YORK UNIVERSITY

Faculty of Arts
Faculty of Pure and Applied Science
January - April 2003
AS/SC/MATH 2320 3.0 P
Term Test 2b
SOLUTIONS

1. (7 marks) Use the Principle of Mathematical Induction to prove that $9^n - 5^n \equiv 0 \pmod 4$, for every integer $n \geq 0$.
   Answer:
   Prove: $(\forall n \in \mathbb{N})\ P(n)$, where $P(n)$ is the proposition $4 \mid 9^n - 5^n$.
   Basis Step: $P(0)$ is true, since $9^0 - 5^0 = 1 - 1 = 0$ and $4 \mid 0$.
   Induction Hypothesis: Assume that $P(k)$ is true, i.e. $4 \mid 9^k - 5^k$.
   Induction Step: Then $P(k+1)$ must be true also.
   $9^{k+1} - 5^{k+1} = 9 \cdot 9^k - 9 \cdot 5^k + 4 \cdot 5^k = 9(9^k - 5^k) + 4 \cdot 5^k$,
   where the second addend is a multiple of 4 and the first addend is divisible by 4 by the induction hypothesis. So, the sum is divisible by 4, and consequently,
   $4 \mid 9^{k+1} - 5^{k+1}$, that is $P(k+1)$ is true.

   Hence, $9^n - 5^n \equiv 0 \pmod 4$, for every integer $n \geq 0$.

2. (5 marks) Write a pseudocode description for a recursive algorithm that finds the reversal of a binary string.
   Answer:
   procedure $reverse((b_1, b_2, \cdots, b_n)$: binary string)
   if $n = 0$ then $reverse(b_1, b_2, \cdots, b_n) := \lambda$
   else $reverse(b_1, b_2, \cdots, b_n) := b_n\, reverse(b_1, b_2, \cdots, b_{n-1})$

3. (10 marks)

   (a) Show that if f is a function from S to T where S and T are finite sets with $|S| > |T|$, then f is not one-to-one.
   Hint: Use the pigeonhole principle.
   Answer:
   Show that
   $|S| < \infty\ \&\ |T| < \infty \implies (\exists s_1, s_2 \in S) :\ s_1 \neq s_2\ \&\ f(s_1) = f(s_2)$.
   Let members of S be pigeons, members of T be pigeonholes.
   $|S| > |T|$, means that there are more pigeons than pigeonholes. Hence, by the pigeonhole principle there exists at least one pigeonhole containing two or more pigeons, i.e. $(\exists s_1, s_2 \in S) :\ s_1 \neq s_2\ \&\ f(s_1) = f(s_2)$.

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(b) A conference room contains eight tables and 105 chairs. What is the smallest possible number of chairs at the table having the most seats?
Answer:

$\lceil \frac{105}{8} \rceil = 14$(chairs).

4. (9 marks)

(a) How many binary strings of length 12 begin with 11 and end with 10?
Answer:
2^8

(b) Canadian postal codes consist of three letters alternated with three non-zero digits, starting with a letter (e.g. M2R 1C3). How many postal codes are possible if the digits are all different?
Answer:
$26 \cdot 26 \cdot 26 \cdot 9 \cdot 8 \cdot 7 = 26^3 \cdot 9\underline{3}$.

(c) In a club with 20 women and 17 men needs to form a committee of size six. How many committees are possible if the committee must have at least two men?
Answer:
$C(17,2) \cdot C(20,4) + C(17,3) \cdot C(20,3) + C(17,4) \cdot C(20,2)$
$+C(17,5) \cdot C(20,1) + C(17,6)$.

5. (9 marks)

(a) Find n if $C(n,5) = C(n,2)$.
Answer:
$C(n,k) = C(n,n-k) \implies k = 5 \;\&\; n-k = 2 \implies n = 7$.
Moreover, there is no other solutions, since $C(n,k)$ is an increasing function of k for $0 \le k \le \frac{n}{2}$.
Hence, $n = 7$.

(b) Find the coefficient of x^8 in the expansion of $(x^2+2)^{13}$.
Answer:
$C(13,9) \cdot 2^9$

(c) Show that $C(n,0) - C(n,1) + C(n,2) - \cdots + (-1)^n C(n,n) = 0$, for every integer $n > 0$.
Answer:
$0 = [1 + (-1)]^n = \Sigma_{k=0}^{n} (-1)^k C(n,k)$
$= C(n,0) - C(n,1) + C(n,2) - \cdots + (-1)^n C(n,n)$.

The end