York University

Faculty of Arts
Faculty of Pure and Applied Science

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AS/SC/MATH 1515 3.0 B
Midterm Test 1a

SOLUTIONS

1. Find the exact value (i.e. not as a number found by a calculator) of:

   (a) $\tan(-\frac{2\pi}{3})$;

   (b) $(\sin\frac{\pi}{8} - \cos\frac{\pi}{8})^2$.

   Answers:

   (a) $\tan(-\frac{2\pi}{3}) = -\tan\frac{2\pi}{3} = -\tan(\pi - \frac{\pi}{3}) = -(-\tan\frac{\pi}{3}) = \tan\frac{\pi}{3} = \sqrt{3}$.

   (b) $(\sin\frac{\pi}{8} - \cos\frac{\pi}{8})^2 = \sin^2\frac{\pi}{8} - 2\sin\frac{\pi}{8}\cos\frac{\pi}{8} + \cos^2\frac{\pi}{8}$
   $= 1 - \sin\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{\sqrt{2}}$.

2. Solve the equation $2\cos^2 2x + \sin 2x = 2$ for x, where $0 < x < \pi$.
   Hint: Write the equation in the terms of $\sin 2x$ first.

   Answer:
   $2\cos^2 2x + \sin 2x = 2$,
   $2(1 - \sin^2 2x) + \sin 2x = 2$,
   $2 - 2\sin^2 2x + \sin 2x = 2$,
   $2\sin^2 2x - \sin 2x = 0$,
   $\sin 2x(2\sin 2x - 1) = 0$.
   $\sin 2x = 0$ has no solutions on the interval $0 < x < \pi$.
   $2\sin 2x - 1 = 0 \Longrightarrow \sin 2x = \frac{1}{2} \Longrightarrow 2x = \frac{\pi}{6}, \frac{5\pi}{6} \Longrightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$.

3. Evaluate the following limits:

   (a) $\lim_{x\to\frac{\pi}{8}} \cos 4x$;

   (b) $\lim_{x\to 0} \frac{5\sin x - 3x}{6x}$.

   Answers:

   (a) $\lim_{x\to\frac{\pi}{8}} \cos 4x = \cos(4\frac{\pi}{8}) = \cos\frac{\pi}{2} = 0$.

(b) $\lim\limits_{x\to 0}\dfrac{5\sin x - 3x}{6x} = \lim\limits_{x\to 0}(\dfrac{5\sin x}{6x} - \dfrac{3x}{6x}) = \lim\limits_{x\to 0}(\dfrac{5}{6}\dfrac{\sin x}{x} - \dfrac{1}{2}) = \dfrac{5}{6} - \dfrac{1}{2} = \dfrac{1}{3}.$

4. Use the properties of limits to find the following limits if they exist, or state that they do not exist. Justify your answer.

   (a) $\lim\limits_{x\to 0}\dfrac{3-\sqrt{x}}{x-9}$

   (b) $\lim\limits_{x\to 9}\dfrac{3-\sqrt{x}}{x-9}$

   (c) $\lim\limits_{x\to 5^-}\sqrt{x(5-x)}.$

   Answers:

   (a) $\lim\limits_{x\to 0}\dfrac{3-\sqrt{x}}{x-9} = \dfrac{3-0}{0-9} = -\dfrac{1}{3}.$

   (b) $\lim\limits_{x\to 9}\dfrac{3-\sqrt{x}}{x-9} = \lim\limits_{x\to 9}\dfrac{-(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$
   $= \lim\limits_{x\to 9}\dfrac{-1}{(\sqrt{x}+3)} = -\dfrac{1}{3+3} = -\dfrac{1}{6}.$

   (c) $\lim\limits_{x\to 5^-}\sqrt{x(5-x)} = \lim\limits_{x\to 5^-}\sqrt{5x-x^2} = 0$, since whenever $x \to 5^-$, the expression under the square root remains positive.

5. Let $f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

   (a) Where is $f(x)$ discontinuous? Why?

   (b) Find $\lim\limits_{x\to -1^-} f(x)$ and $\lim\limits_{x\to -1^+} f(x)$.

   Answers:

   (a) $f(x)$ is discontinuous at $x = -1$, because $\lim\limits_{x\to -1^-} f(x) \neq \lim\limits_{x\to -1^+} f(x)$.

   (b) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-}(x-1) = -1-1 = -2.$
   and $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} x^2 = (-1)^2 = 1.$

The end