

MATH 2320 3.0 A
Solutions to Term Test 2 (White)

1. (5 marks) Show that if $a \equiv b \pmod{m}$ and $d \mid m$, then $a \equiv b \pmod{d}$.

Solution:

$$a \equiv b \pmod{m} \Rightarrow m \mid a - b$$

$$\Rightarrow \exists k \in \mathbb{Z} : a = km + b.$$

On the other hand, $d \mid m \Rightarrow m = dt$ for some integer t .

$$\text{Hence, } a = d(kt) + b.$$

Therefore, $d \mid a - b$.

2. (7 marks) Find the number that congruent to 5 modulo 12 and also congruent to 13 modulo 19.

Solution:

$$\begin{cases} x \equiv 5 \pmod{12} \\ y \equiv 13 \pmod{19} \end{cases}$$

$$\gcd(12, 19) = 1,$$

$$m = 12 \cdot 19 = 228$$

$$M_1 = \frac{m}{m_1} = \frac{228}{12} = 19,$$

$$M_2 = \frac{m}{m_2} = \frac{228}{19} = 12.$$

The solution

$$x = a_1 M_1 y_1 + a_2 M_2 y_2,$$

where y_1 is an inverse of $M_1 = 19$ modulo 12, and y_2 is an inverse of $M_2 = 12$ modulo 19.

Using the Euclidean Algorithm, we find that -5 is an inverse of 19 modulo 12, and 8 is an inverse of 12 modulo 19.

But $-5 \equiv 7 \pmod{12}$.

Hence, the solution

$$x \equiv 5 \cdot 19 \cdot 7 + 13 \cdot 12 \cdot 8 \equiv 1913 \pmod{228}.$$

Since $89 \equiv 1913 \pmod{228}$, the solution $x \equiv 89 \pmod{228}$.

3. (8 marks) Prove that $f_n \leq (\frac{7}{4})^n$ for every positive integer n .
Hint: Use the second principle of Mathematical Induction.

Solution:

$\forall n \in \mathbb{Z}^+ P(n)$ with
 $P(n)$ to be the statement $f_n \leq (\frac{7}{4})^n$

Base Cases:

$n = 1 : f_1 = 1 \leq \frac{7}{4} = (\frac{7}{4})^1$; so, $P(1)$ is true.

$n = 2 : f_2 = 2 \leq \frac{49}{16} = (\frac{7}{4})^2$; so, $P(2)$ is true.

Induction Hypothesis:

Assume that $P(i)$ is true for all integers i from 1 to k , i.e.
 $f_i \leq (\frac{7}{4})^i$ for all $i = 1, 2, \dots, k$.

Induction Step:

$$\begin{aligned} &P(k+1), \text{ i.e. } f_{k+1} \leq (\frac{7}{4})^{k+1}? \\ &LHS = f_{k+1} = f_k + f_{k-1} \leq \\ &\text{by Induction Hypothesis} \\ &\leq (\frac{7}{4})^k + (\frac{7}{4})^{k-1} = (\frac{7}{4})^k (1 + \frac{4}{7}) = \\ &= (\frac{7}{4})^k (\frac{11}{7}) < (\frac{7}{4})^k (\frac{7}{4}) = \\ &= (\frac{7}{4})^{k+1} = RHS. \end{aligned}$$

Conclusion:

$\forall n \in \mathbb{Z}^+ f_n \leq (\frac{7}{4})^n$.

4. (6 marks) Show that the set S defined by $1 \in S$ and $(s + t) \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.

Solution:

$$S = \mathbb{Z}^+ \iff S \subseteq \mathbb{Z}^+ \text{ and } \mathbb{Z}^+ \subseteq S.$$

1) $n \in S \Rightarrow n = 1$ or n is the sum of two positive integers. Hence, $n \in \mathbb{Z}^+$.

2) To show that $\mathbb{Z}^+ \subseteq S$ we proceed by Mathematical Induction.

$1 \in S$. Assuming that $k \in S$, we get that $(k + 1) \in S$ by the recursive part of the definition of S when $s = k$ and $t = 1$.

5. (7 marks) Prove that if seven integers are selected from the first ten positive integers, there must be at least two pairs of these integers with the sum 11.

Hint: Use the pigeonhole principle.

Solution:

We group integers $1, 2, \dots, 10$ (objects) into five subsets (boxes) of two integers each, each subset adding up to 11:

$[1, 10], [2, 9], [3, 8], [4, 7], [5, 6]$.

Then by the generalized pigeonhole principle at least $\lceil \frac{7}{5} \rceil = 2$ pairs of these integers will be in the same box, i.e. will be added up to 11.

6. (7 marks)

(a) How many different strings are there of the letters of the word "number"?

Answer:

It is the number of 6-permutations of the set with 6 element:

$$P(6, 6) = 6! = 720.$$

(b) How many different committees consisting of three representatives of management and two representatives of labor can be formed from among six representatives of management and five representatives of labor?

Answer:

By the product rule the number of different committees is

$$C(6, 3) \cdot C(5, 2) = \frac{6!}{3! \cdot 3!} \cdot \frac{5!}{2! \cdot 3!} = 200.$$

The end