York University

Faculty of Arts

Faculty of Pure and Applied Science

September - December 2002

AS/SC/MATH 2221 3.0 C
Midterm Test 1
SOLUTIONS

1. **(a)** Given the system of linear equations

$$\begin{cases} x_1 & - & x_2 & - & 4x_3 & = & -4 \\ x_1 & + & 2x_2 & + & 5x_3 & = & 2 \\ x_1 & + & x_2 & + & 2x_3 & = & 0 \end{cases}$$

Write the system in matrix form $AX = B$ and express the general solution to the system as a sum of its particular solution and the general solution of the associated homogeneous system $AX = 0$.

Hint: Use the Gaussian Algorithm to solve the linear system.

Answer:

$$\begin{bmatrix} 1 & -1 & -4 \\ 1 & 2 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -4 & -4 \\ 1 & 2 & 5 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -4 & -4 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -4 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 6 & 4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -4 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding linear system

$$\begin{cases} x_1 & & & - & x_3 & = & -2 \\ & & x_2 & + & 3x_3 & = & 0 \\ & & & & 0 & = & 0 \end{cases}$$

has the general solution $\begin{cases} x_1 & = & -2 & + & s \\ x_2 & = & 2 & - & 3s \\ x_3 & = & & & s \end{cases}$,

which can be written as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + = s \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = X_0 + sX_1,$$

where X_0 is a particular solution of $AX = B$, and sX_1 is the general solution of $AX = 0$.

(b) If a linear system of 7 equations in 9 variables has a solution, explain why there are more than one solutions.

Answer:
There are more than one solutions since the reduced row-echelon form of the augmented matrix may have at most 7 leading 1's, and consequently, the corresponding system will have at least $2 = 9 - 7$ non-leading (free) variables.

2. A man must take 4 units of vitamin A, 17 units of vitamin B and 25 units of vitamin C each day. Three brands fo vitamin pills are available and the number of units of each vitamin per pill are given in the following table:

Brand	A	B	C
1	1	2	4
2	1	1	3
3	0	1	1

Find all combinations of pills that provide the exact daily requirements.

Write the problem as a system of linear equations but do NOT solve it.

Answer:
Let
$x_1 =$ the number of pills of brand 1 taken each day,
$x_2 =$ the number of pills of brand 2 taken each day,
$x_3 =$ the number of pills of brand 3 taken each day.
Then problem can be formulated as the following system of linear equations:

$$\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$$

3. **(a)** If A is a matrix and the system $AX = 0$ has a non-trivial solution, show that there is no matrix B, such that $BA = I$.

Answer:
Assume that the system $AX = 0$ has a non-trivial solution X_0, and there is a matrix B, such that $BA = I$. Then multiplying both sides of the equation $AX_0 = 0$ by B on the left, we obtain, $(BA)X_0 = B0$, $IX_0 = 0$, or $X_0 = 0$, i.e. we arrive to the contradiction.

(b) Suppose that A is a 7×7 matrix and $AX = 0$ has a unique solution. Is $AX = B$ consistent for each column B of 7×1? If so, is the solution unique? Justify your answer.

Answer:
Yes, the system $AX = B$ is consistent. Moreover, since A is a square matrix, $AX = 0$ has a unique solution $\iff$ the system has only the trivial solution $\iff$ A is invertible. Therefore,
$\forall B[7 \times 1]\ \exists! X\ :\ X = A^{-1}B.$

4. **(a)** Write a strategy to solve a linear system $AX = B$ if A is an invertible matrix.

Answer:
To solve the linear system $AX = B$ we need to find the inverse A^{-1} of the coefficient matrix A, and then determine the solution to the system as $X = A^{-1}B$.

(b) Solve the system given below by finding the inverse of the coefficient matrix if possible.

$$\begin{cases} 3x_1 + 5x_2 = -5 \\ x_1 + 2x_2 + x_3 = 1 \\ 3x_1 + 7x_2 + x_3 = 0 \end{cases}$$

Answer:

$$[A \;\; I] = \left[\begin{array}{ccc|ccc} 3 & 5 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 7 & 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 & -3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & -1/5 & 6/5 & -1/5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1/5 & -1/5 & 1/5 \\ 0 & 1 & 0 & -2/5 & -3/5 & 3/5 \\ 0 & 0 & 1 & -1/5 & 6/5 & -1/5 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -2/5 & -3/5 & 3/5 \\ 0 & 0 & 1 & -1/5 & 6/5 & -1/5 \end{array}\right] = [I \;\; A^{-1}].$$

Hence, $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -2/5 & -3/5 & 3/5 \\ -1/5 & 6/5 & -1/5 \end{bmatrix}$, and consequently,

$$X = A^{-1}B = \begin{bmatrix} 1 & 1 & -1 \\ -2/5 & -3/5 & 3/5 \\ -1/5 & 6/5 & -1/5 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 7/5 \\ 11/5 \end{bmatrix}.$$

5. **(a)** While trying to invert a matrix A, the double matrix $[A \;\; I]$ is carried to $[P \;\; Q]$. Show that $P = QA$.

Answer:
Assume that a sequence of ERO's that transforms $[A \;\; I]$ to $[P \;\; Q]$ consists of k ERO's. Then there exist elementary matrices $E_1, E_2, \cdots, E_k$, such that $E_k E_{k-1} \cdots E_1 A = P$ and $E_k E_{k-1} \cdots E_1 I = Q$, that is $QA = P$.

(b) Given the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$. Find an invertible matrix U, such that $UA = R$ is the reduced row-echelon form of A.

Answer:

$$A = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$
$= I = R$.

So, the ERO's that transform A to R are:
– interchange the rows;
– subtract 3 times row 1 from row 2;
– multiply the second row by $\frac{1}{8}$;

– add 3 times row 2 to row 1.
The corresponding elementary matrices are:
$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.
Hence, $U = E_4(E_3(E_2E_1)) = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$. The matrix U is invertible as it is a product of elementary matrices.

The end