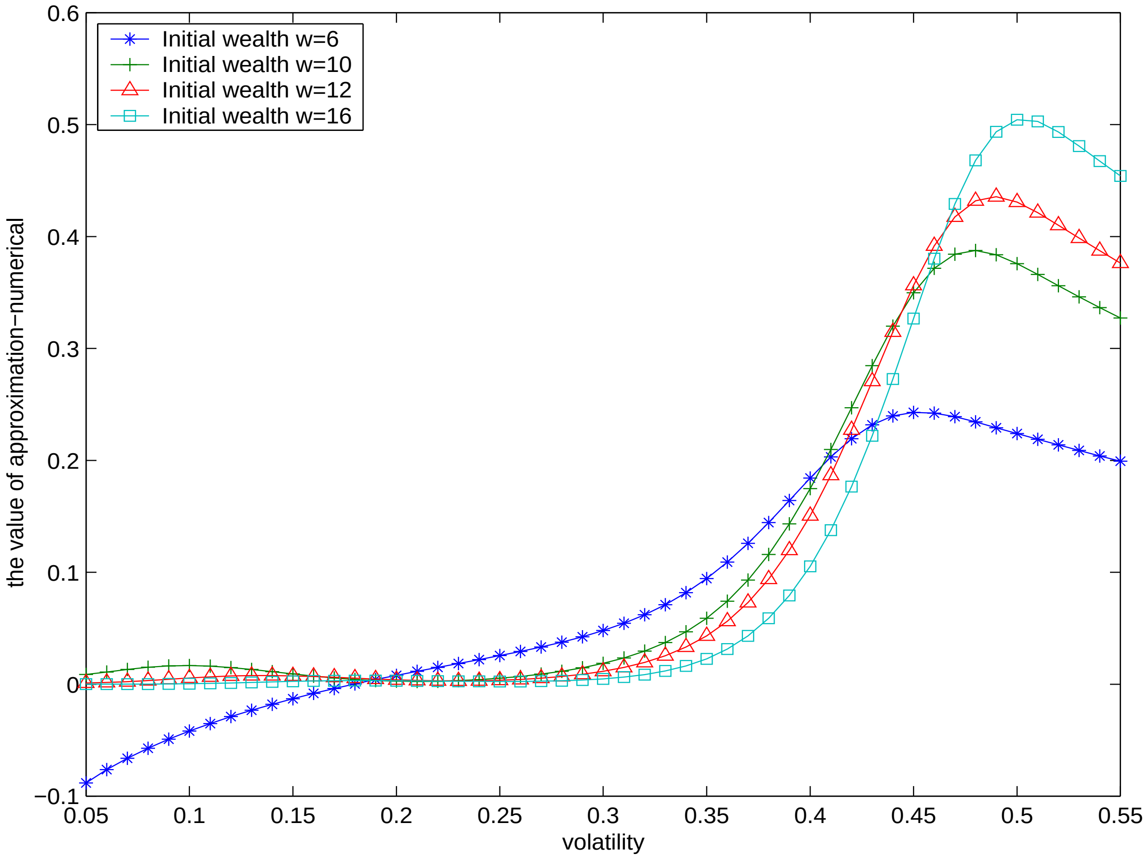

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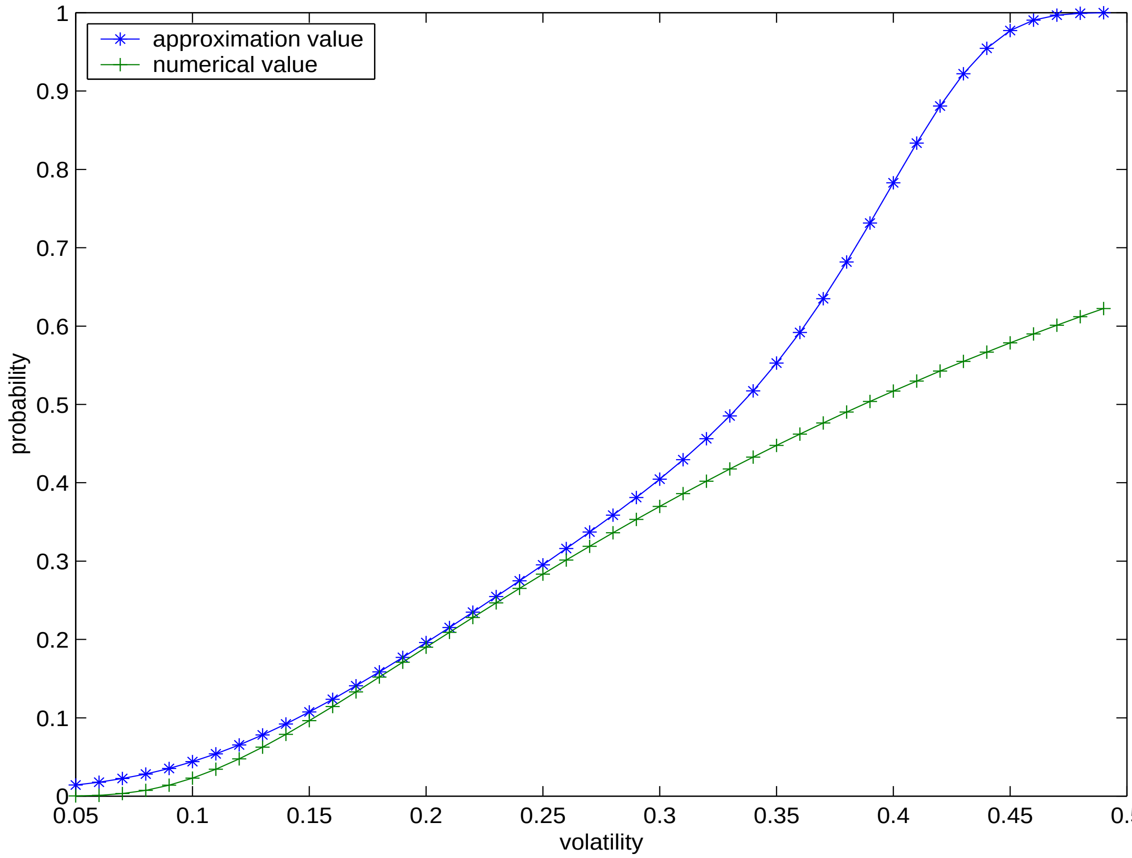

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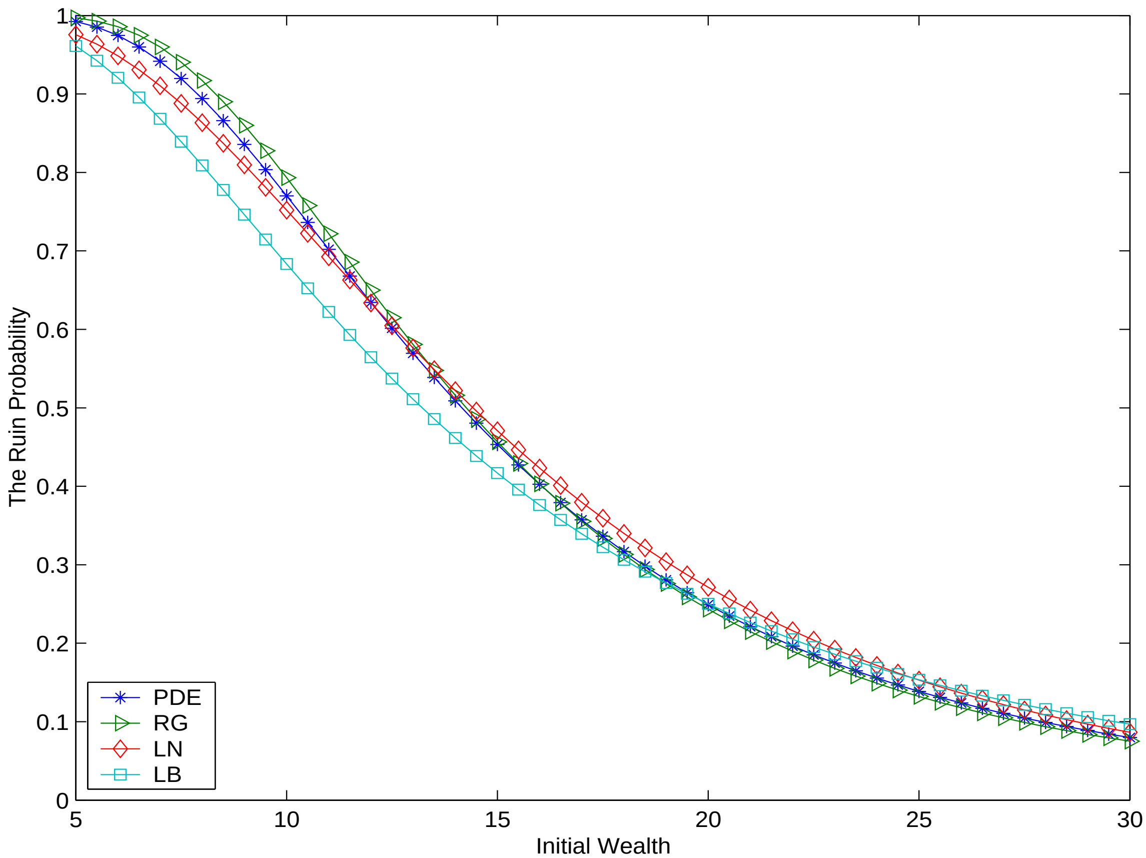

Figure 11: The figure compares the results from a variety of methods for computing the ruin probability (P_2) as a function of initial wealth, assuming a $T = 25$ year time horizon. The capital market assumptions are based on historical estimates of real (after-inflation) returns, which are $\mu = 7\%$ and $\sigma = 20\%$. Notice that the LB method understates the ruin probability at low levels of initial wealth, and then overstates at higher levels.

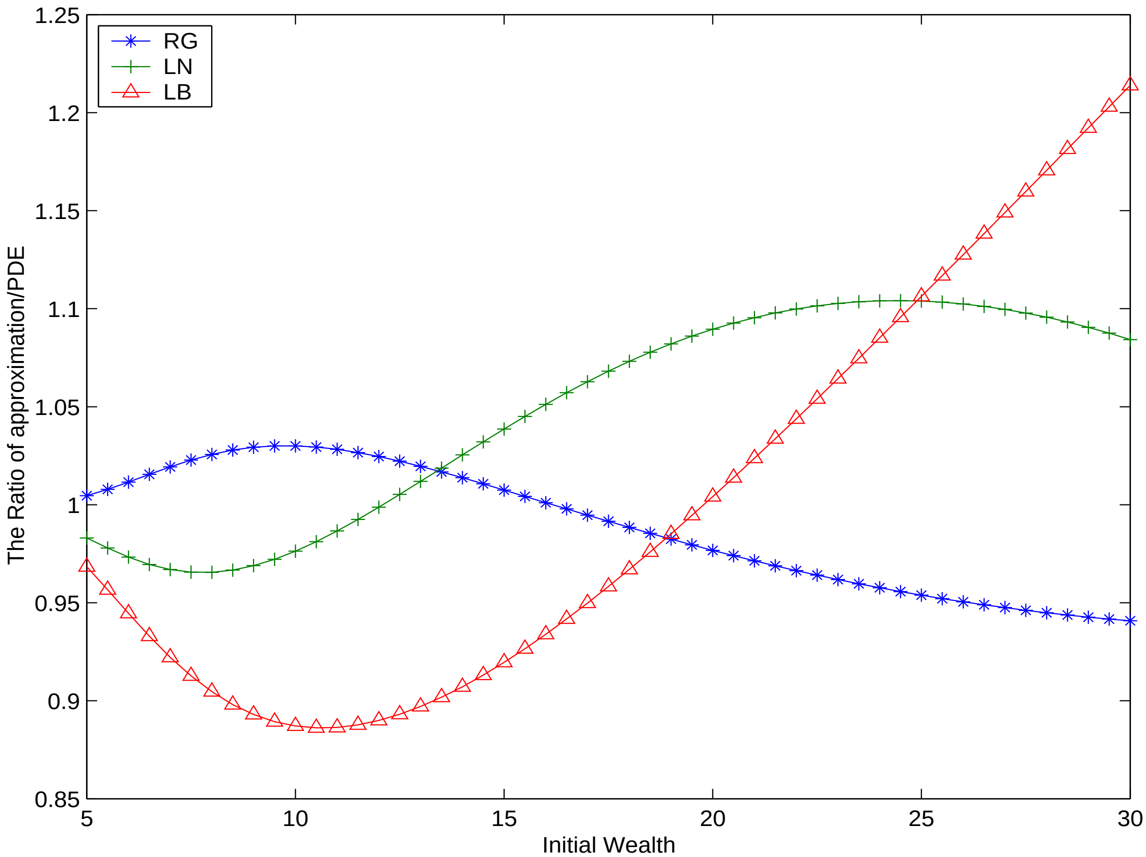

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Table 1: The table displays the probability that an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined within 30 years (P_2) or at the end of 30 years (P_1), where ruin is defined as wealth hitting a level of y. The market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$.

y	Probability P_1	Probability P_2
0	.3517019	.3517019
1	.3639941	.3642634
2	.3749395	.3759978
3	.3875812	.390536
4	.3972156	.4028596
5	.4092463	.4207916
6	.417814	.4358777
7	.4310415	.4637261
8	.4385038	.4821129
9	.4508516	.5170691
10	.4598995	.5464339
11	.469635	.5817383
12	.474769	.6019616
13	.485585	.6483504
14	.4912722	.6748824
15	.5032159	.7356768
16	.5094758	.770427
17	.5159293	.8084472
18	.5225771	.8500499
19	.5294195	.8955809
20	.53645613	1

Table 2: The table compares the probability than an individual with an initial wealth of $w = 20$ dollars who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. The deviation of the three approximation methods from the PDE value is listed in brackets. Note that the market parameters for the stochastic process driving wealth is a mean return of $\mu = 7\%$ and a volatility of $\sigma = 20\%$, which correspond to long-run historical values for these parameters in real (after-inflation) terms.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.259859	0.519114	0.675381	0.861087
	RG	0.254278 (-2.15%)	0.530307 (+2.16%)	0.805383 (+19.25%)	0.999999 (+16.13%)
	LN	0.268334 (+3.26%)	0.527981 (+1.71%)	0.621477 (-7.98%)	0.720074 (-16.38%)
	LB	0.264812 (+1.91%)	0.483851 (+6.79%)	0.611100 (-9.25%)	0.784282 (-8.91%)
0.07	PDE	0.030627	0.282757	0.514435	0.796587
	RG	0.029375 (-4.09%)	0.277442 (-1.88%)	0.578085 (+12.37%)	0.999999 (+25.54%)
	LN	0.026378 (-13.87%)	0.305567 (+8.07%)	0.492238 (-4.31%)	0.636515 (-20.09%)
	LB	0.04426 (+44.51%)	0.279380 (-1.19%)	0.464903 (-9.63%)	0.709220 (-10.97%)
0.09	PDE	0.004070	0.163098	0.405108	0.745403
	RG	0.003537 (-13.10%)	0.157593 (-3.38%)	0.430867 (+6.36%)	0.999999 (+34.16%)
	LN	0.002096 (-48.50%)	0.179819 (+10.25%)	0.405212 (0.03%)	0.578785 (-22.35%)
	LB	0.008100 (+99.02%)	0.173647 (+6.47%)	0.370717 (-8.49%)	0.654208 (+12.23%)
0.11	PDE	0.000360	0.083794	0.304100	0.688520
	RG	0.000241 (-33.06%)	0.080049 (-4.47%)	0.307361 (+1.07%)	0.99988 (+45.22%)
	LN	0.000066 (-81.76%)	0.089511 (+6.82%)	0.320792 (+5.49%)	0.521354 (-24.28%)
	LB	0.000099 (-72.5%)	0.098727 (+17.83%)	0.285099 (+6.25%)	0.596593 (-13.35%)

Table 3: The table compares the probability than an individual with an initial wealth of $w = 15$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. Notice that the ruin probabilities are uniformly higher the lower the level of initial wealth. The capital market parameters are the same as Table 2.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.659908	0.721906	0.792213	0.902623
	RG	0.677698	0.754434	0.919014	0.999999
		(+2.69%)	(+4.50%)	(+16.00%)	(+10.77%)
	LN	0.680396	0.708080	0.717898	0.761642
		(+3.10%)	(-1.92%)	(-9.38%)	(-15.62%)
	LB	0.610218	0.657910	0.716636	0.828508
		(-7.53%)	(-8.86%)	(-9.54%)	(-8.21%)
0.07	PDE	0.220666	0.493114	0.657118	0.851940
	RG	0.220533	0.499339	0.750392	0.999999
		(-0.06%)	(+1.26%)	(+14.19%)	(+17.38%)
	LN	0.231312	0.507041	0.606400	0.684196
		(+4.82%)	(+2.82%)	(-7.72%)	(-19.67%)
	LB	0.221865	0.450932	0.583798	0.763375
		(+0.54%)	(-8.55%)	(-11.16%)	(-10.40%)
0.09	PDE	0.060876	0.338484	0.553410	0.810061
	RG	0.058926	0.336481	0.609348	0.999999
		(-3.20%)	(-0.59%)	(+10.11%)	(+23.45%)
	LN	0.057383	0.361978	0.525135	0.629499
		(-5.74%)	(+6.94%)	(-5.11%)	(-22.29%)
	LB	0.07390	0.319641	0.490438	0.714310
		(+21.39%)	(-5.57%)	(-11.38%)	(-11.82%)
0.11	PDE	0.011049	0.209488	0.447205	0.76193
	RG	0.009542	0.205239	0.471844	0.999967
		(-13.64%)	(-2.03%)	(+5.51%)	(+31.24%)
	LN	0.007144	0.229385	0.442851	0.574163
		(-35.34%)	(+9.50%)	(-0.97%)	(-24.64%)
	LB	0.017200	0.209641	0.399245	0.661740
		(+55.67%)	(+0.07%)	(-10.72%)	(-13.15%)

Table 4: The table compares the probability than an individual with an initial wealth of $w = 10$ dollars (in contrast to Table 2 that examines the case of $w = 20$) who withdraws 1 dollar per annum, will get ruined; where ruin is defined as wealth hitting zero – within 25 years (P_2) using the exact PDE method, the Reciprocal Gamma (RG) approximation, the LogNormal (LN) approximation and the Lower Bound (LB) estimate. The capital market parameters are the same as Table 2.

μ	Methods	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.5$
0.04	PDE	0.976671	0.921083	0.913555	0.947221
	RG	0.987525	0.951954	0.988115	0.887330
		(+1.11%)	(+3.35%)	(+8.16%)	(-6.32%)
	LN	0.977200	0.889043	0.829773	0.814006
		(+0.05%)	(-3.49%)	(-9.17%)	(-14.06%)
	LB	0.947789	0.857179	0.841752	0.881990
		(-2.96%)	(-6.94%)	(-7.86%)	(-6.87%)
0.07	PDE	0.819309	0.798142	0.833526	0.915111
	RG	0.840389	0.823716	0.924784	0.999999
		(+2.57%)	(+3.20%)	(+10.95%)	(+9.28%)
	LN	0.831504	0.776120	0.749075	0.746374
		(+1.49%)	(-2.76%)	(-10.13%)	(-18.44%)
	LB	0.733420	0.712207	0.743665	0.831776
		(-10.48%)	(-10.77%)	(-10.78%)	(-9.11%)
0.09	PDE	0.570904	0.674834	0.760512	0.886989
	RG	0.583943	0.689864	0.839804	0.999999
		(+2.28%)	(+2.23%)	(+10.43%)	(+12.74%)
	LN	0.593166	0.670089	0.687930	0.697144
		(+3.90%)	(-0.70%)	(-9.54%)	(-21.40%)
	LB	0.489219	0.590833	0.666023	0.792516
		(-14.31%)	(-12.45%)	(-12.42%)	(-10.65%)
0.11	PDE	0.288923	0.530327	0.673947	0.853074
	RG	0.291877	0.536036	0.729814	0.999999
		(+1.02%)	(+1.08%)	(+8.29%)	(+17.22%)
	LN	0.303727	0.543343	0.621613	0.646185
		(+5.12%)	(+2.45%)	(-7.77%)	(-24.25%)
	LB	0.253385	0.462657	0.582280	0.749126
		(-12.30%)	(-12.76%)	(-13.6%)	(-12.19%)

Table 5: Results from the LB approximation assuming different discretization schemes. We start with the case where each period is exactly one year – which is the situation reported by Goovaerts et. al. (2000) – and then show the results for more quarterly, monthly and weekly compounding. Note that as one would expect, as n gets large the probabilities converge.

n	Probability P_2
25×1	0.450932
25×4	0.462180
25×12	0.464746
25×52	0.465740
25×365	0.465900