York University

Faculty of Arts

Faculty of Pure and Applied Science

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AS/SC/MATH 2221 3.0 C
Midterm Test 2a
SOLUTIONS

1. Indicate whether the statement is always true or sometimes false. Justify each answer by giving a logical argument or a counter example.

   (a) The sum of two triangular matrices is a triangular matrix.

   Answer:

   Flase. For instance, the sum of an upper triangular and lower triangular matrices is not a triangular matrix.

   (b) AA^T is a symmetric matrix.

   Answer:

   True. Let A be an $m \times n$ matrix. Then A^T is an $m \times n$ matrix and AA^T will be a square matrix of $m \times m$. On the other hand,
   $(AA^T)^T = (A^T)^T(A)^T = AA^T$.

   (c) If AA^T is singular, then so is A.

   Answer:

   True. By contradiction, let's assume that AA^T is singular, and A is not, i.e. A is invertible. But, if A were invertible, then A^T would be invertible, and consequently, AA^T would also be invertible as the product of two invertible matrices.

   (d) If A^2 is symmetric, then so is A.

   Answer:

   Flase. For example, consider a matrix $A = \begin{bmatrix} a & 0 \\ a & -a \end{bmatrix}$, $\quad \forall a \in \mathbb{R}$. $A^2 = AA$
   $= \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$ is a symmetric matrix, while A is not.

2. Given the system
$$\begin{cases} 5x_1 & - & 3x_2 & = & \lambda x_1 \\ x_1 & + & x_2 & = & \lambda x_2 \end{cases}$$

(a) Write the system in the matrix form $(\lambda I - A)X = 0$.

Answer:

$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix} = - \begin{bmatrix} \lambda - 5 & 3 \\ -1 & \lambda - 1 \end{bmatrix}.$

Hence, we obtain

$\begin{bmatrix} \lambda - 5 & 3 \\ -1 & \lambda - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

(b) Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$.

Answer:

The characteristic equation is

$det(\lambda I - A) = (\lambda - 5)(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0.$

Hence, the eigenvalues of the matrix A are $\lambda_1 = 2$ and $\lambda_2 = 4$.

For $\lambda_1 = 2$, we have the augmented matrix

$\begin{bmatrix} -3 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

So, we obtain $x_2 = s$ and $x_1 - s = 0$, i.e. $x_1 = s$.

Hence, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the corresponding to $\lambda_1 = 2$ eigenvector.

For $\lambda_2 = 4$, the augmented matrix

$\begin{bmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

So, we obtain $x_2 = t$ and $x_1 - 3t = 0$, i.e. $x_1 = 3t$.

Hence, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ will be the eigenvector corresponding to $\lambda_2 = 4$.

3. **(a)** Let A be an $n \times n$ matrix. Express $det(A + A)$ in terms of $det(A)$.

Answer:

$det(A + A) = det(2A) = 2^n det(A)$,

since A is an $n \times n$ matrix.

(b) Let B be an invertible matrix of $n \times n$. Show that $det(adj(B)) = (det(B))^{n-1}$.

Answer:

Since B is invertible, $B\, adj(B) = det(B)\, I_n$.

Then $det(B\, adj(B)) = det(det(B)\, I_n)$,

$det(B)\, det(adj(B)) = (det(B))^n\, det(I_n)$,

$det(B)\, det(adj(B)) = (det(B))^n \cdot (1)$,

$det(adj(B)) = \frac{(det(B))^n}{det(B)} = (det(B))^{n-1}$.

4. **(a)** Explain in your own words how to find the determinant of a matrix using cofactor expansion.

   Answer:

   Choose a row or column having the largest number of zeros. Multiply each entry of that row (or column) by its cofactor and then add up obtained products.

   (b) Evaluate the determinants of the matrices given below using cofactor expansion:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix}.$$

   Answer:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3.$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = (2 - (-2)) + (4 + 0) = 8.$$

   (c) Use the Cramer's Rule to solve the system

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_2 - x_3 = 1 \\ x_2 + 2x_3 = 2 \end{cases}$$

   for x_2 without solving for x_1 and x_3.

   Answer:

   From part (b), the determinant of the coefficient matrix A is equals 3, and $det(A_2) = 8$. Hence,

$$x_2 = \frac{det(A_2)}{det(A)} = \frac{8}{-3} = -\frac{8}{3}.$$

5. Consider a triangle with vertices A, B and C, and let E and F be the midpoints of the sides AB and BC. Show that the line joining E and F is parallel to AC and the length of EF is a half of the length of AC.

   Answer:

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \tfrac{1}{2}\overrightarrow{AC}.$$

   Hence, $\overrightarrow{EF}$ is parallel to $\overrightarrow{AC}$ and the length
   $|| \overrightarrow{EF} || = || \frac{1}{2}\overrightarrow{AC} || = | \frac{1}{2} | \, || \overrightarrow{AC} || = \frac{1}{2} \, || \overrightarrow{AC} ||$.

The end