York University,
Department of Mathematics and Statistics
Math 1014N
Mar. 2, 2018
Test 2

First name (please write as legibly as possible within the boxes)

Last name

Solutions

Student ID number

Instructions.

1) NO calculators or other aids.
2) There 6 questions on 6 pages and 4 extra space pages. Let the grader know, where your work is by page number, when it is on an "extra space"page.
3) Show all work for full credit.

1. (15 pts.) Calculate the length of the arc defined in polar coordinates by $\boldsymbol{r(\theta) = \theta^2}$**, where** $\boldsymbol{0 \leq \theta \leq \pi}$**.**

$$ds = \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

5

length:

$$S = \int_0^{\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

5

$$= \int_0^{\pi} |\theta| \sqrt{\theta^2 + 4}\, d\theta$$

$$= \int_0^{\pi} \theta (\theta^2 + 4)^{1/2}\, d\theta$$

$$= (\theta^2 + 4)^{3/2} \frac{1}{3} \Big|_0^{\pi}$$

$$= \frac{1}{3}\left[(\pi^2 + 4)^{3/2} - 8\right]$$

5

Hao

2. (15 pts.) Calculate the surface area of the surface obtained by revolving the arc $\gamma(t) = (t^2, t^6)$, where $0 \leq t \leq 1$, about the x-axis.

$$ds = \sqrt{4t^2 + 36t^{10}}\, dt$$

5

$$dA = 2\pi t^6 \sqrt{4t^2 + 36t^{10}}\, dt$$

$$= 2\pi t^7 \sqrt{4 + 36t^8}\, dt$$

5

$$A = 2\pi \int_0^1 t^7 \sqrt{4 + 36t^8}\, dt$$

$$= 2\pi \cdot (4 + 36t^8)^{3/2} \frac{2/3}{(36)\cdot 8} \Big|_0^1$$

$$= \frac{\pi}{(72)(3)} \left[(40^{3/2} - 8 \right]$$

5

Richard

3. (10 pts.) Solve the differential equation $e^{-x^2}y'-y=0$, with $y(0)=1$.

$y' e^{-x^2} = y$ ③

$\frac{y'}{y} = e^{x^2} \Rightarrow \int y'/y = \int e^{x^2}$ ②

$\ln y = \int_0^x e^{t^2} dt + C$ ③

$y(0)=1 \Rightarrow 0 = \ln(y(0))$
$= \ln(1) = C.$

$C=0$

$\ln y = \int_0^x e^{t^2} dt$

$y = e^{\int_0^x e^{t^2} dt}$ ②

Mahmuda

4. (20pts.) Solve the differential equation $y' + y/x = e^{x}$ with $y(1) = 2$.

Integrating factor

$\frac{y'}{y} = 1/x. \quad \ln y = \ln x$

$y = x$

or $e^{\int dx/x} = x$

5

$x y' + y = x e^{x}$

$\frac{d}{dx}(yx) = xe^{x}.$

5

$yx = \int xe^{x}dx = xe^{x} - \int e^{x} = xe^{x} - x + c$

$u = x. \quad dv = e^{x}dx$

$v = ex$

7

$y = e^{x} - e^{x}/x + c/x$

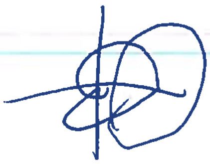
$2 = e^{1} \cancel{+} e^{1} + c \Rightarrow c = 2.$

$y = e^{x} - e^{x}/x + 2/x$

5

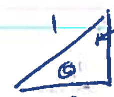
Justin

5. (20 pts.) Calculate the area outside the cardiod $r = 1 + \frac{\cos(\theta)}{\sqrt{3}}$ and inside the circle $r = \frac{3}{\sqrt{3}}\cos(\theta)$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.



5

$1 + \frac{\cos\theta}{\sqrt{3}} = \frac{3}{\sqrt{3}}\cos\theta$ iff $\cos\theta = \frac{\sqrt{3}}{2}$.



$\pi/3$, 1, $1/2$, $\sqrt{3}/2$, $\theta = \pm \pi/6$

$\theta = \pm \pi/6$

$dA = r\,dr\,d\theta$ $\quad A = \int_{-\pi/6}^{\pi/6} \int_{1+\frac{\cos\theta}{\sqrt{3}}}^{\sqrt{3}\cos\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} r^2/2 \Big|_{1+\frac{\cos\theta}{\sqrt{3}}}^{\sqrt{3}\cos\theta}$

$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} 3\cos^2\theta - 1 - 2/\sqrt{3}\cos\theta - \frac{\cos^2\theta}{3}$ 5

$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{8}{3}\cos^2\theta - 1 - 2/\sqrt{3}\cos\theta$ 5

$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} 8/3 \cos\frac{2\theta+1}{2} - 1 - \frac{2}{\sqrt{3}}\cos\theta$

$= 1/2\left(\frac{4}{3}\frac{\sin 2\theta}{2}\Big|_{-\pi/6}^{\pi/6} + \frac{4}{3}\theta\Big|_{-\pi/6}^{\pi/6} - \theta\Big|_{-\pi/6}^{\pi/6} - \frac{2}{\sqrt{3}}\sin\theta\Big|_{-\pi/6}^{\pi/6}\right)$

$= 1/2\left(\frac{2\sqrt{3}}{3} + 1/3\,\frac{2\pi}{6} - \frac{2}{\sqrt{3}}\right) = \frac{\pi}{18}$ 5

Masoud.

Work Space

6. (15pts.) A tank contains 30 kgs. of salt dissolved in 5000 liters of water. Brine with 0.4kg/liter enters at a rate of 10 liters/min. The solution is kept at a uniform concentration of salt and drains at a rate of 10 liters/min. How long will it take to reduce the amount of salt to 15 grams ?

$A(t)$ = amount of salt at time t 2

$A'(t)$ = amount in/min − amount out/min

$= (.4)10 - \frac{A}{5000}\cdot(10)$

$= 4 - A/500$ 4

$A' + \frac{A}{500} = 4$

Integrating factor $e^{\int dt/500} = e^{t/500}$ 2

$e^{t/500}A' + e^{t/500}A/500 = 4e^{t/500}$ 1

$\frac{d}{dt}(Ae^{t/500}) = 4e^{t/500}$ ~~1~~

$A = 4e^{t/500}(500) + C$ 1

Kanchan

Work Space

$A(t) = 2000 + c/e^{x/500}$

$30 = 2000 + c \Rightarrow c = -1970$

$A(t) = 2000 - 1970/e^{t/500}$ kg.

2

$\frac{15}{1000}$ kg $= 2000 - 1970/e^{t/500}$

$\frac{1970}{e^{t/500}} = 2000 - .015$

$e^{t/500} = \cancel{\frac{(2000-.015)}{19}} \frac{1970}{(2000-.015)}$

$t/500 = \ln\left(\frac{1970}{2000-.015}\right)$

$t = 500 \ln\left(\frac{1970}{2000-.015}\right)$ min

3

Work Space