MATH 2320 P
QUIZ #2b
March 11, 2003
SOLUTIONS

1. (6 marks) Consider the following list of sets:

$$\emptyset,\ \{1,2,3,4\},\ \{5,6,7,8\},\ (0,1),\ \mathbb{Z}^+,\ \mathbb{Q},\ \mathbb{R},\ \mathbb{R}-\mathbb{Q}.$$

   (a) Identify three sets in the list with the same cardinality or explain why this is impossible.
   Answer: The sets $(0,1)$, $\mathbb{R}$ and $\mathbb{R}-\mathbb{Q}=\mathbb{I}$ have the same cardinality, they are infinite uncountable.

   (b) Find set(s) in the list with the same cardinality as $\mathbb{N}$, or explain briefly why none exists.
   Answer:
   The sets $\mathbb{Z}^+$ and $\mathbb{Q}$ have the same cardinality as $\mathbb{N}$, they are infinite countable.

2. (7 marks) Use the Principle of Mathematical Induction to prove that $2n+3 \le 2^n$, for all integers $n \ge 4$.
Answer:
Prove: $(\forall n \in \mathbb{N})$ with $n \ge 4$, $P(n):\ \ 2n+3 \le 2^n$.
Basis Step: $P(4)$ is true, since
$LHS = 2\cdot 4+3 = 11$, while $RHS = 2^4 = 16$.
Induction Hypothesis: Assume that $P(k)$ is true, i.e. $2k+3 \le 2^k$.
Induction Step: Then $P(k+1)$ must be true also.
$LHS = 2(k+1)+3 = (2k+3)+2$
$\le 2^k + 2$ (by the induction hypothesis)
$\le 2^k + 2^k \quad (\because k \ge 4)$
$= 2\cdot 2^k = 2^{k+1} = RHS.$

3. (7 marks)

   (a) Define recursively the set of binary strings of even length each of which starts with 1.
   Answer:
   Basis Step: $10, 11 \in S$.
   Recursive Step: $b \in S \implies b00 \in S \ \&\ b01 \in S \ \&\ b10 \in S \ \&\ b11 \in S$.

   (b) Give a recursive definition of the sequence $\{a_n\}_{n=1}^{\infty}$, if $a_n = n^2 - n + 2$.
   Answer:
   Basis Step: $a_1 = 2$.
   Recursive Step: $a_n = a_{n-1} + 2(n-1), \ \ \forall n \ge 2$.

The end