

**MATH 2320 A-F**  
**QUIZ #2 SOLUTIONS**  
November 05, 2001

1. (5 marks) Use the Euclidean algorithm to find  $\gcd(1356, 414)$ .

*Solution:*

$$1356 = 414 \cdot 3 + 114, \text{ so } \gcd(1356, 414) = \gcd(414, 114);$$

$$414 = 114 \cdot 3 + 72, \text{ so } \gcd(414, 114) = \gcd(114, 72);$$

$$114 = 72 \cdot 1 + 42, \text{ so } \gcd(114, 72) = \gcd(72, 42);$$

$$72 = 42 \cdot 1 + 30, \text{ so } \gcd(72, 42) = \gcd(42, 30);$$

$$42 = 30 \cdot 1 + 12, \text{ so } \gcd(42, 30) = \gcd(30, 12);$$

$$30 = 12 \cdot 2 + 6, \text{ so } \gcd(30, 12) = \gcd(12, 6);$$

$$12 = 6 \cdot 2 + 0, \text{ so } \gcd(12, 6) = \gcd(6, 0) = 6.$$

Hence,  $\gcd(1356, 414) = 6$ .

2. (5 marks) Prove that  $x \equiv y \pmod{m}$  if and only if  $x$  and  $y$  have the same remainder when divided by  $m$ .

*Solution:*

$$x \equiv y \pmod{m}$$

$$\iff m \mid (x - y)$$

$$\iff \exists k \in \mathbb{Z}, \text{ such that } x - y = km$$

$$\iff x = km + y$$

$$\iff x \bmod m = y \bmod m, \text{ since } m \mid km.$$

3. (5 marks) Solve the congruence  $20x \equiv 21 \pmod{29}$ .

*Solution:*

Since  $\gcd(20,29)=1$ , there exists an inverse of 20 modulo 29.

Using the Euclidean algorithm, we find that an inverse of 20 modulo 29 is -13.

But,  $-13 \equiv 16 \pmod{29}$ .

Hence, 16 is also an inverse of 20 modulo 29. Multiplying both sides of the congruence by 16, we obtain

$$x \equiv 16 \cdot 21 \pmod{29}$$

$$x \equiv 336 \pmod{29}.$$

Since  $336 \equiv 17 \pmod{29}$ , the general solution of the congruence will be

$$x \equiv 17 \pmod{29}.$$

4. (5 marks) Prove that  $(1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{n+1}) = \frac{1}{n+1}$   
for every positive integer.

*Solution:*

$\forall n \in \mathbb{Z}^+ \quad P(n)$ , where

$$P(n) : (1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{n+1}) = \frac{1}{n+1}.$$

*Base Case:*  $n = 1$

$$LHS = 1 - \frac{1}{1+1} = \frac{1}{2},$$

$$RHS = \frac{1}{1+1} = \frac{1}{2}.$$

*Induction Hypothesis:*

$$P(k) : (1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{k+1}) = \frac{1}{k+1}.$$

*Induction Step:*  $P(k+1)$  ?

$$LHS = (1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{k+1})(1 - \frac{1}{k+2}) =$$

by the Induction Hypothesis

$$= \frac{1}{k+1} (1 - \frac{1}{k+2}) = \frac{1}{k+1} \cdot \frac{k+1}{k+2} = \frac{1}{k+2} = RHS.$$

$$Conclusion: \forall n \in \mathbb{Z}^+ \quad (1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{n+1}) = \frac{1}{n+1}.$$

**The end**