York University

Faculty of Arts

Faculty of Pure and Applied Science

September - December 2002

AS/SC/MATH 1515 3.0 A
Midterm Test 1b
SOLUTIONS

1. Find the exact value (i.e. not as a number found by a calculator) of:

   (a) $\sin \frac{3\pi}{4}$;

   (b) $\cos^2 \frac{\pi}{16} - \sin^2 \frac{\pi}{16}$.

   Answers:

   (a) $\sin(\frac{3\pi}{4}) = \sin(\pi - \frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

   (b) $\cos^2 \frac{\pi}{16} - \sin^2 \frac{\pi}{16} = \cos(2\frac{\pi}{16}) = \cos \frac{\pi}{8}$.
   But $\cos \frac{\pi}{4} = 2\cos^2 \frac{\pi}{8} - 1 \Longrightarrow \frac{1}{\sqrt{2}} = 2\cos^2 \frac{\pi}{8} - 1 \Longrightarrow \cos^2 \frac{\pi}{8} = \frac{1}{2\sqrt{2}} + \frac{1}{2} \Longrightarrow$
   $\cos \frac{\pi}{8} = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$.

2. Solve the equation $\sin \frac{x}{2} + \cos \frac{x}{2} = \sqrt{2}$ for x, where $0 < x < 2\pi$.
   Hint: Square both sides of the equation first.

   Answer:
   $(\sin \frac{x}{2} + \cos \frac{x}{2})^2 = (\sqrt{2})^2$,
   $\sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2} + \cos^2 \frac{x}{2} = 2$,
   $\sin x = 1 \Longrightarrow x = \frac{\pi}{2}$.

3. Evaluate the following limit:

   (a) $\lim_{x \to \frac{\pi}{4}} \sin 2x$;

   (b) $\lim_{x \to 0} \frac{x}{\tan 3x}$.

   Answers:

   (a) $\lim_{x \to \frac{\pi}{4}} \sin 2x = \sin(2\frac{\pi}{4}) = \sin \frac{\pi}{2} = 1$.

(b) $\lim_{x\to 0} \dfrac{x}{\tan 3x} = \lim_{x\to 0} \dfrac{x\cos 3x}{\sin 3x} = \lim_{x\to 0} \dfrac{1}{3}\dfrac{3x}{\sin 3x}\cos 3x$
$= \dfrac{1}{3}\lim_{x\to 0} \dfrac{1}{\frac{\sin 3x}{3x}}\cos 3x = \dfrac{1}{3}\dfrac{1}{\lim_{x\to 0}\frac{\sin 3x}{3x}}\lim_{x\to 0}\cos 3x = \dfrac{1}{3}\cdot 1\cdot 1 = \dfrac{1}{3}.$

4. Use the properties of limits to find the following limits if they exist, or state that they do not exist. Justify your answer.

(a) $\lim_{x\to 0} \dfrac{x^2+x-2}{x^2-4x+3}$

(b) $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-4x+3}$

(c) $\lim_{x\to 2^-} \dfrac{|\,x-2\,|}{x-2}.$

Answers:

(a) $\lim_{x\to 0} \dfrac{x^2+x-2}{x^2-4x+3} = \dfrac{0+0-2}{0-0+3} = -\dfrac{2}{3}.$

(b) $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-4x+3} = \lim_{x\to 1} \dfrac{(x-1)(x+2)}{(x-1)(x+3)}$
$= \lim_{x\to 1} \dfrac{x+2}{x+3} = \dfrac{1+2}{1+3} = \dfrac{3}{4}.$

(c) $\lim_{x\to 2^-} \dfrac{|\,x-2\,|}{x-2} = \lim_{x\to 2^-} \dfrac{-(x-2)}{x-2} = \lim_{x\to 2^-}(-1) = -1.$

5. Let $f(x) = \begin{cases} (x+1)^2 & \text{if } x < 0 \\ x-2 & \text{if } x \geq 0 \end{cases}$

(a) Where is $f(x)$ discontinuous? Why?

(b) Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Answers:

(a) $f(x)$ is discontinuous at $x = 0$, because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1)^2 = (0+1)^2 = 1.$
and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-2) = 0-2 = -2.$

The end