York University

Faculty of Arts

Faculty of Pure and Applied Science

September - December 2002

AS/SC/MATH 1300 3.0 B
Midterm Test 1b
SOLUTIONS

1. (10 marks) Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{\sqrt{x-1}}{x+1}$. Find:

   (a) $(f \circ g)(8)$;

   (b) $Dom(f \circ g)$.

   Answers:

   (a) $(f \circ g)(x) = f(g(x)) = \sqrt{\dfrac{\sqrt{x-1}}{x+1}}$. Hence, $(f \circ g)(8) = \frac{\sqrt[4]{8-1}}{\sqrt{8+1}} = \frac{\sqrt[4]{7}}{3}$.

   (b) $Dom(f) = [0, \infty)$.
   $Dom(g) = ((-\infty, -1) \cup (-1, -\infty)) \cap [1, \infty) = [1, \infty)$, $\because$ $g(x)$ is defined only if $x \neq -1$ & $x - 1 \geq 0$.
   At the same time, from part (a) we have seen that $f(g(x))$ is defined $\forall x \in [1, \infty)$, i.e. $Range(g) \subseteq Dom(f)$.
   Hence, $Dom(f \circ g) = Dom(g) = [1, \infty)$.

2. (10 marks)

   (a) Determine whether the function $f(x) = \dfrac{3x-5}{x+1}$, $x \neq -1$, is an invertible function?

   (b) If it is, find the inverse; if not, determine its restriction which is invertible.

   (c) Find $Range(f)$.

   Answers:

   (a) We need to show that the function f is one-to-one.
   $f(x_1) = f(x_2) \implies \dfrac{3x_1 - 5}{x_1 + 1} = \dfrac{3x_2 - 5}{x_2 + 1} \implies (3x_1 - 5)(x_2 + 1) = (3x_2 - 5)(x_1 + 1) \implies 3x_1x_2 + 3x_1 - 5x_2 - 5 = 3x_1x_2 + 3x_2 - 5x_1 - 5 \implies 8x_1 = 8x_2 \implies x_1 = x_2$.
   Hence, f is one-to-one and consequently, invertible.

   (b) $y = f(x) = \dfrac{3x-5}{x+1} \implies y(x+1) = 3x - 5 \implies yx - 3x = -y - 5 \implies x = \dfrac{y+5}{3-y}$.
   Hence, $f^{-1}(y) = \dfrac{y+5}{3-y}$ or $f^{-1}(x) = \frac{x+5}{3-x}$.

(c) $Range(f) = Dom(f^{-1}) = (-\infty, 3) \cup (3, \infty)$,
$\because$ at $x = 3$ the denominator equals zero and the value of f^{-1} is not defined.

3. (10 marks)

(a) Give the $(\epsilon - \delta)$ definition of $\lim_{x \to c} f(x) = \infty$.

(b) Use the $(\epsilon - \delta)$ definition to prove that $\lim_{x \to 16} \sqrt{x} = 4$.

Answers:

(a) $\lim_{x \to c} f(x) = \infty \iff (\forall N > 0)\ (\exists \delta_N > 0)$:
$0 <\mid x - c \mid < \delta_N \implies f(x) > N$.
Alternatively, $\lim_{x \to c} f(x) = \infty \iff (\forall N > 0)\ (\exists \delta_N > 0)$:
$Graph(f(x) \mid I_{\delta_N}(c)) \subset S_N(\infty)$, where
$I_{\delta_N}(c) = \{x \in \mathbb{R} \mid -\delta_N < x - c < \delta_N\}$
and $S_N(\infty) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > N\}$.

(b) $\lim_{x \to c} f(x) = L \iff (\forall \epsilon > 0)\ (\exists \delta_\epsilon > 0)$:
$0 <\mid x - c \mid < \delta_\epsilon \implies \mid f(x) - L \mid < \epsilon$.
We have $c = 16,\ L = 4$.
$0 <\mid x - 16 \mid < \delta_\epsilon \implies \mid \sqrt{x} - 4 \mid < \epsilon$,
$-\delta_\epsilon < x - 16 < \delta_\epsilon \implies -\epsilon < \sqrt{x} - 4 < \epsilon$,
$16 - \delta_\epsilon < x < 16 + \delta_\epsilon \implies 4 - \epsilon < \sqrt{x} < 4 + \epsilon$,
$16 - \delta_\epsilon < x < 16 + \delta_\epsilon \implies (4 - \epsilon)^2 < x < (4 + \epsilon)^2$,
$16 - \delta_\epsilon < x < 16 + \delta_\epsilon \implies 16 - 8\epsilon + \epsilon^2 < x < 16 + 8\epsilon + \epsilon^2$,
$16 - \delta_\epsilon < x < 16 + \delta_\epsilon \implies 16 - \epsilon(8 - \epsilon) < x < 16 + \epsilon(8 + \epsilon)$.
Hence, $\delta_\epsilon \in (\epsilon(8 - \epsilon), \epsilon(8 + \epsilon))$.

4. (5 marks) Find a formula for $g(x) = \cos(\arcsin(4x - 1))$, which does not involve any trigonometric functions.

Answer:
$g(x) = \sqrt{1 - \sin^2(\arcsin(4x - 1))} = \sqrt{1 - (4x - 1)^2}$
$= \sqrt{1 - (16x^2 - 8x + 1)} = 2\sqrt{2x - 4x^2}$.

5. (15 marks) Evaluate each of the following limits if it exists; if the limit does not exist, explain, why.

(a) $\lim_{x \to 2} (\frac{1}{x - 2} - \frac{4}{x^2 - 4})$

(b) $\lim_{x \to 1^+} \frac{1 - \sqrt{x}}{1 - x}$

(c) $\lim_{\phi \to \pi} \frac{\tan \phi}{1 + \cos \phi}$.

Answers:

(a) We have the undefined form $\infty - \infty$.
$\lim_{x\to 2}(\frac{1}{x-2} - \frac{4}{x^2-4}) = \lim_{x\to 2}\frac{x+2-4}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$.

(b) We have the undefined form $\frac{0}{0}$.
$\lim_{x\to 1^+}\frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1^+}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1^+}\frac{1}{1+\sqrt{x}} = \frac{1}{1+\sqrt{1}} = \frac{1}{2}$.

(c) We have the undefined form $\frac{0}{0}$.
Let $\theta = \pi - \phi$, then $\phi = \pi - \theta$, and $\phi \to \pi \Longrightarrow \theta \to 0$.
So, $\lim_{\phi\to\pi}\frac{\tan\phi}{1+\cos\phi} = \lim_{\theta\to 0}\frac{\tan(\phi-\theta)}{1+\cos(\phi-\theta)} = \lim_{\theta\to 0}\frac{-\tan\theta}{1-\cos\theta}$
$= -\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta(1-\cos\theta)} = -\lim_{\theta\to 0}(\frac{\sin\theta}{\theta}\frac{\theta}{1-\cos\theta}\frac{1}{\cos\theta})$
$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{\theta}{1-\cos\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}) = -\lim_{\theta\to 0}\frac{\theta}{1-\cos\theta}$.

But $\lim_{\theta\to 0}\frac{\theta}{1-\cos\theta}$ does not exist, since $\lim_{\theta\to 0^+}\frac{\theta}{1-\cos\theta} = \infty$,
while $\lim_{\theta\to 0^-}\frac{\theta}{1-\cos\theta} = -\infty$.
Therefore, $\lim_{\phi\to\pi}\frac{\tan\phi}{1+\cos\phi}$ does not exist also.

The end