

# Extensional flows with viscous heating

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We consider the role played by viscous heating in extensional flows of viscous threads with temperature-dependent viscosity. The results show that there exists an interesting interplay between the effects of viscous heating, which accelerates thinning, and inertia, which prevents pinch-off. We consider steady drawing of a thread that is fed through a fixed aperture at given speed and pulled with a constant force at a fixed downstream location. For pulling forces above a critical value, we show that inertialess solutions cannot exist and investigate the role played by inertia in controlling the dynamics. We also consider the unsteady stretching of a thread that is fixed at one end pulled with a constant force at the other end. In contrast to the case in which inertia is neglected, the thread will always pinch at the end where the force is applied. Viscous heating can have a profound effect on the total extension at pinching.

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## 1. Introduction

Viscous heating plays an important role in a number of applications and is particularly relevant to the polymer processing industry. Many of the fluids used in such applications have a viscosity that varies rapidly with temperature and this can give rise to strong feedback effects that can lead to profound changes in the flow structure. In recent years there has been a renewed interest in viscous heating as a number of authors have shown that its effects can dramatically destabilize viscous flows (Al-Mubaiyedh *et al.* 2002, White & Muller 2002). Vasilyev *et al.* (2001) have shown that viscous gravity currents can propagate faster as a result of viscous heating. In channel flows, Pearson (1977) and Ockendon (1979) showed that plug flows can occur, whereas Costa & Macedonia (2005) have shown that interesting secondary flows can be triggered.

To our knowledge, the effect of viscous heating in the pulling of viscous threads has received relatively little attention. For high speed optical fibre pulling, Yin and Jaluria (2000) showed that the heat generated by viscous dissipation becomes important in the neck down region. Another application in which viscous heating can be significant in the latter stages of the pulling of glass microelectrodes in which a glass tube is pulled by an external force (Huang *et al.* 2003). The extensional velocity increases dramatically during stretching, resulting in a large strain rate and significant viscous heating.

For low-speed glass fiber pulling the inertia of glass is frequently neglected. However, if the thread becomes sufficiently narrow and the extensional velocity sufficiently large, the inertia can no longer be neglected, as shown by Wilson (1988), Kaye (1991) and Stokes & Tuck (2004).

In this paper, we examine two distinct types of extensional flows and show that apparently weak viscous heating can have a dramatic effect on the dynamics. The first type of

flow is that of a thread that is fed into an apparatus with a fixed speed and is pulled at a fixed downstream location. The results show that inertialess solutions can only exist for values of the pulling force below a critical value. Beyond this critical value inertia is crucial in controlling the flow. The second type of flow is that of a thread that is fixed at one end and pulled at the other end with a fixed force. When inertia is included the thread always pinches at the location where the force is applied. Viscous heating can have a dramatic effect on the total extension that occurs before pinching.

## 2. Model for thread pulling

We consider an axisymmetric thread with a cross-sectional area  $A'$ , velocity  $u'$ , and temperature  $\theta'$ . We define  $x'$  as the distance along the thread measured from a fixed reference point and  $t'$  as the time. The thread is pulled with an external force  $F = O(1)$  N. The density, specific heat capacity and surface tension coefficient are assumed to be constants and are given by  $\rho = O(10^3)$  kg/m<sup>3</sup>,  $c_p = O(10^3)$  J/kg/K,  $\gamma = O(10^{-1})$  kg/s<sup>2</sup>, respectively. The thread has characteristic cross-sectional area  $A_0 = O(10^{-6})$  m<sup>2</sup>, length  $L = O(10^{-2})$  m and viscosity  $\mu_0 = O(10^4)$  kg/m/s. The viscosity is assumed to be a function only of the temperature. A temperature change of  $\Theta = O(50)$  K is required to cause a significant change in the viscosity.

The governing equations for this type of extensional flows have been derived by a number of authors, see for example Forest *et al.* (2000) and Yin & Jaluria (2000). Using the natural scales

$$u' = Uu, \quad A' = A_0A, \quad x' = Lx, \quad t' = Lt/U, \quad \theta' = \Theta\theta, \quad \text{with } U = \frac{FL}{3\mu_0A_0}$$

the long-wavelength dimensionless equations are

$$A_t + (uA)_x = 0, \tag{2.1}$$

$$R \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = \frac{1}{A} \frac{\partial}{\partial x} \left( \mu A \frac{\partial u}{\partial x} + \lambda \sqrt{A} \right), \tag{2.2}$$

$$\theta_t + u\theta_x = \mathcal{H}\mu u_x^2, \tag{2.3}$$

where

$$R = \frac{\rho UL}{3\mu_0} = O(10^{-4}), \quad \lambda = \frac{\gamma\sqrt{\pi A_0}}{F} = O(10^{-4}), \quad \mathcal{H} = \frac{F}{\rho c_p \Theta A_0} = O(10^{-2}). \tag{2.4}$$

We will adopt the widely used exponential viscosity law that, in dimensionless form, is given by

$$\mu(\theta) = \exp(-\theta). \tag{2.5}$$

This viscosity law can give excellent agreement with experimental data for viscosity variations in excess of five orders of magnitude (Huang *et al.* 2003). The boundary condition at the location where the force is applied becomes

$$\mu A u_x + \lambda A^{1/2} = 1. \tag{2.6}$$

Since  $\lambda$  is small we will neglect surface tension. The values of  $\mathcal{H}$  and  $R$  are typically small, but if the thread thins significantly, we will show that the effects of viscous heating and inertia will become important.

We will consider two related problems that share a number of similar features. The first problem is motivated by continuous drawing with a constant feeding rate, whereas the second problem is motivated by the unsteady extension of a thread. The first problem

has steady states and these allow us to gain important insights into the role of inertia in the second problem.

### 3. Steady drawing

We consider a device that feeds a cylindrical thread of viscous fluid through a fixed aperture with cross-sectional area  $A_0$  at speed  $u_0 = O(1)$  m/s. At a position  $L$  downstream a fixed force  $F$  is applied that stretches the thread.

The dimensionless equations are given by (2.1)–(2.3). The boundary condition at the exit  $X = 1$  is given by (2.6) with  $\lambda = 0$ . The boundary conditions at the entry  $x = 0$  are  $A = 1$ ,  $\theta = 0$  and  $\mathcal{F}u = 1$ , where

$$\mathcal{F} = \frac{FL}{3\mu_0 A_0 u_0} = O(0.3) \quad (3.1)$$

measures the degree of thinning experienced by a constant viscosity thread whilst it remains in the domain. Small values of  $\mathcal{F}$  imply that the thread passes through the device sufficiently quickly that it is thinned weakly by the imposed force, whereas large values of  $\mathcal{F}$  imply that the thread will experience significant thinning.

If we consider steady states the equation of conservation of mass can be integrated along with the boundary conditions at  $x = 0$  to yield

$$\mathcal{F}uA = 1. \quad (3.2)$$

Using (3.2) to eliminate  $A$  from the time-independent version of (2.2), integrating and applying the boundary condition (2.6) at  $x = 1$  yields

$$u_x = \mu^{-1}u[\mathcal{F} - R(u(1) - u)], \quad (3.3)$$

where  $u(1)$  is the velocity at the exit which must be solved for. Substituting (3.3) into the time-independent version of the heat equation (2.3) yields

$$\theta_x = \mu^{-1}\mathcal{H}u[\mathcal{F} - R(u(1) - u)]^2. \quad (3.4)$$

Dividing (3.4) by (3.3), integrating and applying the boundary condition at  $x = 0$  gives

$$\theta = \mathcal{H} \left[ (\mathcal{F} - Ru(1))(u - \mathcal{F}^{-1}) + \frac{1}{2}R(u^2 - \mathcal{F}^{-2}) \right]. \quad (3.5)$$

Substituting (3.5) into (3.3), integrating and applying the boundary condition at  $x = 0$  yields

$$\int_1^{\mathcal{F}u} \frac{\exp(-\mathcal{H}[(1 - \mathcal{P}D_r)(v - 1) + \frac{1}{2}\mathcal{P}(v^2 - 1)])}{v[(1 - \mathcal{P}D_r) + \mathcal{P}v]} dv = \mathcal{F}x, \quad (3.6)$$

where  $D_r = u(1)/u(0) = \mathcal{F}u(1)$  is the draw ratio and

$$\mathcal{P} \equiv R\mathcal{F}^{-2} = \frac{\rho A_0 u_0^2}{F} = O(10^{-3}) \quad (3.7)$$

is the ratio of the rate of change of momentum due to the feeding and the rate of change of momentum from the stretching force. In order to evaluate this integral one needs to know  $D_r$ . This can be obtained by using the condition  $\mathcal{F}u = D_r$  at  $x = 1$  to obtain

$$\int_1^{D_r} \frac{\exp(-\mathcal{H}[(1 - \mathcal{P}D_r)(v - 1) + \frac{1}{2}\mathcal{P}(v^2 - 1)])}{v[(1 - \mathcal{P}D_r) + \mathcal{P}v]} dv = \mathcal{F}. \quad (3.8)$$

For any parameter values this can be solved numerically for  $D_r$  using straightforward

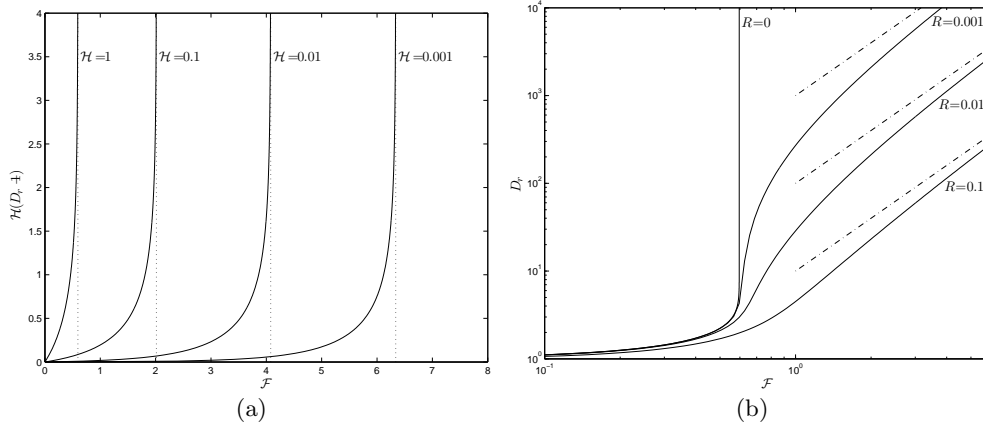


FIGURE 1. Steady drawing. a) The scaled draw ratio  $\mathcal{H}(D_r - 1)$  is plotted against  $\mathcal{F}$  for  $R = 0$  for various values of  $\mathcal{H}$ . The dotted lines show the critical value of  $\mathcal{F}$  above which no steady solutions exist for  $R = 0$ . b) The draw ratio is plotted against  $\mathcal{F}$  on a log-log scale for  $\mathcal{H} = 1$  and various values of  $R$ . The dashed lines represent the large  $\mathcal{F}$  asymptotic limit.

quadrature and Newton-Rhapson methods. However, since  $\mathcal{P}$  is small we begin by setting it to zero which corresponds to neglecting inertia. In this case the integral (3.6) can be computed without first computing  $D_r$  and the solution is given by

$$E_1(\mathcal{F}\mathcal{H}u) = E_1(\mathcal{H}) - \mathcal{F}e^{-\mathcal{H}}x, \quad (3.9)$$

where  $E_1$  is the exponential integral

$$E_1(u) = \int_u^\infty \frac{e^{-v}}{v} dv. \quad (3.10)$$

We can therefore obtain the draw ratio  $D_r$  as a function of  $\mathcal{F}$  by solving

$$E_1(\mathcal{H}D_r) = E_1(\mathcal{H}) - \mathcal{F}e^{-\mathcal{H}}. \quad (3.11)$$

This is plotted in Figure 1a. We immediately see that as  $\mathcal{F}$  approaches a critical value from below, that  $D_r$  tends to infinity. For values of  $\mathcal{F}$  above the critical value, no steady state solutions exist. The critical value,  $\mathcal{F}_c = e^{\mathcal{H}}E_1(\mathcal{H})$  can be obtained by letting  $D_r \rightarrow \infty$  in (3.11).

For sufficiently small  $\mathcal{F}$ , the amount of stretching that can occur before material elements exit the device is small. Therefore the heat generated by viscous heating will only weakly affect the viscosity and solutions will be close to the isothermal case. As  $\mathcal{F}$  is increased, the time that the thread remains in the device increases and hence the amount of stretching increases. This causes extra viscous heating which makes the viscosity decrease and hence reduces the resistance to stretching. This allows more stretching to occur which generates more temperature and so can lead to a runaway effect.

When  $\mathcal{H} \rightarrow 0$  the critical value of  $\mathcal{F}_c \rightarrow \infty$  and so the runaway phenomena disappears. However, we note that as  $\mathcal{H} \rightarrow 0$  the critical value of  $\mathcal{F}$  tends to infinity very slowly as  $\mathcal{F}_c = O(\ln(1/\mathcal{H}))$ . Therefore, even apparently very small values of the parameter  $\mathcal{H}$  can still give rise to this runaway phenomena at relatively moderate values of  $\mathcal{F}$ .

In practice such runaway effects will be saturated in one of two ways: either the temperature will increase sufficiently that the exponential viscosity law (2.5) will not be valid or alternatively inertia will become important. In this paper we will focus on the role of inertia, since treatment of the former case is straightforward.

In Figure 1b we present the results of numerical integration of (3.8) that show how

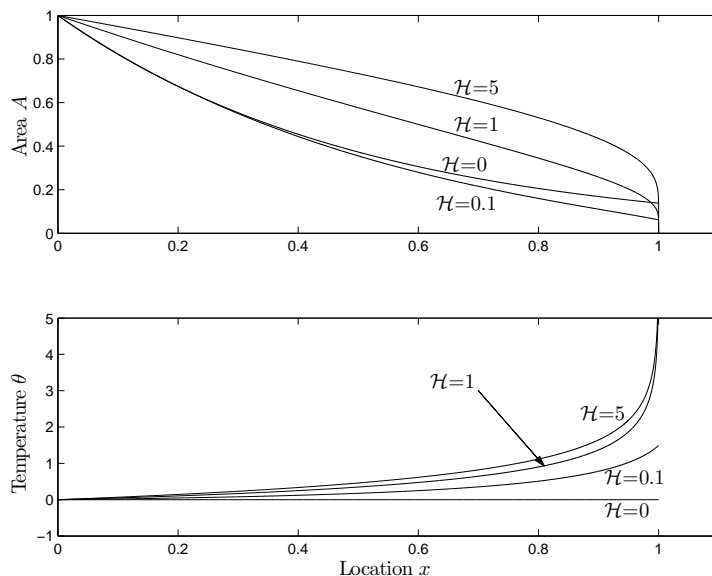


FIGURE 2. The cross-sectional area and temperature profiles for  $R = 0.01$  and  $\mathcal{F} = 2$  and various values of  $\mathcal{H}$ .

inertia modifies the dynamics. When  $\mathcal{F}$  is below the critical value the inertia plays a weak role in modifying the draw ratio. For values of  $\mathcal{F}$  above the critical value inertia becomes crucial. The draw ratio is an increasing function of  $\mathcal{F}$ , and  $D_r \rightarrow \mathcal{F}^2/R$  as  $\mathcal{F} \rightarrow \infty$ . This corresponds to the case in which the thermal runaway is very strong and so the viscous resistance to stretching is small. In this case the draw ratio is inertially controlled. In Figure 2 we present the cross-sectional area and temperature profiles for a fixed value of  $\mathcal{F}$ , a fixed small value of  $R$  and varying values of  $\mathcal{H}$ . As the heating rate increases the critical value of  $\mathcal{F} = \mathcal{F}_c$ , for which inertialess solutions can exist, decreases. When the heating rate is small ( $\mathcal{H} = 0.1$ ), inertialess solutions can exist and are similar to the  $\mathcal{H} = 0$  situation with slightly increased thinning due to the viscous heat generation. However, for sufficiently large values of  $\mathcal{H} = 1$  or  $5$ , inertialess solutions do not exist. In this case, the solutions thin very rapidly near the exit and experience much weaker thinning over the bulk of the thread. The role of inertia can be understood by considering equation (3.3). This equation shows how the effective pulling force, which we define to be  $\mathcal{F} - R(u(1) - u)$ , is reduced by inertia. At the exit of the device the force is prescribed. Near the exit,  $u$  is close to  $u(1)$ , and so inertia plays a weak role. Since viscous heating can lead to very small values of the viscosity, very rapid thinning can occur near the exit. As one moves away from the exit, part of force is required to accelerate the thread. This reduces the effective force as one moves away from the exit and thus leads to weaker thinning in the bulk.

#### 4. Extension of thread with fixed force

We now turn our attention to the case of a thread that is fixed at one end and pulled with a constant force at the other end. As shown in Stokes & Tuck (2004), the system of equations becomes significantly simpler if expressed in Lagrangian coordinates  $(\xi, \tau)$ . The relationship between the Eulerian coordinates  $(x, t)$  and the Lagrangian coordinates  $(X, \tau)$  is given by  $\tau = t$  and  $X_\tau = u$ . This implies that the Lagrangian variable  $X$  is the

spatial coordinate of a material point which was at the location  $x = \xi$  at the initial time  $\tau = 0$ .

In Lagrangian coordinates, (2.1) becomes  $(AX_\xi)_\tau = 0$ . Integrating and applying the initial conditions  $A(\xi, 0) = A_i(\xi)$  and  $X(\xi, 0) = \xi$ , gives  $X_\xi = A_i/A$ , and the total extension of the thread at time  $\tau$  is given by

$$X(1, \tau) = \int_0^1 \frac{A_i(\xi)}{A(\xi, \tau)} d\xi. \quad (4.1)$$

The equations (2.1)–(2.3) become

$$X_\xi = \frac{A_i}{A}, \quad (4.2)$$

$$A_\tau = -\frac{A^2 u_\xi}{A_i} \quad (4.3)$$

$$RA_i u_\tau = \left[ \frac{\mu A^2 u_\xi}{A_i} \right]_\xi \quad (4.4)$$

$$\theta_\tau = \mathcal{H} \frac{\mu(\theta) A^2 u_\xi^2}{A_i^2}. \quad (4.5)$$

The boundary conditions are  $u = 0$  at  $\xi = 0$  and  $\mu A^2 u_\xi / A_i = 1$  at  $\xi = 1$ . The initial conditions are  $u = 0$ ,  $\theta = \theta_i(\xi)$  and  $A = A_i(\xi)$  at  $\tau = 0$ . Since we are using the long wavelength approximation we will assume that  $A_i$  and  $\theta_i$  have continuous second derivatives.

#### 4.1. Zero viscous heating

If we neglect both viscous heating and inertia, then the equations (4.3–4.5) can readily be solved to obtain

$$A = A_i(\xi) - \tau / \mu_i(\xi). \quad (4.6)$$

Therefore, pinching occurs at  $\tau_p = \min_\xi \{\mu_i A_i\}$  at the location  $\xi = \xi_p$ . If the thread does not pinch at one of the end points, that is  $\xi_p \in (0, 1)$ , then since  $\xi_p$  is a minimum of  $\mu_i A_i$ , we have  $(\mu_i A_i)'|_{\xi_p} = 0$  and generically,  $(\mu_i A_i)''|_{\xi_p} > 0$ . Therefore, as time approaches the pinching time, the total extension is dominated by the contribution to the integral (4.1) near  $\xi = \xi_p$ . Asymptotically, we obtain

$$X(1, \tau) \sim \frac{\sqrt{2\pi\tau_p}}{\sqrt{(\mu_i A_i)''|_{\xi_p}(\tau_p - \tau)}}. \quad (4.7)$$

If the thread pinches at one of the end points a similar calculations shows that

$$X(1, \tau) \sim \left| \frac{\tau_p \ln(\tau_p - \tau)}{(\mu_i A_i)'|_{\xi_p}} \right|. \quad (4.8)$$

In both cases, the total extension of the thread tends to infinity as the pinching time is approached. As the area tends to zero, the velocity tends to infinity and so we expect inertial terms to become important. When inertial terms are retained we were unable to obtain a general analytic solution, but an exact solution can be obtained for the area at the end at which the force is applied. This is because the force is prescribed at the end and so the inertial terms do not reduce the effective pulling force. Hence, (4.3) can be used to rewrite the applied force boundary boundary condition as  $A_\tau = -1/\mu_i(1)$  at  $\xi = 1$ . Integrating with respect to  $\tau$  and applying the initial condition we obtain  $A(1, \tau) = A_i(1) - \tau/\mu_i(1)$ . Results obtained using a standard explicit time-stepping

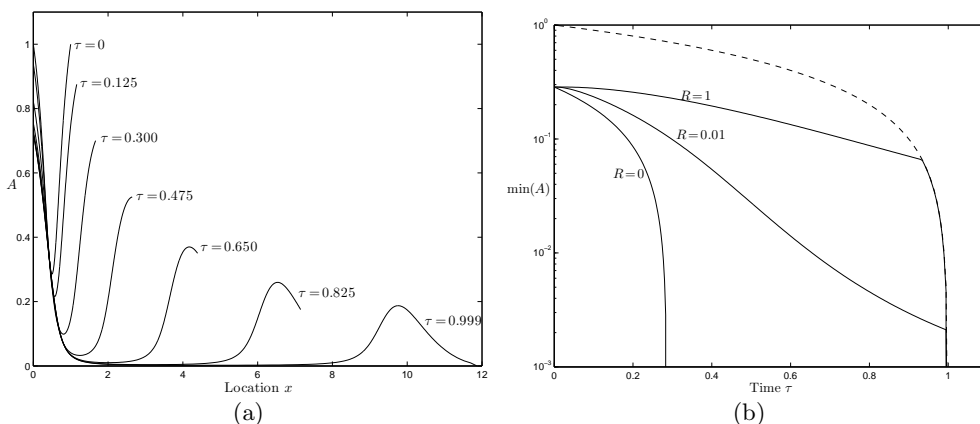


FIGURE 3. Stretching without viscous heating ( $\mathcal{H} = 0$ ). a) The cross-sectional area is plotted against  $x$  at various times for  $R = 0.1$ . Pinching occurs at the end point at  $\tau = 1$ . b) The minimum area is plotted against time for various values of  $R$ . The dotted line represents the area at the end point. The initial condition is given by (4.9) with  $K = 10$ .

technique are shown in Figure 3a. For the examples presented in this paper we take the simple initial condition given by

$$\theta_i = 0 \quad \text{and} \quad A_i = \frac{1}{1 + K(\frac{1}{2})^2} + \frac{K(\xi - \frac{1}{2})^2}{1 + K(\xi - \frac{1}{2})^2} \quad (4.9)$$

that is symmetric about  $\xi = 1/2$  and has unit area at the end points  $\xi = 0$  and  $\xi = 1$ . We note that other initial conditions give similar behavior. Although in applications,  $R$  is typically on the order of  $10^{-4}$ , for purposes of visualization, we use somewhat large values of  $R$ .

If  $R$  is small, then initially the inertial correction to the effective pulling force is small and so the solution will be well approximated by (4.6). Therefore, the area decreases at an approximately uniform rate at all  $x$  locations. However, when the area becomes small the inertial terms become important. Our numerical results show that inertia always prevents pinching in the middle of the thread and the thread always pinches at the end point where the force is applied. The area remains bounded away from zero everywhere except at the end point, so the total extension (4.1) can diverge only if the contribution from the neighborhood of  $\xi = 1$  diverges. In order to investigate the local behavior one must consider the similarity solution of the form  $A = A(1, \tau)a(\eta)$ , where  $\eta = (1 - \xi)A_i(1)R^{1/2}/[\mu_i(1)A(1, \tau)^{3/2}]$ . Then, at leading order the equations (4.3)–(4.5) give  $2a^2a'' - 3\eta a' + 2a = 0$  with boundary conditions  $a(0) = 1$  and  $a'(\infty) = 0$ . This similarity solution remains bounded away from zero and so the contribution to (4.1) near  $\xi = 1$  is of order  $\mu_i(1)R^{-1/2}A(1, \tau)^{1/2}$ . So near pinching,  $A(1, \tau) \rightarrow 0$ , the extension will remain finite.

In Figure 3b we show the result of pulling a thread that has an initial profile given by (4.9). We plot the minimum area and the area at the end of the thread against time for different values of  $R$ . For  $R = 0$  the thread pinches in the middle, whereas for  $R \neq 0$  the area in the middle thins to a small value, but is ultimately overtaken by the value at the end point. The thread pinches at the end point in a finite time.

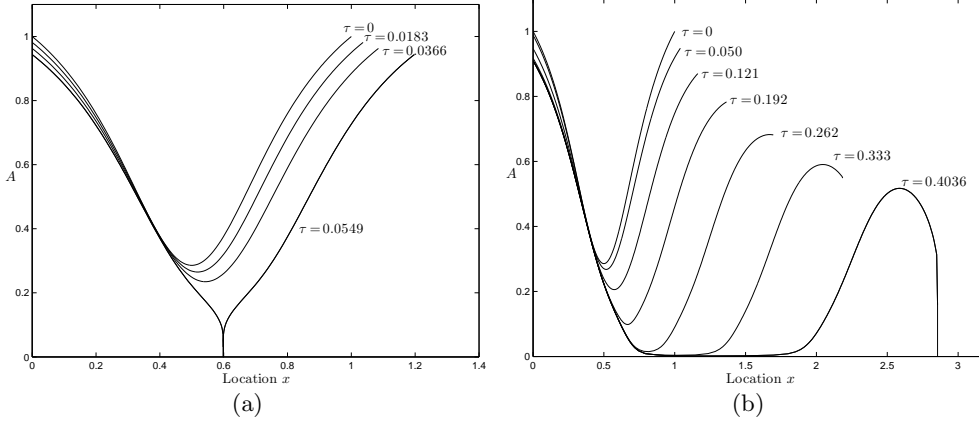


FIGURE 4. Pulling with viscous heating ( $\mathcal{H} = 1$ ). The cross-sectional area is plotted against  $x$  for various values of the time for a)  $R = 0$  which pinches in the middle and b)  $R = 10^{-2}$  which pinches at the end point. The initial condition is given by (4.9) with  $K = 10$ .

#### 4.2. The role of viscous heating

Initially the inertial terms are small and therefore we neglect them. Equation (4.4) can be integrated to give

$$\mu A^2 u_\xi = A_i \quad (4.10)$$

We can therefore eliminate  $u$  from equations (4.3) and (4.5) to obtain

$$\mu A_\tau = -1 \quad (4.11)$$

and

$$\mu A^2 \theta_\tau = \mathcal{H}. \quad (4.12)$$

Dividing (4.12) by (4.11), integrating and applying the initial condition yields

$$\theta - \theta_i = \mathcal{H}(A^{-1} - A_i^{-1}). \quad (4.13)$$

Substituting into (4.11), integrating and using the initial condition gives

$$\frac{A}{\mathcal{H}} e^{-\mathcal{H}/A} - E_1\left(\frac{\mathcal{H}}{A}\right) = -e^{-\mathcal{H}/A_i + \theta_i} \frac{\tau}{\mathcal{H}} + \frac{A_i}{\mathcal{H}} e^{-\mathcal{H}/A_i} - E_1\left(\frac{\mathcal{H}}{A_i}\right). \quad (4.14)$$

The cross-sectional area  $A$  will go to zero at time

$$\tau_p = \min_{\xi} \left\{ e^{-\theta_i} \left[ A_i - \mathcal{H} e^{\mathcal{H}/A_i} E_1\left(\frac{\mathcal{H}}{A_i}\right) \right] \right\}. \quad (4.15)$$

The solution of this equation is shown in Figure 4a. One can clearly see that the increased viscous heating leads the pinching to become highly localised.

When the thread is close to pinching, that is  $A \ll 1$ , (4.14) can be approximated by

$$\frac{A^2}{\mathcal{H}} e^{-\mathcal{H}/A} = e^{-\mathcal{H}/A_i + \theta_i} (\tau_p - \tau). \quad (4.16)$$

We can therefore see that  $A \rightarrow 0$  as  $-\mathcal{H}/\ln(\tau_p - \tau)$  as  $\tau \rightarrow \tau_p$ .

In this case, in order to determine the total extension of the thread we define

$$p(\xi) = e^{-\theta_i} \left[ A_i - \mathcal{H} e^{\mathcal{H}/A_i} E_1\left(\frac{\mathcal{H}}{A_i}\right) \right]. \quad (4.17)$$

Since  $\xi_p$  is a minimum, we have  $p'(\xi_p) = 0$  and so  $p(\xi) = \tau_p + \frac{1}{2}p''(\xi_p)(\xi - \xi_p)^2 + O((\xi - \xi_p)^3)$  and at  $\tau = \tau_p$  (4.16) becomes

$$\frac{A^2}{\mathcal{H}} e^{-\mathcal{H}/A} = \frac{1}{2} e^{-\mathcal{H}/A_i(\xi_p) + \theta_i(\xi_p)} p''(\xi_p) (\xi - \xi_p)^2 + O((\xi - \xi_p)^3). \quad (4.18)$$

Therefore

$$A \rightarrow \frac{\mathcal{H}}{2 \ln(\xi - \xi_p)} \quad \text{as } \xi \rightarrow \xi_p \quad (4.19)$$

and the contribution to the extension (4.1) from the region that is close to the pinching point is integrable. Hence the total extension will be finite even at pinching.

However, as the area tends to zero the velocity tends to infinity and so inertial terms will become important. The solution of (4.3)–(4.5) with non-zero inertia is shown in Figure 4b. Initially, the thread thins approximately uniformly until the minimum area becomes sufficiently small that inertia is important and prevents the thread from pinching at this location. As in the case without viscous heating, the inertia does not affect the end point and the area at the end point satisfies (4.14). Therefore, the thread will pinch at the end point at time  $\tau_p = e^{-\theta_i(1)} \left[ A_i(1) - \mathcal{H} e^{\mathcal{H}/A_i(1)} E_1 \left( \frac{\mathcal{H}}{A_i(1)} \right) \right]$ . Using a similar calculation to that for  $\mathcal{H} = 0$ , we can see that the total extension will remain finite.

In Figure 5 we show the result of pulling a thread that has an initial profile given by (4.9). We plot the minimum area and the area at the end of the thread against time for two different values of  $R$  and various values of  $\mathcal{H}$ . For the case with the larger inertia ( $R = 1$ ), the inertial effects prevent significant stretching in the middle of the thread and so the viscous heating plays a weak role. Ultimately, the thread will pinch at the end point where the inertia does not play a role and so viscous heating effects are important. For the case with weaker inertia ( $R = 0.1$ ), significant stretching occurs and initially cases with large viscous heating thin much more rapidly than those with weak viscous heating. Ultimately, the thread will also pinch at the end point, but the thinning in the bulk is much greater than that for  $R = 1$ .

## 5. Discussion

We have considered the role played by viscous heating in controlling extensional flows of viscous threads. We have shown that even small amounts of viscous heating can lead to a fundamental changes in the dynamics. For steady drawing, we have derived exact solutions for the case of zero inertia and have shown that there exists a critical pulling force above which inertialess solutions cannot exist. For pulling forces exceeding the critical value, the inclusion of inertia gives solutions that thin weakly over the bulk of the thread, but thin rapidly in a narrow region near the location where the pulling force is applied.

For an extending thread with zero inertia, we have derived exact solutions and shown that the total extension at pinching is infinite when viscous heating is neglected, but is finite when viscous heating is included. However, if inertia is present, no matter how small, it will eventually become important and the thread will always pinch at the end where the force is applied. Nevertheless, viscous heating can have a profound effect on the profile and lead to very rapid thinning in the bulk of the thread.

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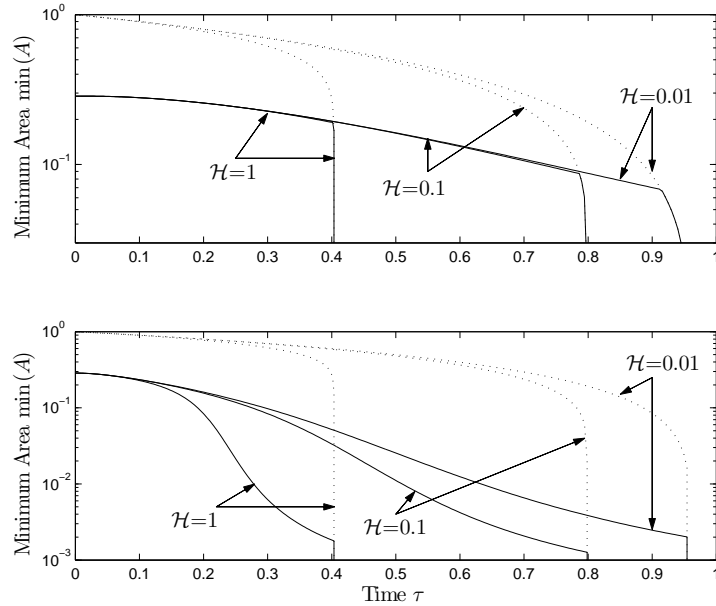


FIGURE 5. The minimum cross-sectional area is plotted against the time for various values of  $\mathcal{H}$  for a)  $R = 1$  and b)  $R = 0.1$ . The initial condition is given by (4.9) with  $K = 10$ . The dotted line represents the cross-sectional area at the end point.

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