Fall 2009

YORK UNIVERSITY

Differential Calculus with Applications, Math 1300-B

First Midterm

Justify all your answers. You have 50 minutes. No calculator, no books.

REMARK: I have put some partial credit along the solution. That doesn't mean it applies to all possible ways of solving the problem. This is just a guide line and the grading may be done differently.

1) [10 points] Is the function

$$f(x) = \begin{cases} \dfrac{\cos\theta - 1}{\sin\theta} & x \neq 0 \\ \\ 1 & x = 0 \end{cases}$$

continuous at $x = 0$.

SOLUTION First remark that I announced that there was a typo and that all θ's should be x's. Now the function is continuous at $x = 0$ exactly if

$$\lim_{x\to 0} f(x) = f(0).$$

[2 points for stating something like this] To verify this, we have to compute both side and make sure we get the same thing. First

$$f(0) = 1$$

[2 point to make that clear]. Second

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$$

We **cannot** just plug $x = 0$ in the last limit [1 points to realize this]. We have to use Theorem 1.6.11 in the book

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0} \frac{\cos x - 1}{x}}{\frac{\lim_{x\to 0} \sin x}{x}} = \frac{0}{1} = 0$$

[2 points to compute this limit properly]. Since $0 \neq 1$ the function is not continuous [3 points for the conclusion using proper arguments].

2) [10 points] Evaluate $\lim_{x\to -1} \frac{\sqrt{x}(x^2+3x+2)}{x^2+2x+1}$.

SOLUTION Well this one was easy! when $x < 0$, the function $\sqrt{x}$ is not defined. So the limit does not exists [10 points for saying this properly]

I allow partial credit [3 points] if someone does not realize this and compute the limit of $\lim_{x\to -1} \frac{(x^2+3x+2)}{x^2+2x+1}$ properly as a partial step.

3) [10 points] Explain why $x^3 - 7$ has a root between 1 and 2. [Bonus 5 points, can you evaluate this root up to the first decimal?]

SOLUTION This is a typical application of the IVT (Intermediate Value Theorem) [3 points to at least realize that you have to use this]. We let $f(x) = x^3 - 7$. This is a continuous function [2 points to say that, IT IS IMPORTANT] At $x = 1$, we have $f(1) = 1 - 7 = -6$ and at $x = 2$ we have $f(2) = 8 - 7 = 1$. By the IVT applied to the continuous function $f(x)$ on the domain $[1, 2]$ we have that zero $0 \in [-6, 1] \subseteq f([1, 2])$. In other words, 0 is in the range of f on the domain $[1, 2]$. That is $f(x)$ has a root between 1 and 2. [5 points for using the IVM correctly].

Now for the bonus. One can cut the interval $[1, 2]$ in 10. We pick a value in the middle, say 1.5, and compute $f(1.5) = 3.375 - 7 < 0$. This means a root is between 1.5 and 2. We pick a value in the middle of 1.5 and 2, say 1.8. We get $f(1.8) = 5.832 - 7 < 0$. This means a root is between 1.8 and 2. We do it again with 1.9. We get $f(1.9) = 6.859 - 7 < 0$. So a root of $f(x)$ is between 1.9 and 2 but is not 2, so the root is at $x = 1.9$...

4) [10 points] Determine the largest domain for which the following formula determines a function:

$$f(x) = \frac{x^2 - 1}{\sqrt{4 - x^2} - \sqrt{3}}.$$

SOLUTION This formula make sense as long as (1) all square roots are positive, and (2) no division by zero. [1 points] For (1) we must have

$$4 - x^2 > 0 \text{ and this happen when } -2 \le x \le 2$$

[3 points for this]. For (2) we must **not** have

$$\sqrt{4 - x^2} - \sqrt{3} = 0\,,$$

$$\sqrt{4 - x^2} = \sqrt{3}\,,$$

$$4 - x^2 = 3\,,$$

$$x^2 = 1\,.$$

That is $x \neq 1$ and $x \neq -1$ [3 points]. Now we combine (1) and (2) to say that the largest domain is

$$[-2, 2] \text{ minus } \{-1, 1\} = [-2, -1) \cup (-1, 1) \cup (1, 2].$$

[3 points to put it all together correctly.]

5) [10 points] Consider the function $g(x) = 9 - x^2$ with domain $[-1, 3]$.

(a) Is $g(x)$ one-to-one? If not, what is the largest domain on which g restricts to a one-to-one function?

(b) Find a formula for g^{-1} (or its restriction) and gives its domain of definition.

SOLUTION

(a) at $x = -1$ we have $g(-1) = 8$, but also at $x = 1$ we have $g(1) = 8$. Since $g(-1) = g(1)$ and both 1 and -1 are in the domain. So $g(x)$ is not one to one as defined [3 points]. If we restrict $g(x)$ to $[-1, 0]$ it is one-to-one, but if we restrict $g(x)$ to the larger domain $[0, 3]$ it is also one to one. so the correct answer is

- The largest on which $g(x)$ is one-to-one is $[0, 3]$ [2 points]
- If you give any other answer **that make sense** [1 point]

(b) We have

$$y = g(x) = 9 - x^2\,,$$

$$x^2 = 9 - y\,,$$

$$x = \sqrt{9 - y}\,.$$

So a formula for $g^{-1}(y) = \sqrt{9 - y}$. [3 points] Now we have to use the restriction that we used in (a). If in (a) you used the domain $[0, 3]$ for $g(x)$, then your answer here should be $[0, 9]$. [2 points for the right answer]. Now if you give an answer here that is **compatible** with your answer in (a) you also get full mark [2 points]. If it is not compatible but still make sense... [1 points].