# York University

## Faculty of Arts

## Faculty of Pure and Applied Science

**September - December 2002**

## AS/SC/MATH 1300 3.0 B

## Midterm Test 3b

**SOLUTIONS**

1. Let $f(x) = \dfrac{2x}{x^2+1}$.

   **(a)** Find the $x$ and $y-$intercepts of $f(x)$.

   **(b)** Determine the equations of all asymptotes.

   **(c)** Where is the function increasing and where decreasing?

   **(d)** Where is the function concave up and where concave down?

   **(e)** Locate any local extrema and any inflection points.

   **(f)** Sketch the curve $y = f(x)$ on the space provided.

   *Answers:*

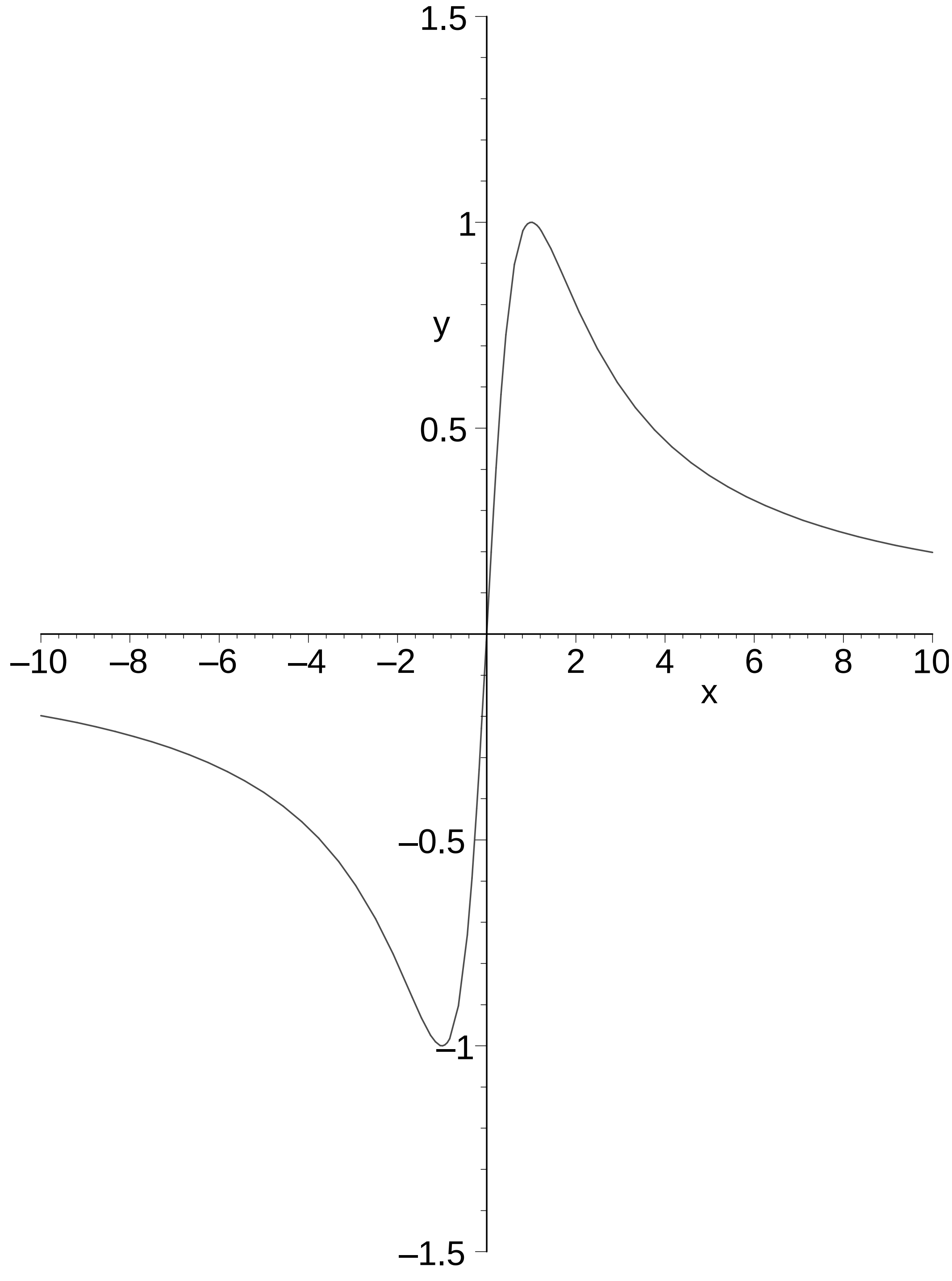
   **(a)** $f(0) = 0$ and $f(x) \neq 0$ $(\forall x \neq 0)$. Hence, $(0,0)$ is the only $x$ and $y-$intercept of $f(x)$.

   **(b)** $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 0$. Hence, $y = 0$ is a horizontal asymptote on the right and on the left.
   The denominator $x^2+1 > 0$ $(\forall x)$, hence the function has no vertical asymptotes.

   **(c)** $f'(x) = -\dfrac{2(x+1)(x-1)}{(x^2+1)^2}$, and $f'(x) = 0 \iff x = \pm 1$. Hence, $x = \pm 1$ are critical points.
   $f'(x) < 0$ whenever $x < -1$. Hence, $f(x)$ is decreasing on $(-\infty, -1)$.
   $f'(x) > 0$ whenever $-1 < x < 1$. Hence, $f(x)$ is increasing on $(-1, 1)$.
   $f'(x) < 0$ whenever $x > 1$. Hence, $f(x)$ is decreasing on $(1, \infty)$.

   **(d)** $f''(x) = \dfrac{4x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$, and $f''(x) = 0 \iff x = \pm\sqrt{3}$. Hence, $x = \pm\sqrt{3}$ are possible inflection points.
   $f''(x) < 0$ whenever $x < -\sqrt{3}$. Hence, $f(x)$ is concave down on $(-\infty, -\sqrt{3})$.
   $f'(x) > 0$ whenever $-\sqrt{3} < x < 0$. Hence, $f(x)$ is concave up on $(-\sqrt{3}, 0)$.
   $f'(x) < 0$ whenever $0 < x < \sqrt{3}$. Hence, $f(x)$ is concave down on $(0, \sqrt{3})$.
   $f'(x) > 0$ whenever $x > \sqrt{3}$. Hence, $f(x)$ is concave up on $(\sqrt{3}, \infty)$.

**(e)** The point $(-1, f(-1)) = (-1, -1)$ is a local minima since $f'(x) < 0$ whenever $x < -1$, and $f'(x) > 0$ whenever $-1 < x < 1$.
The point $(1, f(1)) = (1, 1)$ is a local maxima since $f'(x) > 0$ whenever $-1 < x < 1$, and $f'(x) < 0$ whenever $x > 1$.
The point $(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, -\frac{\sqrt{3}}{2})$ is an inflection point since $f''(x) < 0$ whenever $x < -\sqrt{3}$, and $f''(x) > 0$ whenever $-\sqrt{3} < x < 0$, i.e. at $x = -\sqrt{3}$ the concavity of the function changes.
The point $(0, f(0)) = (0, 0)$ is an inflection point since $f''(x) > 0$ whenever $-\sqrt{3} < x < 0$, and $f''(x) < 0$ whenever $0 < x < \sqrt{3}$, i.e. at $x = 0$ the concavity of the function changes.
The point $(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, \frac{\sqrt{3}}{2})$ is an inflection point as well, since $f''(x) < 0$ whenever $0 < x < \sqrt{3}$, and $f''(x) > 0$ whenever $x > \sqrt{3}$, i.e. at $x = \sqrt{3}$ the concavity of the function changes.
Observe that $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function and its graph is symmetric about the origin.




**(f)**

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2. Evaluate each of the following limits:

**(a)** $\lim_{x\to 0} \frac{\arcsin x - x}{\sin x - x}$

**(b)** $\lim_{x\to 1} \frac{x^m - 1}{x^n - 1}$

**(c)** $\lim_{x\to \frac{\pi}{2}} (\tan x - \sec x)$.

*Answers:*

**(a)** Applying L'Hopital's rule twice,

$$\lim_{x\to 0} \frac{\arcsin x - x}{\sin x - x} = \lim_{x\to 0} \frac{(1-x^2)^{-\frac{1}{2}} - 1}{\cos x - 1}$$
$$= \lim_{x\to 0} \frac{(-\frac{1}{2})(1-x^2)^{-\frac{3}{2}}(-2x)}{-\sin x} = \lim_{x\to 0}[\frac{x}{-\sin x}(1-x^2)^{-\frac{3}{2}}]$$
$$= -\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}} \cdot \lim_{x\to 0}(1-x^2)^{-\frac{3}{2}} = -\frac{1}{1}\cdot 1 = -1.$$

**(b)** Applying L'Hopital's rule,

$$\lim_{x\to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x\to 1} \frac{mx^{m-1}}{nx^{n-1}}$$
$$= \frac{m}{n} \lim_{x\to 1} \frac{x^{m-1}}{x^{n-1}} = \frac{m}{n}\frac{1}{1} = \frac{m}{n}.$$

**(c)** First we write indefinite form as a ratio, and then apply L'Hopital's rule,

$$\lim_{x\to \frac{\pi}{2}} (\tan x - \sec x) = \lim_{x\to \frac{\pi}{2}} (\frac{\sin x}{\cos x} - \frac{1}{\cos x})$$
$$= \lim_{x\to \frac{\pi}{2}} \frac{\sin x - 1}{\cos x} = \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

3. An object moves along the $x$-axis, its position at each time $t \geq 0$ is given by $s(t) = 5t^4 - t^5$.
Determine the time interval(s), if any, during which the object is speeding up to the right.

*Answer:*
$v(t) = \frac{ds}{dt} = 5t^3(4-t) > 0 \iff 0 < t < 4$. Hence, the object moves to the right when $0 < t < 4$.
$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = 20t^2(3-t) > 0 \iff 0 < t < 3$. Hence, the object is speeding up when $0 < t < 3$.
Thus, the object is speeding up to the right on the time interval $(0, 3)$, since
$v(t) > 0 \;\&\; a(t) > 0 \quad (\forall x \in (0,3))$.

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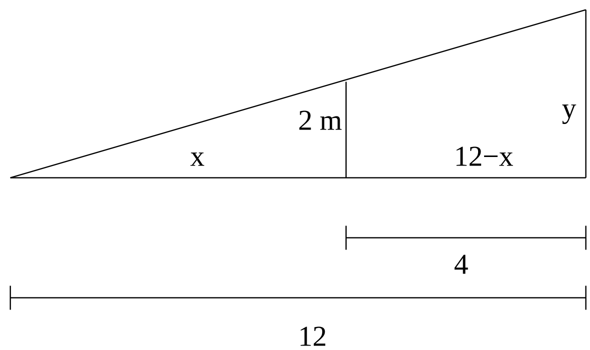
4. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed 1.6 m/sec, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

*Answer:*




Let $x$ be a distance of the man from the spotlight and $y$ be the length of his shadow on the building.

$\dfrac{dx}{dt} = 1.6\dfrac{m}{sec}$.

We need to determine $\dfrac{dy}{dt}$ at the instant when $12 - x = 4$, i.e. $x = 8$.

From the similar triangles,

$\dfrac{y}{12} = \dfrac{2}{x}$,

$y = \dfrac{24}{x} = 24x^{-1}$.

Hence, $\dfrac{dy}{dt} = \dfrac{d}{dt}(24x^{-1}) = -24x^{-2}\dfrac{dx}{dt} = -\dfrac{24}{x^2} \cdot \dfrac{dx}{dt}$.

When $x = 8$, and $\dfrac{dx}{dt} = 1.6$, we obtain

$\dfrac{dy}{dt} = -\dfrac{24}{8^2} \cdot (1.6) = -0.6$.

Thus, when $12 - x = 4$, the shadow of the man on the building is decreasing at a rate of 0.6 m/sec.

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5. A rectangular box with a square base is to be made using two different materials. The material for the top and sides of the box costs $\$1/\,m^2$, while for the base costs $\$2/\,m^2$. Find the dimensions of such a box of the greatest volume subject to the condition that $144.00 is spent on the material to make it.

*Answer:*
Let $x$ be the length of the base and $h$ be the length of the hight of the box. Then the volume of the box $V = x^2h$.
The areas of the base and the top of the box are $x^2$, the area of its sides is $4xh$.
We are given that $2 \cdot x^2 + 1 \cdot (x^2 + 4xh) = 144$, from which we find, $h = \dfrac{144 - 3x^2}{4x}$.
Hence, the volume of the box
$V(x) = \dfrac{144x - 3x^3}{4}$.
Thus, we obtain the optimization problem
$V(x) = \dfrac{144x - 3x^3}{4} \Longrightarrow max, \quad x \in [0, 12]$.
$V'(x) = \dfrac{1}{4}(144 - 9x^2) = 0 \iff x = \pm 4$.
But $(-4) \notin Dom(V)$, hence $x = 4$ is only the critical point.
$V(0) = V(12) = 0, \;\; V(4) = \dfrac{144(4) - 3(4)^3}{4} = 96$.
Hence, for the box of the greatest volume $x = 4$ and $h = \dfrac{144 - 3(4)^2}{4(4)} = 6$.
Alternatively, $V'(x) > 0 \quad (\forall x < 4)$ & $V'(x) < 0 \quad (\forall x > 4)$ implies that $x = 4$ is the absolute maximum point of the function.

**The end**