Fall 2009

YORK UNIVERSITY
DIFFERENTIAL CALCULUS WITH APPLICATIONS, MATH 1300-B
SECOND MIDTERM

Justify all your answers. You have 50 minutes. No calculator, no books.

1) [10 points] Find the derivative of $(2x^2+3x+1)(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)$.
Solution Using the product rule:

$$D\left((2x^2+3x+1)(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)\right)$$

$$= \left(D(2x^2+3x+1)\right)(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)+(2x^2+3x+1)\left(D(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)\right)$$

$$= (4x+3)(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)+(2x^2+3x+1)\left(3\cdot\frac{1}{2}\frac{1}{\sqrt{x}}-2\left(\frac{-1}{2}\right)\frac{1}{\sqrt{x}^3}\right)$$

$$= (4x+3)(3\sqrt{x}-2\frac{1}{\sqrt{x}}-2)+(2x^2+3x+1)\left(\frac{3}{2\sqrt{x}}+\frac{1}{\sqrt{x}^3}\right)$$

2) [10 points] Find the second derivative of $arcsin(3x)$. **Solution**
Let $f(x) = arcsin(3x)$. We have

$$f'(x) = \frac{1}{\sqrt{1-(3x)^2}}\cdot 3 = \frac{3}{\sqrt{1-9x^2}}$$

Hence

$$f''(x) = D\left(\frac{3}{\sqrt{1-9x^2}}\right) = 3\cdot\left(\frac{-1}{2}\right)\frac{1}{\sqrt{1-9x^2}^3}\cdot(18x) = \frac{-27x}{\sqrt{1-9x^2}^3}$$

3) [10 points] Define the notion of *the derivative of $f(x)$ at $x=c$.*
Solution The derivative of $f(x)$ at $x=c$ is the slope of the tangent line of $f(x)$ at $x=c$. More formally, the derivative of $f(x)$ at $x=c$ is

$$\lim_{h\to c}\frac{f(c+h)-f(h)}{h}$$

if this limit exists and is not equal to $\pm\infty$.

4) [10 points] Find the equation of the tangent line to the implicitly define function $x^3+2x^2y-2xy^2-8y^3=2x$ at the point $(x,y)=(-2,1)$

Solution Let us first make sure that $(-2,1)$ is indeed a solution of the equation:

$$(-2)^3 + 2\cdot(-2)^2\cdot 1 - 2\cdot(-2)\cdot 1 - 8 = -4 = 2\cdot(-2)$$

The equation of the line that is tangent to the the function $y = f(x)$ implicitly defined by $x^3 + 2x^2y - 2xy^2 - 8y^3 = 2x$ at $(x,y) = (-2,1)$ is given by

$$y = f'(-2)(x+2) + f(-2) = y'(x+2) + 1$$

To find the implicit derivative of $y = f(x)$ in term of (x,y) we differentiate both side of the equation:

$$D(x^3 + 2x^2y - 2xy^2 - 8y^3) = D(2x) = 2$$

$$3x^2 + (4xy + 2x^2y') - (2y^2 + 2x\cdot 2yy') - 24y^2y' = 2$$

Grouping together all term that contains y' and isolating we get

$$(3x^2 + 4xy - 2y^2) + (2x^2 - 4xy - 24y^2)y' = 2$$

$$(2x^2 - 4xy - 24y^2)y' = 2 - (3x^2 + 4xy - 2y^2)$$

$$y' = \frac{2 - (3x^2 + 4xy - 2y^2)}{(2x^2 - 4xy - 24y^2)}$$

At $(x,y) = (2,1)$, this gives:

$$y' = \frac{2 - (3\cdot(-2)^2 + 4\cdot(-2) - 2)}{(2\cdot(-2)^2 - 4\cdot(-2) - 24)} = 0$$

Hence the equation of the tangent line at $(x,y) = (2,1)$ is

$$y = y'(x+2) + 1 = 0(x+2) + 1 = 1$$

5) [10 points] Find the critical points of the functions $f(x) = ((x-2)(x+1))^3$ with domain $[-3,3]$. Use the first derivative test to identify each of them as a local maximum, a local minimum or neither.

Solution

First we compute $f'(x)$:

$$f(x) = 3((x-2)(x+1))^2((x+1) + (x-2)) = 3(x-2)^2(x+1)^2(2x-1)$$

The critical point of $f(x)$ is the set of point where $f'(x) = 0$ or is not defined. Since $f'(x)$ is defined everywhere we only need to find the point where $f'(x) = 0$. That is

$$\left\{-1, \frac{1}{2}, 2\right\}$$

Next we need to find the sign of $f'(x)$ in the region $x < -1$, $-1 < x < \frac{1}{2}$, $\frac{1}{2} < x < 2$, and $x > 2$. By continuity, we only need to pick a point in each region:

$$f(-2) = 3 \cdot (-4)^2 \cdot (-1)^2 \cdot (-5) < 0$$

$$f(0) = 3 \cdot (-2)^2 \cdot (1)^2 \cdot (-1) < 0$$
$$f(1) = 3 \cdot (-1)^2 \cdot (2)^2 \cdot (1) > 0$$
$$f(3) = 3 \cdot (1)^2 \cdot (4)^2 \cdot (5) > 0$$

That is

$$- - - - -(-1) - - - - - (\frac{1}{2}) + + + + + (2) + + + ++$$

So $x = \frac{1}{2}$ is a local minimum and the other two critical point are neither.