

YORK UNIVERSITY

Faculty of Arts

Faculty of Pure and Applied Science

January - April 2003

MATH 2280 3.0 W M

Term Test 1

SOLUTIONS

1. (10 points)

- (a) What simple interest rate per year is equivalent to $i^{(4)} = 12\%$ if money is invested for three years.

Answer:

Let the unknown rate of interest be x , then

$$P(1 + 3x) = P\left(1 + \frac{0.12}{4}\right)^{12}.$$

Solving for x , we obtain $x = 0.14192$ or 14.192% .

- (b) A invests \$1000 at $i^{(1)} = 6\%$, and B invests \$1250 at $i^{(1)} = 3\%$. At the end of n years A has twice as much as B . Find n .

Answer:

$$1000(1.06)^n = 2 \cdot 1250(1.03)^n$$

$$\left(\frac{1.06}{1.03}\right)^n = 2.5, \quad n \ln\left(\frac{1.06}{1.03}\right) = \ln 2.5$$

Hence, $n \approx 31.93$ (years).

2. (8 points) Find an expression for the fraction of a period at which the excess of accumulated values calculated at simple interest over compound interest is a maximum.

Answer:

$$\text{Let } y = f(t) = (1 + ti) - (1 + i)^t = 1 + ti - e^{\ln(1+i)t} = 1 + ti - e^{t \ln(1+i)}, \quad 0 < t < 1.$$

$$\text{Then } \frac{dy}{dt} = i - e^{t \ln(1+i)} \ln(1+i) = i - (1+i)^t \ln(1+i).$$

$$\text{So, } \frac{dy}{dt} = 0 \implies (1+i)^t = \frac{i}{\ln(1+i)} \implies t \ln(1+i) = \ln\left(\frac{i}{\ln(1+i)}\right).$$

Hence, the function $f(t)$ has a critical point at

$$t = \frac{1}{\ln(1+i)} \ln\left(\frac{i}{\ln(1+i)}\right) = \frac{\ln i - \ln[\ln(1+i)]}{\ln(1+i)} = \frac{\ln i - \ln \delta}{\delta},$$

$$\because e^\delta = 1+i \implies \delta = \ln(1+i).$$

$$\frac{d^2y}{dt^2} = -\ln(1+i) \ln(1+i) e^{t \ln(1+i)} = -[\ln(1+i)]^2 e^{t \ln(1+i)} < 0 \quad \forall t.$$

Thus, $y = f(x)$ is concave downward, and consequently, it has the absolute maximum

$$\text{at } t = \frac{\ln i - \ln \delta}{\delta}.$$

3. (8 points) \$5000 is invested at $i = 11.8\%$ on May 10, 2000. What will be the accumulated amount on April 05, 2003 if the contract says that simple interest is calculated for a fraction of a year on the 30/360 basis?

Answer:

$$\begin{aligned} \text{Investment period} &= 360(3 - 0) + 30(4 - 5) + (5 - 10) = 1045 \\ &= 2 \cdot 360 + 325 \text{ (days), or 2 years and 325 days.} \end{aligned}$$

The accumulated amount will be

$$5000[(1 + i)^2 + \frac{325}{360}i] = 5000[(1.118)^2 + \frac{65}{72}(0.118)] = 5000[(1.118)^2 + 0.10653] = \$6782.27.$$

4. (8 points) You are given $\delta_t = \frac{2}{t-1}$ for any $t \in [2, 10]$. For any one year interval between n and $n + 1$, with $2 \leq n \leq 9$, calculate the equivalent nominal rate of discount $d^{(2)}$.

Answer:

$$a(t) = a(m)e^{\int_m^t \delta_r dr}, \text{ where } a(t) \text{ is the accumulation function.}$$

Substituting $\delta_r = \frac{2}{r-1}$ and $m = 2$, we obtain

$$\frac{a(t)}{a(2)} = e^{\int_2^t \frac{2}{r-1} dr} = e^{2 \ln(r-1)|_2^t} = e^{2 \ln(t-1) - 2 \ln 1} = e^{2 \ln(t-1)} = e^{\ln(t-1)^2} = (t-1)^2.$$

$$\text{Hence, } a(t) = (t-1)^2, \therefore a(2) = e^0 = 1.$$

$$\text{On the other hand, } a(n) = a(n+1)\left(1 - \frac{d^{(2)}}{2}\right)^2.$$

$$\text{So, when } t = n \text{ and } t = n+1 \text{ we obtain, } (n-1)^2 = n^2\left(1 - \frac{d^{(2)}}{2}\right)^2$$

$$\text{or } n-1 = n\left(1 - \frac{d^{(2)}}{2}\right), \text{ since the both sides are positive.}$$

$$\text{Solving for } d^{(2)}, \text{ we obtain } d^{(2)} = \frac{2}{n}.$$

5. (8 points) Louis has an obligation to pay a sum of \$3000 in four years from now and a sum of \$5000 in six years from now. His creditor permits him to discharge these debts by paying \$ X in two years from now, \$1000 in three years from now, and a final payment of \$ $2X$ in nine years from now. Assuming an effective rate of interest of 0.06 per year, find X .

Answer:

Taking a focal point at $t = 2$, an equation of value can be written

$$X + 1000 \cdot v^1 + 2X \cdot v^7 = 3000 \cdot v^2 + 5000 \cdot v^4,$$

$$\text{where } v = \frac{1}{1+i} = (1+i)^{-1} = (1.06)^{-1}.$$

$$\text{So, we have } X + 1000(1.06)^{-1} + 2X(1.06)^{-7} = 3000(1.06)^{-2} + 5000(1.06)^{-4}.$$

$$\text{Solving for } X, \text{ we obtain } X = \$2440.80.$$

6. (8 points) The sum of the accumulated value of 1 at the end of three years at a certain effective rate of interest i , and the present value of 1 to be paid at the end of three years at the effective rate of discount numerically equal to i , is 2.0096. Find the rate.

Answer:

$$1 \cdot (1 + i)^3 + 1 \cdot (1 - d)^3 = 2.0096, \text{ where } i = d.$$

So, we have

$$1 \cdot (1 + i)^3 + 1 \cdot (1 - i)^3 = 2.0096,$$

$$(1 + 3i + 3i^2 + i^3) + (1 - 3i + 3i^2 - i^3) = 2.0096,$$

$$2 + 6i^2 = 2.0096,$$

$$i^2 = \frac{0.0096}{6} = 0.0016,$$

$$i = 0.04 \text{ or } 4\%.$$

The end