

STRONGLY REINFORCED PÓLYA URNS WITH GRAPH-BASED COMPETITION

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We introduce a class of reinforcement models where, at each time step t , one first chooses a random subset A_t of colours (independently of the past) from n colours of balls, and then chooses a colour i from this subset with probability proportional to the number of balls of colour i in the urn raised to the power $\alpha > 1$. We consider stability of equilibria for such models and establish the existence of phase transitions in a number of examples, including when the colours are the edges of a graph; a context which is a toy model for the formation and reinforcement of neural connections. We conjecture that for any graph G and all α sufficiently large, the set of stable equilibria is supported on so-called whisker-forests, which are forests whose components have diameter between 1 and 3.

1. Introduction. Random processes with reinforcement have been studied mathematically since at least the early 1900s, and have connections to applied problems such as the design of clinical trials, and the formation of networks such as neural networks, the Internet and social networks. One of the most simple and elegant of these models is known as *Pólya's urn*, where (starting with one black and one red ball in an urn) we repeat the following procedure indefinitely: select a ball uniformly at random from the urn, replace it and add another of the same colour. The proportion X_t of black balls in the urn after t balls have been added is a bounded martingale, and has a discrete uniform distribution for each t , whence there is a random variable $X \sim U[0, 1]$ such that $\mathbb{P}(X_t \rightarrow X) = 1$. Various generalisations of this model have been studied in the last hundred years or so; see, for example, [18, 22]. In recent times, reinforced random walks and preferential attachment models continue to be studied extensively.

One direction of generalisation of Pólya's urn is to modify this selection probability (the probability of selecting a ball of a given colour). Fix $W : \mathbb{N} \rightarrow (0, \infty)$,

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and if $N_t^{(i)}$ is the number of balls of colour i in the urn at time t , then at time $t + 1$ we select a ball of colour i from the urn with probability $W(N_t^{(i)}) / \sum_j W(N_t^{(j)})$. In Pólya's urn, there are two colours and $W(x) = x$. A beautiful construction due to Rubin [7] shows that if $\sum_{x=1}^{\infty} W(x)^{-1} < \infty$ (sometimes called the *strong* reinforcement regime) then only one colour is chosen infinitely often. Otherwise, each colour is chosen infinitely often, and if W grows sufficiently slowly [e.g., $W(x) = x^\alpha$ for some $\alpha \in (0, 1)$] then the proportions of each colour are equal in the limit.

A further direction of generalisation involves having multiple interacting urns, where colours may be present in more than one urn and where multiple balls may be added to one or more urns depending on what colour is selected. See, for example, the Ph.D. thesis (and related papers) of Launay [13–15], recent work of Launay and Limic [16], and of Benaïm and coauthors [3, 4]. In such settings, colours may not be competing with each other on every iteration of the process, and Rubin's construction need not apply.

In this paper, we introduce a large class of “interacting urn”-type models, which are inspired by neuronal processing in the brain. It is estimated that at birth the human brain possesses tens of billions of neurons, with thousands of synapses (connections between the neurons) per neuron. It is believed that in the first 2–3 years the number of synapses per neuron increases, with pathways becoming smoother and stronger via a process called myelination. Subsequently, due to environmental factors and learning, synaptic pruning takes place: the brain removes connections which are seldom or never used and reinforces those which are stimulated (see, e.g., [9, 24]).

A subclass (that are defined on graphs) of the aforementioned models corresponds to the following simplistic model for neuronal processing: A signal enters the brain at some (randomly) chosen neuron and is transmitted to a (random) single neighbouring neuron with probability depending on the relative efficiency of the synapses connecting the neurons, and in doing so the efficiency of the synapse is improved/reinforced. We are interested in the structures (or architectures) and relative efficiency of the neuronal networks that can arise from repeating this process a very large number of times, in a strong reinforcement regime. Neuronal architecture has been related to IQ [25].

In the simplistic model described above, each signal is transmitted between a single pair of neighbouring neurons. More realistic models might allow a random motion (with or without branching of the signal). Without branching, this could be modelled using edge-reinforced random walks (see, e.g., [7, 8, 17, 19, 21, 22] and the references therein) on graphs, killed at certain vertices. With branching, this would give rise to a certain kind of branching reinforced walk with killing.

Let us define our models more precisely. Suppose that we have n colours of balls. Let $\alpha > 1$ and let $\{A_s\}_{s \in \mathbb{N}}$ be an i.i.d. sequence of nonempty subsets of $[n] = \{1, 2, \dots, n\}$. Let $N_t^{(i)}$ be the number of balls of colour i in our “urn” at time

$t \in \mathbb{Z}_+$, with $N_0^{(i)} = 1$ for each $i \in [n]$. The process $\vec{N}_t = (N_t^{(i)} : i \in [n])$ evolves as follows. At time $t \in \mathbb{N}$, we select a colour i from the balls of colours in A_t according to their current weights in the urn, that is, given A_t , we select a ball of colour $i \in A_t$ with probability

$$(1) \quad \frac{(N_{t-1}^{(i)})^\alpha}{\sum_{j \in A_t} (N_{t-1}^{(j)})^\alpha},$$

then we replace that ball and add another of the same colour, so that $N_t^{(j)} = N_{t-1}^{(j)} + \mathbb{1}_{\{j=i\}}$. For a fixed n , the law of such a model is then completely specified by the parameter α and the law of A_1 [i.e., the collection of probabilities $(p_A)_{A \subset [n]}$ where $p_A \equiv \mathbb{P}(A_1 = A)$ and $p_\emptyset = 0$]. Since this construction is completely specified by a function W [where $W(m) = m^\alpha$ in the present paper] and the law of A_1 , we refer to our models as WARMs (W - A Reinforcement Models).

We are interested in the random vectors $\vec{X}_t = \vec{N}_t / (t + n)$ of proportions of balls of each colour, and more precisely their limits as $t \rightarrow \infty$. Note that any model with $p_\emptyset \in (0, 1)$ can be considered as a random time change of a model with $p_\emptyset = 0$, which does not affect the possible limits of \vec{X}_t so we have lost nothing in assuming that A_1 is almost surely nonempty (i.e., $p_\emptyset = 0$).

The following subclass of WARMs will be studied extensively in this paper: the colours $i \in [n]$ are the edges (synapses) of a connected graph G (brain) with n edges and n_v vertices (neurons). In this setting, we will assume that the sets A_t are chosen as follows.

CONDITION 1 (Graph-based WARMs). A_t is the set of edges incident to V_t , where V_t is a single vertex, chosen uniformly at random from the vertices of G .

WARMs where the law of A_1 corresponds to Condition 1 on some graph G will be called *graph-based* WARMs. When G is specified, we will call the graph-based WARM a *G-WARM*.

Many interesting examples of WARMs have various symmetries in terms of the colour labellings, and in this case we often consider the ordered vector $[\vec{X}_t]$, having the same elements as \vec{X}_t , but listed in decreasing order. Most of our examples satisfy the following symmetry property, which implies that $\mathbb{P}(|A_1| = m) = nm^{-1} a_m p_m$.

CONDITION 2 (Weak symmetry). There exist $(p_\ell)_{\ell=1}^n$ and $(a_\ell)_{\ell=1}^n$ such that for every $m \geq 0$:

- (i) $p_A \in \{0, p_m\}$ whenever $|A| = m$, and
- (ii) $\#\{A \ni i : |A| = m, p_A = p_m\} = a_m$ for every $i \in [n]$.

Condition 2 is somewhat unpalatable, so let us point out that many of the models considered in this paper satisfy the following stronger symmetry property, which implies that $\mathbb{P}(|A_1| = m) = \binom{n}{m} p_m$, and also that (almost surely) at least $n - \underline{m} + 1$ colours are drawn a positive proportion of the time, where $\underline{m} = \min\{m \geq 1 : p_m > 0\}$.

CONDITION 3 (Strong symmetry). There exist $(p_i)_{i=1}^n$ such that $p_A = p_m$ whenever $|A| = m$.

Below, we will give simple examples of graph-based WARMS satisfying: (i) Condition 3; (ii) Condition 2 but not Condition 3; and (iii) neither of the symmetries in Conditions 2 and 3. We begin however with two natural examples of WARMS that are in general not graph-based WARMS.

EXAMPLE 1 (Uniform, fixed m). Fix $m \in [n]$ (the model becomes relatively trivial when $m = 1$ or $m = n$) and choose A_t with $|A_t| = m$ uniformly at random from $[n]$. Then $|A_t| = m$ almost surely and $\mathbb{P}(A_t = A) = m!(n - m)!/n!$ when $|A| = m$. This is the special case of Condition 3 with $p_r = 0$ for all $r \neq m$. At least $n - m + 1$ colours are each drawn a positive proportion of the time (as discussed before Condition 3).

EXAMPLE 2 [Bernoulli(p)]. Fix $p \in (0, 1)$, and independently choose each colour to be in A_t with probability p . After a parameter change [due to $p_\emptyset = (1 - p)^n > 0$], this is the special case of Condition 3 with $p_m = p^m(1 - p)^{n-m}(1 - p_\emptyset)^{-1}$ for all $m \geq 1$. All n colours are drawn a positive proportion of the time.

A natural extension of Example 2 would be to have a different p for each colour. Turning to graph-based WARMS (i.e., assuming Condition 1 hereafter), observe that the special case of Example 2 with $n = 2$ and $p = 1/2$ is the same as the G -WARM when G is the star-graph on 2 edges.

EXAMPLE 3 (Star graph-WARM). Let G be the star-graph on $n_v = n + 1$ vertices consisting of a central vertex connected by n edges to n leaves (vertices of degree 1). Then the G -WARM is the special case of Condition 3 with $p_1 = p_n = 1/(n + 1)$ and $p_m = 0$ otherwise.

In the next two examples, G is regular with degree $d = d(n)$ (so $|A_t| = d$ almost surely), so the G -WARM satisfies Condition 2 with $p_A = 0$ if $|A| \neq d$, and with $p_d = 1/n_v$ and $a_d = 2$ since any one of the n_v vertices is equally likely to be V_t and every edge is incident to 2 vertices. On the other hand, there exist subsets of size d that are chosen with probability 0 (so Condition 3 is not satisfied).

EXAMPLE 4 (Cycle graph-WARM). Let G be the cycle graph with n edges and n vertices. Each vertex is of degree $d = 2$.

EXAMPLE 5 (Complete graph-WARM). Let G be the complete graph on n_v vertices, with $n = n_v(n_v - 1)/2$ edges. Each vertex is of degree $d = n_v - 1$.

Note that Examples 1 (with $m = 2$), 4 and 5 are all identical when $n = 3$, and correspond to the *triangle graph-WARM* which is studied extensively in Section 3.2. All of the above examples satisfy the symmetry property Condition 2. Let us now give a simple example that does not satisfy Condition 2(ii).

EXAMPLE 6 (Line/Path graph-WARM). Let G be the line segment with n edges (and $n + 1$ vertices). The two leaves have degree 1, while all interior vertices have degree 2.

Star graphs and the line graph with $n = 3$ are special cases of *whisker graphs* (which also fail to satisfy Condition 2 in general) defined as follows.

EXAMPLE 7 (Whisker graph-WARM). A whisker graph is defined as a tree with a diameter between 1 and 3. This includes star graphs (diameter between 1 and 2). When the diameter is 3 the graph consists of a distinguished edge e with $r \geq 1$ leaves incident to one endvertex of e and $s = n - (r + 1) \geq 1$ leaves incident to the other endvertex (i.e., G is constructed by connecting two star graphs by a single edge, e). A whisker-graph with $r = s$ is called a *symmetric whisker-graph*.

We believe that whisker graphs play a central role in the graph setting (see Conjecture 2 below).

REMARK 1 (More general models). Our models can be generalised in several ways. One possibility is to consider different reinforcement functions W [i.e., other than $W(m) = m^\alpha$], the second possibility is to drop the i.i.d. assumption on the $\{A_s\}_{s \in \mathbb{N}}$.

For fixed n and $\vec{v} \in \Delta_n \equiv \{\vec{u} \in \mathbb{R}^n : u_i \geq 0, \sum_{i=1}^n u_i = 1\}$, let $F: \Delta_n \rightarrow \mathbb{R}^n$ be defined (for a given WARM) by

$$(2) \quad F(\vec{v})_i = -v_i + \sum_{A \ni i} p_A \cdot \frac{v_i^\alpha}{\sum_{j \in A} v_j^\alpha} \quad \text{for each } i \in [n].$$

Observe that $\sum_{i=1}^n F(\vec{v})_i = 0$ since

$$\sum_{i=1}^n \sum_{A \ni i} p_A \frac{v_i^\alpha}{\sum_{j \in A} v_j^\alpha} = \sum_{A \neq \emptyset} \sum_{i \in A} p_A \frac{v_i^\alpha}{\sum_{j \in A} v_j^\alpha} = \sum_{A \neq \emptyset} p_A = 1.$$

DEFINITION 1 (Equilibrium distributions). For fixed n , a vector $\vec{v} \in \Delta_n$ is an *equilibrium distribution* for the WARM if $F(\vec{v}) = \vec{0}$. We let \mathcal{E} denote the set of equilibria for a given WARM.

Note that $F(\vec{v}) = \vec{0}$ can be rewritten as

$$(3) \quad v_i = \sum_{A \ni i} p_A \cdot \frac{v_i^\alpha}{\sum_{j \in A} v_j^\alpha} \quad \text{for each } i \in [n].$$

Intuitively, this says that the proportion of balls of colour i in the urn is equal to the probability that the next ball drawn is of colour i .

Let the partial derivatives of F at \vec{v} be denoted by $D_{i,k} = \partial F(\vec{v})_i / \partial v_k$, and let $\mathbf{D}(\vec{v})$ denote the matrix with (i, k) entry $D_{i,k}$ evaluated at the point \vec{v} .

DEFINITION 2 (Linear stability of equilibria). An equilibrium distribution \vec{v} [i.e., a \vec{v} satisfying $F(\vec{v}) = \vec{0}$ in (2)] is a *linearly-stable equilibrium* if all eigenvalues of $\mathbf{D}(\vec{v})$ have negative real parts, a *linearly-unstable equilibrium* if some eigenvalue of $\mathbf{D}(\vec{v})$ has positive real part, and a *critical equilibrium* otherwise. Let \mathcal{S} denote the set of linearly-stable equilibria for a given WARM.

For a given WARM, let \mathcal{A} denote the (random, nonempty) set of accumulation points of the sequence \vec{X}_t . The main reason that we are interested in linearly-stable equilibria is because of the following theorem (and conjecture) whose proof relies on Theorem 2 below together with the general theory of the dynamical system approach to studying stochastic approximation algorithms, established by Benaïm and coauthors. See, for example, [4], Proposition 3.5, Theorems 3.9 and 3.11.

THEOREM 1. *For a given WARM, the set of accumulation points satisfies:*

- (i) *almost surely $\mathcal{A} \subset \mathcal{E}$ and \mathcal{A} is a connected subset of Δ_n ,*
- (ii) *$\mathbb{P}(\vec{X}_t \rightarrow \vec{v}) > 0$ for every $\vec{v} \in \mathcal{S}$.*

It follows from Theorem 1(i) that if $|\mathcal{E}| < \infty$ then $|\mathcal{A}| = 1$ and \vec{X}_t converges almost surely. Moreover, if $|\mathcal{E}| = 1$ then \vec{X}_t converges almost surely to this unique equilibrium. We shall see that when $n = 2$ and $\alpha = 3$ in Example 3 there is a unique equilibrium $[\mathcal{E} = \{(1/2, 1/2)\}]$, whence \vec{X}_t almost surely converges to $(1/2, 1/2)$ that is not linearly stable ($\mathcal{S} = \emptyset$). It is an open problem to prove nonconvergence to linearly-unstable equilibria in our general setting.

CONJECTURE 1. For any WARM there exists a random vector $\vec{X} = (X_1, \dots, X_n)$, supported on the set of linearly-stable and critical equilibria such that $\mathbb{P}(\vec{X}_t \rightarrow \vec{X}) = 1$.

1.1. Main results. Our main results describe the set \mathcal{S} of linearly-stable (and critical) equilibria in various situations, and hence (assuming Conjecture 1) the possible limiting proportions of balls of each colour. We are particularly interested in phase transitions in the set $\mathcal{S} = \mathcal{S}(\alpha)$ (including whether each colour can be chosen equally often) as $\alpha > 1$ varies, with $[n]$ and $(p_A)_{A \subset [n]}$ fixed. The following theorem is our first main result.

THEOREM 2. *The set \mathcal{S} of linearly-stable equilibria is finite.*

As noted after Theorem 1, \mathcal{S} may be empty. However, in many cases the existence of at least one linearly-stable equilibrium is given by the following result, when $\alpha > 1$ is sufficiently small.

PROPOSITION 1. (i) *If weak symmetry (Condition 2) holds then $\vec{1}/n$ is an equilibrium.*

(ii) *Assume strong symmetry (Condition 3). Then $\vec{1}/n$ is linearly stable if and only if*

$$(4) \quad \alpha < \frac{1}{n^2 \sum_{m=2}^n p_m / m^2 \binom{n-2}{m-2}}.$$

Moreover, $\vec{1}/n$ is critical if and only if equality holds in (4).

Note that $\vec{1}/n$ is not an equilibrium for Example 6, which does not satisfy Condition 2. The right-hand side of (4) is equal to 1 when $p_n = 1$ (so in this well-known case $\vec{1}/n$ is not stable for any $\alpha > 1$), while it is strictly larger than 1 when $p_n < 1$. So when $p_n < 1$, under the assumptions of Proposition 1, $\vec{1}/n \in \mathcal{S}$ for $\alpha > 1$ but sufficiently close to 1 (depending on the model), and $\vec{1}/n \notin \mathcal{S}$ for α sufficiently large. In other words, all such models exhibit at least one phase transition, some of which are described in our next result.

PROPOSITION 2. *The equilibrium $\vec{1}/n$ is linearly-stable (critical when equality holds below) for:*

- (i) *Example 1 if and only if $\alpha < \frac{m(n-1)}{n(m-1)}$;*
- (ii) *Example 2 if and only if*

$$\alpha < \frac{1 - (1 - p)^n}{\sum_{m=2}^n p^m (1 - p)^{n-m} n^2 / m^2 \binom{n-2}{m-2}};$$

- (iii) *Example 3 if and only if $\alpha < n + 1$;*
- (iv) *Example 4 if and only if n is odd and $\alpha < \cos(\frac{\pi}{2n})^{-2}$;*
- (v) *Example 5 if and only if $n = 3$ and $\alpha < 4/3$.*

Note that in the graph setting, when $\vec{v} = \vec{1}/n$, the matrix of partial derivatives is related to the edge-adjacency matrix. Typically, $\vec{1}/n$ is not the *only* equilibrium, and indeed we will see many more linearly-stable equilibria for specific models in Section 3.2 (Theorems 6, 7 and 8).

Recall that a WARM is specified by $\alpha > 1$, and $(p_A)_{A \subset [n]}$. Given $(p_A)_{A \subset [n]}$ and $I \subset [n]$ we define for each $A' \subset I$

$$(5) \quad p_{A'}^I = \sum_{A \subset [n]: A \cap I = A'} p_A.$$

The following result often allows one to find stable equilibria in large systems by finding stable equilibria in smaller systems.

PROPOSITION 3. *Fix $\alpha > 1$. Then $\vec{v} = ((v_j)_{j \in I}, (0)_{i \in [n] \setminus I})$ is a (linearly-stable) equilibrium for the WARM $(p_A)_{A \subset [n]}$ if and only if $\vec{v}^I = (v_i)_{i \in I}$ is a (linearly-stable) equilibrium for the WARM $(p_{A'}^I)_{A' \subset I}$.*

For us, the most important consequence of Proposition 3 is in the graph setting. Let G be a graph with vertex set V and edge set E . We denote by \mathcal{G}_G the V -spanning collections of nontrivial connected clusters of G , that is,

$$\mathcal{G}_G = \left\{ \{G_j\}_{j=1}^k : k \leq |V|/2, G_1, \dots, G_k \text{ are connected subgraphs of } G, \right. \\ \left. |V_j| \geq 2 \text{ for each } j, V = \bigcup_{j=1}^k V_j, V_j \cap V_{j'} = \emptyset \text{ for all } j \neq j' \right\},$$

where V_j denotes the vertex set of G_j . For an element $\mathbf{G} = \{G_j\}_{j=1}^k$ of \mathcal{G}_G we write E_j for the edge set of G_j and let $\mathbf{E} = \bigcup_{j=1}^k E_j$. Let \mathcal{E}_G and \mathcal{S}_G denote the equilibria and linearly stable equilibria for a G -WARM.

THEOREM 3. *Assume Condition 1. Fix G , and let*

$$\mathbf{G} = \{G_j\}_{j=1}^k \in \mathcal{G}_G \quad \text{and} \quad \vec{v} = ((v_e)_{e \in E_1}, (v_e)_{e \in E_2}, \dots, (v_e)_{e \in E_k}, (0)_{e \in E \setminus \mathbf{E}}).$$

Then, for any G -WARM:

- (1) $\vec{v} \in \mathcal{E}_G$ if and only if $\frac{|V|}{|V_j|}(v_e)_{e \in E_j} \in \mathcal{E}_{G_j}$ for each $j = 1, \dots, k$,
- (2) $\vec{v} \in \mathcal{S}_G$ if and only if $\frac{|V|}{|V_j|}(v_e)_{e \in E_j} \in \mathcal{S}_{G_j}$ for each $j = 1, \dots, k$.

DEFINITION 3. Given a graph G , $\alpha > 1$ and $\mathbf{G} \in \mathcal{G}_G$, we say that \mathbf{G} admits a (G, α) -stable allocation if there exists \vec{v} with $v_{e'} > 0$ for all $e' \in \mathbf{E}$ and $v_e = 0$ for all $e \in E \setminus \mathbf{E}$ such that $\vec{v} \in \mathcal{S}_G$ or \vec{v} is critical.

An element \mathbf{G} of \mathcal{G}_G is said to be a whisker-forest if each component G_j is a whisker graph. We will show in Section 3.2 that when G is the triangle graph and $\alpha > 4/3$, any stable equilibrium has some $v_i = 0$. We believe that the same is true (for $\alpha > \alpha_G$) when G is the line graph on 4 edges. Assuming that this can be verified, it is reasonable to expect (and Monte Carlo simulations suggest) that for any fixed G , and all α sufficiently large depending on G , the only linearly-stable equilibria are those admitted by whisker-forests.

CONJECTURE 2. Let G be any graph. There exists α_G such that, for all $\alpha > \alpha$:

- (i) any whisker-forest \mathbf{G} on G admits a (G, α) -stable allocation;
- (ii) for any $\vec{v} \in \mathcal{S}_G$, there exists a whisker-forest \mathbf{G} on G such that $v_e > 0$ if and only if $e \in \mathbf{E} = \bigcup_{j=1}^k E_j$.

What we have proved in this direction is that Conjecture 2(i) holds for forests consisting of star and symmetric-whisker components, and that Conjecture 2(ii) holds for the triangle graph.

THEOREM 4. (i) *For any graph G , any forest \mathbf{G} on G whose components G_j are all stars (resp., all stars or symmetric-whiskers) admits a (G, α) -stable allocation if $\alpha > 3$ (resp., if α is sufficiently large, depending on \mathbf{G}).*

(ii) *For the triangle graph (Example 5 with $n = 3$): for $\alpha > 4/3$ any linearly stable equilibrium (v_1, v_2, v_3) necessarily has $v_i = 0$ for some $i \in \{1, 2, 3\}$.*

REMARK 2. For the star-graph WARM and the triangle-graph WARM we have found and characterised *all* equilibria and hence all phase-transitions in α which occur in these models. We have similarly characterised all symmetric [i.e., of the form (\vec{v}, u, \vec{v})] equilibria on symmetric-whisker graphs. See Theorems 6, 7 and 8 in Section 3.

REMARK 3. Proposition 2(v) tells us that the equilibrium $\vec{1}/3$ is linearly stable for the triangle-graph WARM if and only if $\alpha < 4/3$. Theorem 4 complements this result and exhibits a phase transition at $\alpha = 4/3$: for $\alpha > 4/3$ any stable equilibrium in the triangle-graph WARM has only two nonzero components, and thus it coincides (up to permutation) with the stable equilibrium in the star-graph WARM with two edges.

REMARK 4. While the present paper was under review, significant progress has been made on proving Conjecture 2 (and also Conjecture 1) in [11]. That work is expected to be complete before the present paper is published.

1.2. *Overview of the paper.* The paper is organised as follows: in Section 2, we present the proofs of Theorems 1–3 and Propositions 1–3. In Section 3, we give a complete characterization of the equilibria for the star-graph WARM and the triangle-graph WARM as well as the symmetric equilibria for the symmetric-whisker-graph WARM (see Theorems 6, 7 and 8 in Section 3, resp.). Theorem 4 arises as a consequence of these results and Theorem 3, and is proved at the end of Section 3.

2. Proofs of Theorems 1–3 and Propositions 1–3.

2.1. *Proof of Theorem 1.* The proof of Theorem 1 follows the proof of [3], Theorem 1.2, very closely. We repeat this argument almost exactly, only modifying the expression of the Lyapunov function and some related objects.

The main idea of the proof of Theorem 1 is to interpret the evolution of the WARM as a stochastic approximation algorithm (see [2]). We introduce several definitions and notations. We recall that $N_t^{(i)}$ denotes the number of balls of colour i at time $t \in \mathbb{Z}^+$, $N_0^{(i)} = 1$ and n is the total number of colours. We assume that $p_\emptyset = 0$, therefore, the total number of balls at time t is $n + t$. We denote $X_t^{(i)} := N_t^{(i)} / (n + t)$ to be the proportion of balls of colour i . We define $C_t^{(i)}$ be the number of balls of colour i which is added to the urn at time t , that is, $C_t^{(i)} := N_{t+1}^{(i)} - N_t^{(i)}$. We denote $\mathcal{F}_t := \sigma\{\vec{N}_u : 1 \leq u \leq t\}$. Note that $C_t^{(i)} \in \{0, 1\}$ is a Bernoulli random variable, such that

$$(6) \quad \mathbb{P}(C_t^{(i)} = 1 | \mathcal{F}_t) = \sum_{A:i \in A} p_A \frac{(X_t^{(i)})^\alpha}{\sum_{j \in A} (X_t^{(j)})^\alpha},$$

moreover, we have $\sum_{i=1}^n C_t^{(i)} = 1$ (since only one ball is added to the urn at time t). By definition, we have $N_{t+1}^{(i)} = N_t^{(i)} + C_t^{(i)}$, therefore,

$$(7) \quad X_{t+1}^{(i)} - X_t^{(i)} = \frac{1}{n + t + 1} (-X_t^{(i)} + C_t^{(i)}).$$

Denoting

$$F_i(x_1, x_2, \dots, x_n) := -x_i + \sum_{A:i \in A} p_A \frac{x_i^\alpha}{\sum_{j \in A} x_j^\alpha},$$

and using (6), we can rewrite (7) in the form

$$(8) \quad \vec{X}_{t+1} - \vec{X}_t = \gamma_t (F(\vec{X}_t) + \vec{u}_t),$$

where $F = (F_1, F_2, \dots, F_n)$, $\gamma_t := 1 / (n + t + 1)$ and $u_t^{(i)} := C_t^{(i)} - \mathbb{E}[C_t^{(i)} | \mathcal{F}_t]$. Formula (8) expresses the WARM as a stochastic approximation algorithm. This is a classical approach to studying convergence of generalised Pólya urns, as there exists a well-developed theory for stochastic approximation algorithms (see [2, 6, 12]).

We write $A \sqsubset [n]$ when $A \subset [n]$ and $p_A > 0$. Let us denote $c := \frac{1}{2} \min\{p_A : A \sqsubset [n]\}$. We define Δ to be the set of n -tuples $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ such that:

1. $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$, and
2. for all $A \sqsubset [n]$ we have $\sum_{i \in A} x_i \geq c$.

Clearly, $F : \Delta \mapsto \mathbb{R}^n$ is Lipschitz. The following lemma is an analogue of [3], Lemma 3.4.

LEMMA 1. Δ is positively invariant under the ODE $\frac{d\vec{v}(t)}{dt} = F(\vec{v}(t))$.

PROOF. We need to show that any solution to the ODE $\frac{d\vec{v}(t)}{dt} = F(\vec{v}(t))$ with the initial condition $\vec{v}(0) \in \Delta$ satisfies $\vec{v}(t) \in \Delta$ for all $t > 0$.

If v belongs to the boundary of Δ , then either $v_i = 0$ for some $i \in [n]$, or there exists a set $A \sqsubset [n]$ with $\sum_{i \in A} v_i = c$. In the former case, since $F_i(\vec{v}) = 0$ if $v_i = 0$, it is clear that $v(t)$ will stay on the corresponding boundary. Let us consider the latter case. Given a set A with $p_A > 0$, we have

$$\begin{aligned} \frac{d}{dt} \sum_{i \in A} v_i &= \sum_{i \in A} \left(-v_i + \sum_{B: i \in B} p_B \frac{v_i^\alpha}{\sum_{j \in B} v_j^\alpha} \right) \\ &\geq \sum_{i \in A} \left(-v_i + p_A \frac{v_i^\alpha}{\sum_{j \in A} v_j^\alpha} \right) = -\sum_{i \in A} v_i + p_A. \end{aligned}$$

If v is on the boundary of Δ and there exists a set A such that $\sum_{i \in A} v_i = c$, then

$$\frac{d}{dt} \sum_{i \in A} v_i \geq -\sum_{i \in A} v_i + p_A = -c + p_A > 0,$$

which means that F points inward on the boundary of Δ , therefore, Δ is indeed positively invariant under the ODE $\frac{d\vec{v}(t)}{dt} = F(\vec{v}(t))$. \square

We recall that \mathcal{E} denotes the set of equilibria of the WARM [the set of solutions to $F(\vec{v}) = \vec{0}$]. We say that $\vec{v}(t) = (v_1(t), \dots, v_n(t))$ is an *integral curve* of F if $\frac{d\vec{v}}{dt} = F(\vec{v})$. The vector field F has *unique integral curves* if for any initial condition $\vec{v}(0) \in \Delta$ the ODE $\frac{d\vec{v}(t)}{dt} = F(\vec{v}(t))$ has a unique solution.

DEFINITION 4 (Strict Lyapunov function). A strict Lyapunov function for a vector field F is a continuous map $L : \Delta \mapsto \mathbb{R}$ which is strictly monotone along any integral curve of F outside of \mathcal{E} . In this case, we call F *gradient-like*.

We define a function $L : \Delta \mapsto \mathbb{R}$ as

$$(9) \quad L(x_1, x_2, \dots, x_n) = -\sum_{i=1}^n x_i + \frac{1}{\alpha} \sum_A p_A \ln \left(\sum_{j \in A} x_j^\alpha \right).$$

One can check that

$$(10) \quad x_i \frac{\partial L}{\partial x_i} = -x_i + \sum_{A: i \in A} p_A \frac{x_i^\alpha}{\sum_{j \in A} x_j^\alpha} = F_i(\vec{x}).$$

The following result is an analogue of [3], Lemma 4.1:

LEMMA 2. L is a strict Lyapunov function for F .

PROOF. Assume that $\vec{v}(t)$ is an integral curve of F , then

$$\frac{d}{dt}L(\vec{v}(t)) = \sum_{i=1}^n \frac{\partial L}{\partial x_i} \frac{dv_i}{dt} = \sum_{i=1}^n v_i \left(\frac{\partial L}{\partial x_i} \right)^2 \geq 0.$$

The last expression is zero if and only if $v_i \left(\frac{\partial L}{\partial x_i} \right)^2 = 0$ for all $i \in [n]$, which is equivalent to $F(\vec{v}) = 0$ (or $\vec{v} \in \mathcal{E}$). \square

The following result (see [1, 2] and [3], Theorem 3.3) in which $\|\cdot\|$ denotes the Euclidean norm, will be needed for the proof of Theorem 1.

THEOREM 5. Let $F: \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous gradient-like vector field with unique integral curves, let \mathcal{E} be its set of equilibria, let L be a strict Lyapunov function, and let \vec{X}_t be a solution to the recursion (8), where $(\gamma_t)_{t \geq 0}$ is a decreasing sequence and $(\vec{u}_t)_{t \geq 0} \subset \mathbb{R}^n$. Assume that:

- (i) $(\vec{X}_t)_{t \geq 0}$ is bounded,
- (ii) for each $T > 0$,

$$\lim_{n \rightarrow +\infty} \left(\sup_{\{k: 0 \leq \tau_k - \tau_n \leq T\}} \left\| \sum_{i=n}^{k-1} \gamma_i \vec{u}_i \right\| \right) = 0,$$

where $\tau_n = \sum_{i=0}^{n-1} \gamma_i$, and

- (iii) $L(\mathcal{E}) \subset \mathbb{R}$ has empty interior.

Then the limit set of $(\vec{X}_t)_{t \geq 0}$ is a connected subset of \mathcal{E} .

For each subset $S \subset [n]$, we define

$$\Delta_S := \{v \in \Delta: v_i = 0 \text{ iff } i \notin S\}.$$

We see that Δ_S is a face of Δ , it is also a manifold with corners, and, extending the result of Lemma 1, it is easy to see that Δ_S is positively invariant under the ODE $\frac{d\vec{v}}{dt} = F(\vec{v})$.

DEFINITION 5 (S -singularities). $\vec{v} \in \Delta_S$ is an S -singularity for L if

$$\frac{\partial L}{\partial v_i}(\vec{v}) = 0 \quad \text{for all } i \in S.$$

Let $\mathcal{E}_S \subset \Delta_S$ denote the set of S -singularities for L .

LEMMA 3. $\mathcal{E} = \bigcup_{S \subset [n]} \mathcal{E}_S$.

PROOF. $\vec{v} \in \mathcal{E}$ means that $F(\vec{v}) = 0$, and due to (10) this is equivalent to $v_i \frac{\partial L}{\partial v_i} = 0$. Therefore, $\vec{v} \in \mathcal{E}$ implies that for all $i \in [n]$, either $v_i = 0$ or $\frac{\partial L}{\partial v_i} = 0$. \square

PROOF OF THEOREM 1. We follow the proof of [3], Theorem 1.2, very closely. Note that $\gamma_t = 1/(n + t + 1)$ satisfies

$$\lim_{t \rightarrow +\infty} \gamma_t = 0 \quad \text{and} \quad \sum_{t \geq 0} \gamma_t = +\infty.$$

It is obvious from the definition that $(\vec{X}_t)_{t \geq 0}$ is bounded, thus condition (i) of Theorem 5 is satisfied. Let us verify condition (ii). We define

$$\vec{M}_t := \sum_{s=0}^t \gamma_s \vec{u}_s.$$

It is clear that $(\vec{M}_t)_{t \geq 0}$ is a martingale adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Furthermore, since for any $t \geq 0$,

$$\sum_{s=0}^t \mathbb{E}[\|\vec{M}_{s+1} - \vec{M}_s\|^2 | \mathcal{F}_s] \leq \sum_{s=0}^t \gamma_{s+1}^2 \leq \sum_{s=0}^{\infty} \gamma_s^2 < \infty,$$

the sequence $(\vec{M}_t)_{t \geq 0}$ converges almost surely and in L^2 to a finite random vector. In particular, it is a Cauchy sequence and, therefore, condition (ii) holds almost surely.

Now we need to verify condition (iii) in Theorem 5. We need to distinguish between equilibria lying in the interior of \mathcal{E} and those lying on the boundary.

In order to check condition (iii) of Theorem 5, we need to show that $L(\mathcal{E})$ has empty interior. For any $S \subset [n]$, the function L restricted to Δ_S is a C^∞ function, thus by Sard’s theorem [10] $L(\mathcal{E}_S)$ has zero Lebesgue measure, which implies that $L(\mathcal{E})$ has zero Lebesgue measure, which in turn implies that $L(\mathcal{E})$ has empty interior. This verifies condition (iii) in Theorem 5, and completes the proof of Theorem 1(i).

The proof of Theorem 1(ii) follows at once from equation (8) (which expresses the WARM in an equivalent way as a stochastic approximation algorithm) and [2], Proposition 7.5. \square

2.2. *Proof of Proposition 3.* Fix $\alpha > 1$. Suppose that $\vec{v} = ((v_j)_{j \in I}, (0)_{i \in [n] \setminus I})$ is a (linearly-stable) equilibrium for the WARM $(p_A)_{A \subset [n]}$. Then (3) holds for each $i \in [n]$ (with $v_j = 0$ for $j \in [n] \setminus I$) so for each $i \in I$,

$$\begin{aligned} 0 &= -v_i + \sum_{\substack{A \subset [n]: \\ i \in A}} p_A \frac{v_i^\alpha}{\sum_{j \in A \cap I} v_j^\alpha} = -v_i + \sum_{\substack{A' \subset I: \\ i \in A'}} \sum_{\substack{A \subset [n]: \\ A \cap I = A'}} p_A \frac{v_i^\alpha}{\sum_{j \in A'} v_j^\alpha} \\ (11) \quad &= -v_i + \sum_{\substack{A' \subset I: \\ i \in A'}} p_{A'}^I \frac{v_i^\alpha}{\sum_{j \in A'} v_j^\alpha}, \end{aligned}$$

by definition of $p_{A'}^I$. Thus, $(v_i)_{i \in I}$ is an equilibrium for $(p_{A'}^I)_{A' \subset I}$. Next, for any \vec{v} ,

$$(12) \quad D_{i,i}(\vec{v}) = -1 + \alpha v_i^{\alpha-1} \sum_{\substack{A \subset [n]: \\ i \in A}} p_A \frac{\sum_{j \in A} v_j^\alpha - v_i^\alpha}{(\sum_{j \in A} v_j^\alpha)^2},$$

and for $k \neq i$,

$$(13) \quad D_{i,k}(\vec{v}) = -\alpha v_k^{\alpha-1} v_i^\alpha \sum_{\substack{A \subset [n]: \\ i,k \in A}} p_A \frac{1}{(\sum_{j \in A} v_j^\alpha)^2}.$$

For \vec{v} such that $v_i = 0$ for each $i \in [n] \setminus I$ some of these become trivial:

$$(14) \quad D_{i,k}(\vec{v}) = 0 \quad \text{if } i \neq k \text{ and at least one of } i \text{ or } k \text{ is in } [n] \setminus I; \text{ and}$$

$$(15) \quad D_{i,i}(\vec{v}) = -1 \quad \text{for } i \in [n] \setminus I.$$

For $i, k \in I$, proceeding as in the steps leading up to (11) we obtain

$$(16) \quad \begin{aligned} D_{i,i}(\vec{v}) &= -1 + \alpha v_i^{\alpha-1} \sum_{\substack{A \subset [n]: \\ i \in A}} p_A \frac{\sum_{j \in A} v_j^\alpha - v_i^\alpha}{(\sum_{j \in A} v_j^\alpha)^2} \\ &= -1 + \alpha v_i^{\alpha-1} \sum_{\substack{A' \subset I: \\ i \in A'}} p_{A'}^I \frac{\sum_{j \in A'} v_j^\alpha - v_i^\alpha}{(\sum_{j \in A'} v_j^\alpha)^2}, \end{aligned}$$

and

$$(17) \quad \begin{aligned} D_{i,k}(\vec{v}) &= -\alpha v_k^{\alpha-1} v_i^\alpha \sum_{\substack{A \subset [n]: \\ i,k \in A}} p_A \frac{1}{(\sum_{j \in A} v_j^\alpha)^2} \\ &= -\alpha v_k^{\alpha-1} v_i^\alpha \sum_{\substack{A' \subset I: \\ i,k \in A'}} p_{A'}^I \frac{1}{(\sum_{j \in A'} v_j^\alpha)^2}. \end{aligned}$$

For $i, k \in I$, (16) and (17) are exactly the entries of the Jacobian D^I for the WARM $(p_{A'}^I)_{A' \subset I}$, at the point $\vec{v}^I = (v_i)_{i \in I}$. Combining (14)–(17), we have

$$(18) \quad \det(D(\vec{v}) - \lambda \mathbf{I}_n) = (-(1 + \lambda))^{n-|I|} \det(D^I(\vec{v}^I) - \lambda \mathbf{I}_{|I|}),$$

where $\mathbf{I}_n \in \mathbb{R}^n$ denotes the identity matrix. Thus, except for repeated eigenvalues of -1 the matrices $D(\vec{v})$ and $D^I(\vec{v}^I)$ have the same eigenvalues. It follows immediately that if $((v_j)_{j \in I}, (0)_{i \in [n] \setminus I})$ is a linearly stable equilibrium for $(p_A)_{A \subset [n]}$ then $(v_i)_{i \in I}$ is linearly stable for $(p_{A'}^I)_{A' \subset I}$.

Suppose now that $\vec{v}^I = (v_i)_{i \in I}$ is an equilibrium for $(p_{A'}^I)_{A' \subset I}$ and let $(p_A)_{A \subset [n]}$ be a WARM on n colours such that

$$(19) \quad \sum_{\substack{A \subset [n]: \\ A \cap I = A'}} p_A = p_{A'}^I, \quad \text{for each } A' \subset I.$$

Since \vec{v}^I is an equilibrium, (11) holds for each $i \in I$, and we may reverse the steps leading to (11) to see that $((v_j)_{j \in I}, (0)_{i \in [n] \setminus I})$ is an equilibrium for the WARM $(p_A)_{A \subset [n]}$. Moreover, (18) holds, so if $(v_i)_{i \in I}$ is linearly stable for $(p_{A'}^I)_{A' \subset I}$ then so is $((v_j)_{j \in I}, (0)_{i \in [n] \setminus I})$ for $(p_A)_{A \subset [n]}$.

2.3. *Proof of Theorem 3.* Fix $\alpha > 1$ and let G , and $\mathbf{G} = \{G_j\}_{j=1}^k \in \mathcal{G}_G$, and \vec{v} be as in the statement of the theorem. Under Condition 1, if $e \in E$ is incident to two leaves in G , then $p_{\{e\}} = 2/|V|$. Otherwise, $p_A = 1/|V|$ for every A that is the set of edges incident to some vertex of G , and of course every edge e is an edge in exactly two such A .

By Proposition 3 with $I = \mathbf{E}$ we have that \vec{v} is a (linearly stable) equilibrium for the G -WARM if and only if $\vec{v}^{\mathbf{E}} = (v_e)_{e \in \mathbf{E}}$ is a (linearly stable) equilibrium on $(p_{A'}^{\mathbf{E}})_{A' \subset \mathbf{E}}$, where for $A' \subset \mathbf{E}$,

$$(20) \quad p_{A'}^{\mathbf{E}} \equiv \sum_{\substack{A \subset \mathbf{E}: \\ A \cap \mathbf{E} = A'}} p_A.$$

If A' contains two edges that are not adjacent then $p_A = 0$ for each $A \subset \mathbf{E}$ such that $A \cap \mathbf{E} = A'$, so $p_{A'}^{\mathbf{E}} = 0$. More generally, for $A' \subset \mathbf{E}$, $p_{A'}^{\mathbf{E}} > 0$ if and only if there exists $x \in V$ such that A' is the set of edges incident to x in \mathbf{E} . Since the components of \mathbf{E} are vertex disjoint by assumption, it follows that

$$(21) \quad p_{A'}^{\mathbf{E}} > 0 \iff \exists \text{ a unique } j \leq k \text{ such that } A' \text{ is the set of edges in } E_j \text{ incident to some } x \in V_j.$$

If $p_{A'}^{\mathbf{E}} > 0$ and A' contains more than one element, then there is a unique $A \subset \mathbf{E}$ (with $p_A > 0$) such that $A \cap \mathbf{E} = A'$, namely the set of edges A_x in \mathbf{E} incident to x , so $p_{A'}^{\mathbf{E}} = p_{A_x} = 1/|V|$. Otherwise, A' contains a single element $e = (x, x')$ and so $p_{A'}^{\mathbf{E}} = p_{A_x} + p_{A_{x'}} = 2/|V|$. If E_j contains more than one element, then since it is connected, it contains no edge incident to two leaves, so for every $A' \subset E_j$ such that $p_{A'}^{\mathbf{E}} > 0$ we must have that $p_{A'}^{\mathbf{E}} = 1/|V|$. If $E_j = \{e\}$ then there is precisely one nonempty $A' \subset E_j$ and it satisfies $p_{A'}^{\mathbf{E}} = 2/|V|$. Thus,

$$(22) \quad \text{For every nonempty } A' \subset E_j: \quad p_{A'}^{\mathbf{E}} = \begin{cases} \frac{1}{|V|}, & \text{if } |E_j| > 1, \\ \frac{2}{|V|}, & \text{if } |E_j| = 1. \end{cases}$$

Proposition 3 and (21) and (22) imply that \vec{v} is a (linearly stable) equilibrium for the G -WARM if and only if for each j and each $e \in E_j$,

$$\begin{aligned}
 v_e &= \sum_{\substack{A' \subset \mathbf{E}: \\ e \in A'}} p_{A'}^{\mathbf{E}} \frac{v_e^\alpha}{\sum_{e' \in A'} v_{e'}^\alpha} \\
 (23) \quad &= \sum_{\substack{A' \subset E_j: \\ e \in A'}} p_{A'}^{\mathbf{E}} \frac{v_e^\alpha}{\sum_{e' \in A'} v_{e'}^\alpha} \\
 &= \begin{cases} \sum_{\substack{A' \subset E_j: \\ e \in A'}} \frac{1}{|V|} \frac{v_e^\alpha}{\sum_{e' \in A'} v_{e'}^\alpha}, & \text{if } |E_j| > 1, \\ \frac{2}{|V|}, & \text{if } |E_j| = 1. \end{cases}
 \end{aligned}$$

If $|E_j| = 1$, then $|V_j| = 2$ and we can write (23) as

$$(24) \quad \frac{|V|}{|V_j|} v_e = \frac{2}{|V_j|}.$$

If $|E_j| > 1$, then we can write (23) as

$$(25) \quad \frac{|V|}{|V_j|} v_e = \sum_{\substack{A' \subset E_j: \\ e \in A'}} \frac{1}{|V_j|} \frac{(|V|/|V_j| v_e)^\alpha}{\sum_{e' \in A'} (|V|/|V_j| v_{e'})^\alpha}.$$

Now observe that (24) and (25) are the equilibrium equations for the graph $G_j \ni e$ expressed in terms of the rescaled components $v_e^{(j)} = \frac{|V|}{|V_j|} v_e$. By summing (25) [or using (24) if $|E_j| = 1$], we see that $\sum_{e \in E_j} v_e^{(j)} = 1$. Thus, we have proved that $\vec{v}^{\mathbf{E}}$ is an equilibrium for $(p_{A'}^{\mathbf{E}})_{A' \subset \mathbf{E}}$ if and only if for each $j \leq k$, $\vec{v}^{(j)}$ is an equilibrium for E_j . This proves the first claim of the theorem.

For the second claim, by Proposition 3 [see (16) and (17)] for distinct $e, e^* \in \mathbf{E}$,

$$(26) \quad D_{e,e}^{\mathbf{E}}(\vec{v}) = -1 + \alpha v_e^{\alpha-1} \sum_{\substack{A' \subset \mathbf{E}: \\ e \in A'}} p_{A'}^{\mathbf{E}} \frac{\sum_{e' \in A'} v_{e'}^\alpha - v_e^\alpha}{(\sum_{e' \in A'} v_{e'}^\alpha)^2},$$

$$(27) \quad D_{e,e^*}^{\mathbf{E}}(\vec{v}) = -\alpha v_{e^*}^{\alpha-1} v_e^\alpha \sum_{\substack{A' \subset \mathbf{E}: \\ e, e^* \in A'}} p_{A'}^{\mathbf{E}} \frac{1}{(\sum_{e' \in A'} v_{e'}^\alpha)^2}.$$

Again using (21) we have that if $e \in E_j$ and $e^* \in \mathbf{E} \setminus E_j$ then $D_{e,e^*}^{\mathbf{E}}(\vec{v}) = 0$ since $p_{A'}^{\mathbf{E}} = 0$ for every $A' \ni e, e^*$.

Using both (21) and (22), for $e \in E_j$

$$(28) \quad D_{e,e}^{\mathbf{E}}(\vec{v}) = -1 + \alpha v_e^{\alpha-1} \sum_{\substack{A' \subset E_j: \\ e \in A'}} p_{A'}^{\mathbf{E}} \frac{\sum_{e' \in A'} v_{e'}^\alpha - v_e^\alpha}{(\sum_{e' \in A'} v_{e'}^\alpha)^2}$$

$$(29) \quad = -1 + \mathbb{1}_{\{|E_j|>1\}} \alpha v_e^{\alpha-1} \sum_{\substack{A' \subset E_j: \\ e \in A'}} \frac{1}{|V|} \frac{\sum_{e' \in A'} v_{e'}^\alpha - v_e^\alpha}{(\sum_{e' \in A'} v_{e'}^\alpha)^2},$$

while for distinct $e, e^* \in E_j$,

$$(30) \quad D_{e,e^*}^{\mathbf{E}}(\vec{v}) = -\alpha v_{e^*}^{\alpha-1} v_e^\alpha \sum_{\substack{A' \subset E_j: \\ e, e^* \in A'}} \frac{1}{|V|} \frac{1}{(\sum_{e' \in A'} v_{e'}^\alpha)^2}.$$

Using the fact that $|V_j|^{-1} = |V|^{-1} (|V_j|/|V|)^{2\alpha-1} / (|V_j|/|V|)^{2\alpha}$, we can rewrite (29) and (30) as

$$(31) \quad D_{e,e}^{\mathbf{E}}(\vec{v}) = -1 + \mathbb{1}_{\{|E_j|>1\}} \alpha (v_e^{(j)})^{\alpha-1} \sum_{\substack{A' \subset E_j: \\ e \in A'}} \frac{1}{|V_j|} \frac{\sum_{e' \in A'} (v_{e'}^{(j)})^\alpha - (v_e^{(j)})^\alpha}{(\sum_{e' \in A'} (v_{e'}^{(j)})^\alpha)^2},$$

$$(32) \quad D_{e,e^*}^{\mathbf{E}}(\vec{v}) = -\alpha (v_{e^*}^{(j)})^{\alpha-1} (v_e^{(j)})^\alpha \sum_{\substack{A' \subset E_j: \\ e, e^* \in A'}} \frac{1}{|V_j|} \frac{1}{(\sum_{e' \in A'} (v_{e'}^{(j)})^\alpha)^2},$$

where as above, $v_e^{(j)} \equiv \frac{|V|}{|V_j|} v_e$. These are the entries of the Jacobian $\mathbf{D}^{(j)}(\vec{v}^{(j)})$ for the graph G_j .

The above two paragraphs show that $\mathbf{D}^{\mathbf{E}}(\vec{v}^{\mathbf{E}})$ is a block diagonal matrix of the form

$$\mathbf{D}^{\mathbf{E}}(\vec{v}^{\mathbf{E}}) = \begin{pmatrix} \mathbf{D}^{(1)}(\vec{v}^{(1)}) & 0 & \cdots & 0 \\ 0 & \mathbf{D}^{(2)}(\vec{v}^{(2)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^{(k)}(\vec{v}^{(k)}) \end{pmatrix},$$

where for each $i \leq k$, $\mathbf{D}^{(i)}(\vec{v}^{(i)})$ is the Jacobian matrix for G_i at the point $\vec{v}^{(i)}$. Thus, the eigenvalues of $\mathbf{D}^{\mathbf{E}}(\vec{v}^{\mathbf{E}})$ are simply those of all the $\mathbf{D}^{(i)}(\vec{v}^{(i)})$, $i \in [k]$ combined, and the result follows.

2.4. *Proof of Theorem 2.* Fix $\alpha > 1$. For $n = 1$, the claim is trivial. The proof proceeds via induction over n , assuming that the result holds for all $n' < n$.

Let $\vec{v} = (v_1, \dots, v_n) \in \mathcal{E}$ denote an equilibrium distribution, so that

$$(33) \quad F(\vec{v}) = \vec{0},$$

where

$$(34) \quad F_i(\vec{v}) = -v_i + \sum_{A \ni i} p_A \frac{v_i^\alpha}{\sum_{k \in A} v_k^\alpha}, \quad \text{for } i \in [n].$$

We assume that $v_i \neq 0$ for all $i \in [n]$. If an equilibrium is linearly stable for the system of n equations and there is some $I \neq [n]$ such that $v_i = 0$ for all $i \in [n] \setminus I$, then by Proposition 3 it is linearly stable for the system on I .

Let $A^* \subseteq [n]$ be nonempty. Since $\alpha > 1$, by Hölder’s inequality,

$$(35) \quad \sum_{k \in A^*} v_k^\alpha \geq \left(\sum_{k \in A^*} v_k \right)^\alpha |A^*|^{1-\alpha}.$$

Summing (34) over $i \in A^*$ [subject to (33)] gives

$$(36) \quad \sum_{i \in A^*} v_i = \sum_{i \in A^*} \sum_{A \ni i} p_A \frac{v_i^\alpha}{\sum_{k \in A} v_k^\alpha} \geq p_{A^*} \frac{\sum_{i \in A^*} v_i^\alpha}{\sum_{k \in A^*} v_k^\alpha}.$$

Hence,

$$(37) \quad \sum_{k \in A^*} v_k \geq p_{A^*}.$$

Equations (35) and (37) imply that for every nonempty $A \subseteq [n]$

$$(38) \quad \sum_{k \in A} v_k^\alpha \geq p_A^\alpha |A|^{1-\alpha}.$$

Inserting this into (33), we obtain

$$v_i^{1-\alpha} = \sum_{A \ni i} p_A \frac{1}{\sum_{k \in A} v_k^\alpha} \leq \sum_{A \ni i} p_A p_A^{-\alpha} |A|^{\alpha-1},$$

which is equivalent to

$$v_i \geq \left(\sum_{A \ni i} (p_A/|A|)^{1-\alpha} \right)^{1/(1-\alpha)}.$$

This shows that there exists a (model dependent) $\varepsilon > 0$ such that there is no $\vec{v} \in \mathcal{E}$ satisfying $0 < v_i < \varepsilon$ for some $i \in [n]$. It remains to prove that for any $\varepsilon > 0$ there are only finitely many $\vec{v} \in \mathcal{S}$ satisfying $v_i \geq \varepsilon$ for all $i \in [n]$.

Fix $\varepsilon > 0$, and choose $\delta \in (0, \frac{\pi}{2\alpha})$ and define $H^n \subset \mathbb{C}^n$ to be the Cartesian product of n copies of the open complex domain

$$H := \left\{ z \in \mathbb{C} : \frac{\varepsilon}{2} < |z| < 2, |\arg(z)| < \delta \right\}.$$

Since, for $z \in H$,

$$|\arg(z^\alpha)| = \alpha |\arg(z)| < \alpha \delta < \pi/2,$$

we see that $\operatorname{Re}(z^\alpha) > 0$ for all $z \in H$. Therefore, for nonempty A , $\operatorname{Re}[\sum_{k \in A} v_k^\alpha] > 0$ for $\vec{v} \in H^n$, in particular, all functions $\vec{v} \mapsto \sum_{k \in A} v_k^\alpha$ are analytic and zero-free in H^n , which shows that the functions

$$\vec{v} \mapsto \frac{v_i^\alpha}{\sum_{k \in A} v_k^\alpha}$$

are also analytic in H^n , so finally we conclude that the functions $F_i(\vec{v})$ are analytic in H^n .

Next, define the map $F: H^n \mapsto \mathbb{C}^n$ by $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v}), \dots, F_n(\vec{v}))$ and the set

$$\mathcal{H} := \{\vec{v} \in H^n : F(\vec{v}) = \vec{0} \text{ and } \det[\mathbf{D}(\vec{v})] \neq 0\}.$$

Clearly, $\mathcal{S} \subset \mathcal{H}$. Our goal is to show that (i) \mathcal{H} is a set of isolated points and (ii) it does not have accumulation points in the interior of the domain H^n .

To prove (i), let $\vec{w} \in \mathcal{H}$. Since $F(\vec{w}) = \vec{0}$ and $\det[\mathbf{D}(\vec{w})] \neq 0$, due to the implicit function theorem (see [23], Theorem 2, page 40) there exists a bi-holomorphic map between some neighbourhoods $U \ni \vec{w}$ and $V \ni \vec{0}$ (i.e., a bijective holomorphic function whose inverse is also holomorphic). Since the map is bijective, there are no other solutions to the system $F_i(\vec{v}) = 0, i \in [n]$ in U , which shows that each element of \mathcal{H} must be an isolated point.

To prove (ii), let us assume the converse, that is, there exists a point $\vec{w} \in H^n$ which is an accumulation point of \mathcal{H} . Define

$$\mathcal{Z} := \{\vec{v} \in H^n : F(\vec{v}) = \vec{0}\},$$

so \mathcal{Z} is an *analytic set* in the sense of [23], Definition 1, page 129, and clearly $\mathcal{H} \subseteq \mathcal{Z}$. According to [20], Theorem 2.2, page 52, there exists a neighbourhood $\Delta \subset H^n$ of the point \vec{w} , such that the analytic set $\Delta \cap \mathcal{Z}$ can be decomposed into a *finite* number of pure-dimensional analytic sets (“pure-dimensional” means that the set has the same dimension at each point). One of these pure-dimensional analytic sets must have dimension zero (since we have assumed that \vec{w} is an accumulation point for isolated points in \mathcal{H} , and isolated points are zero-dimensional). It is also clear that this zero-dimensional analytic set must have an accumulation point at \vec{w} . Now we use [23], Theorem 6 on page 135, which says that this is impossible: any zero-dimensional analytic set in Δ cannot have limit points inside Δ . Therefore, we have arrived at a contradiction.

So far we have proved that the set \mathcal{H} consists of isolated points and does not have accumulation points in the interior of H^n . Since the set

$$B := \{\vec{v} \in \mathbb{C}^n : \operatorname{Im}(v_i) = 0, \varepsilon \leq \operatorname{Re}(v_i) \leq 1\}$$

is compact in \mathbb{C}^n , we conclude that the set $B \cap H^n$ is finite. Since stable equilibria are elements of $B \cap H^n$, this shows that we can have only finitely many $\vec{v} \in \mathcal{S}$ satisfying $v_i > \varepsilon$ for each i .

2.5. Proof of Proposition 1.

PROOF OF PROPOSITION 1(i). Assume that Condition 2 holds. Then, for $\vec{v} = \vec{1}/n$, the right-hand side of (3) becomes

$$(39) \quad \sum_{A \ni i} \frac{p_A}{|A|} = \sum_{m=1}^n \sum_{\substack{A \ni i: \\ |A|=m}} \frac{p_A}{m} = \sum_{m=1}^n \frac{a_m p_m}{m},$$

which does not depend on $i \in [n]$. Since these quantities sum to 1, it follows that the right-hand side of (3) is equal to $1/n$ for each i , which proves that $\vec{1}/n$ is an equilibrium. \square

Recall that the adjugate matrix $\text{adj } \mathbf{A}$ of a square matrix \mathbf{A} is given by $\text{adj } \mathbf{A} = \mathbf{C}^T$, that is, the transpose of the cofactor matrix \mathbf{C} of \mathbf{A} . If \mathbf{A} is invertible, then $\mathbf{A}^{-1} \det(\mathbf{A}) = \text{adj}(\mathbf{A})$. Recall that if \mathbf{A} is a diagonal matrix with entries A_{ii} , then its cofactor matrix is a diagonal matrix \mathbf{C} with $C_{ii} = \prod_{j \neq i} A_{jj}$, and its adjugate matrix is a diagonal matrix $\text{adj } \mathbf{A} = \mathbf{C}^T = \mathbf{C}$. In order to prove Proposition 1(ii), we will use the following modification of the matrix determinant lemma, which we have not found in the literature (although we expect that it is well known).

LEMMA 4 (Modified matrix determinant lemma). *If $\mathbf{R} \in \mathbb{R}^{n \times n}$ and $\vec{y}, \vec{w} \in \mathbb{R}^n$ are column vectors then*

$$(40) \quad \det(\mathbf{R} + \vec{y}\vec{w}^T) = \det(\mathbf{R}) + \vec{w}^T \text{adj}(\mathbf{R})\vec{y}.$$

PROOF. If \mathbf{R} is invertible, then the matrix determinant lemma gives

$$\begin{aligned} \det(\mathbf{R} + \vec{y}\vec{w}^T) &= (1 + \vec{w}^T \mathbf{R}^{-1} \vec{y}) \det(\mathbf{R}) \\ &= \det(\mathbf{R}) + \vec{w}^T \mathbf{R}^{-1} \det(\mathbf{R}) \vec{y} = \det(\mathbf{R}) + \vec{w}^T \text{adj}(\mathbf{R}) \vec{y}. \end{aligned}$$

If \mathbf{R} is not invertible, then \mathbf{R} has some eigenvalues that are zero (and possibly some nonzero) and there exists some ε_0 (corresponding to the smallest magnitude-nonzero eigenvalue) such that no $\varepsilon \in (0, \varepsilon_0)$ is an eigenvalue for \mathbf{R} , that is, $\det(\mathbf{R} - \varepsilon \mathbf{I}) \neq 0$ for all such ε . Therefore, $\mathbf{R} - \varepsilon \mathbf{I}$ is invertible for any such ε . It follows that for all $\varepsilon \in (0, \varepsilon_0)$

$$(41) \quad \det(\mathbf{R} - \varepsilon \mathbf{I} + \vec{y}\vec{w}^T) = \det(\mathbf{R} - \varepsilon \mathbf{I}) + \vec{w}^T \text{adj}(\mathbf{R} - \varepsilon \mathbf{I}) \vec{y}.$$

We obtain the desired conclusion by taking the limit as $\varepsilon \downarrow 0$ on both sides of (41), and using the facts that all entries of $\text{adj}(\mathbf{R})$ are just sums and differences of minors (determinants of submatrices), and determinants are continuous functions of \mathbf{R} (in the natural sense). \square

PROOF OF PROPOSITION 1(ii). By (12) and (13),

$$(42) \quad D_{i,k}(\vec{v}) = \begin{cases} -1 + \alpha v_i^{\alpha-1} \sum_{A \ni i} p_A \frac{\sum_{j \in A} v_j^\alpha - v_i^\alpha}{(\sum_{j \in A} v_j^\alpha)^2}, & \text{if } i = k, \\ -\alpha v_k^{\alpha-1} v_i^\alpha \sum_{A \ni i,k} p_A \frac{1}{(\sum_{j \in A} v_j^\alpha)^2}, & \text{if } i \neq k. \end{cases}$$

When $\vec{v} = \vec{1}/n$, this reduces to

$$(43) \quad D_{i,k}(\vec{v}) = \begin{cases} -1 + \alpha n \sum_{A \ni i} p_A \frac{|A| - 1}{|A|^2}, & \text{if } i = k, \\ -\alpha n \sum_{A \ni i,k} p_A \frac{1}{|A|^2}, & \text{if } i \neq k. \end{cases}$$

Assume that Condition 3 holds. Then (43) can be written as

$$\begin{aligned} D_{i,i}(\vec{1}/n) &= -1 + \alpha n \sum_{m=1}^n p_m \sum_{A:|A|=m, i \in A} \frac{m-1}{m^2} \\ &= -1 + \alpha n \sum_{m=2}^n \frac{p_m}{m^2} \binom{n-1}{m-1} (m-1) =: -1 + \beta, \\ D_{i,k}(\vec{1}/n) &= -\alpha n \sum_{m=2}^n \frac{p_m}{m^2} \binom{n-2}{m-2} =: \delta, \quad \text{if } i \neq k. \end{aligned}$$

To compute the eigenvalues of $\mathbf{D}(\vec{1}/n)$, observe that

$$\mathbf{H} := \mathbf{D} - \lambda \mathbf{I} = (-(1 + \lambda) + \beta - \delta) \mathbf{I} + \vec{1}(\delta \vec{1})^T.$$

Hence, by Lemma 4,

$$\det(\mathbf{H}) = (-(1 + \lambda) + \beta - \delta)^n + \sum_{i=1}^n \delta (-(1 + \lambda) + \beta - \delta)^{n-1}.$$

This is equal to zero when $\lambda = \beta - \delta - 1$ or $\lambda = (n - 1)\delta + \beta - 1 = -1$. The first eigenvalue satisfies

$$\begin{aligned} \lambda &= \alpha n \sum_{m=2}^n \frac{p_m}{m^2} \binom{n-1}{m-1} (m-1) + \alpha n \sum_{m=2}^n \frac{p_m}{m^2} \binom{n-2}{m-2} - 1 \\ &= \alpha n^2 \sum_{m=2}^n \frac{p_m}{m^2} \binom{n-2}{m-2} - 1, \end{aligned}$$

thus λ is continuous and increasing in α , and it is negative if and only if (4) is satisfied. \square

2.6. *Proof of Proposition 2.* The WARMs from Examples 1, 2 and 3 all satisfy the strong symmetry Condition 3, thus the statements of parts (i), (ii) and (iii) of Proposition 2 are simple corollaries of Proposition 1(ii). Let us prove parts (iv) and (v) of Proposition 2.

PROOF OF PROPOSITION 2(iv). For the cycle graph with n vertices and edges, we label the edges $\{0, \dots, n - 1\}$ around the circle (in the obvious way) and use addition and subtraction mod $(n - 1)$. Then \vec{v} is an equilibrium if and only if

$$(44) \quad v_i = \frac{1}{n} \frac{v_i^\alpha}{v_i^\alpha + v_{i+1}^\alpha} + \frac{1}{n} \frac{v_i^\alpha}{v_i^\alpha + v_{i-1}^\alpha}.$$

Moreover, equations (43) give us $D_{i,i}(\vec{1}/n) = -1 + \frac{\alpha}{2}$, $D_{i,i+1}(\vec{1}/n) = D_{i,i-1}(\vec{1}/n) = -\frac{\alpha}{4}$ and $D_{i,k}(\vec{1}/n) = 0$ for $|i - k| > 1$. Thus, \mathbf{D} is a circulant matrix with 3 consecutive [mod $(n - 1)$] nonzero entries $-\alpha/4, -1 + \alpha/2, -\alpha/4$. Therefore, its eigenvalues are of the form

$$\lambda_j = -1 + \frac{\alpha}{2} - \frac{\alpha}{4} e^{2\pi i j/n} - \frac{\alpha}{4} e^{-2\pi i j/n} = -1 + \frac{\alpha}{2} - \frac{\alpha}{2} \cos(2\pi j/n),$$

for $j = 0, \dots, n - 1$. All of these eigenvalues are negative if and only if for every $j = 0, \dots, n - 1$,

$$(45) \quad \alpha [1 - \cos(2\pi j/n)] < 2.$$

When n is even, the left-hand side of (45) attains its maximum of 2α at $j = n/2$ for which the stability criterion is $\alpha < 1$. When n is odd, the left-hand side of (45) attains its maximum at $j = (n + 1)/2$ for which the stability criterion becomes

$$\alpha < \frac{2}{1 - \cos(\pi(1 + 1/n))} = \frac{2}{1 + \cos(\pi/n)} = \frac{1}{\cos(\pi/2n)^2}. \quad \square$$

REMARK 5. Note that for n even, the vector $\vec{v}_{\text{alt}} = 2(1, 0, 1, 0, \dots, 1, 0)/n$ is a linearly-stable equilibrium for the cycle-graph WARM for all $\alpha > 1$ (by Theorem 3).

PROOF OF PROPOSITION 2(v). The case of $n_v = 3$ (the triangle graph) is the same as Example 1 with $n = 3$ and $m = 2$, thus this case is covered by Proposition 2(i). Let us assume that $n_v \geq 4$. Let K_{n_v} be the complete graph on n_v vertices. We recall that the line-graph $L = L(K_{n_v})$ is defined by considering edges of K_{n_v} as vertices of L , and the vertices of L are adjacent if and only if the corresponding edges of K_{n_v} are both incident to some vertex in K_{n_v} . Equations (43) give us

$$(46) \quad D_{i,j}(\vec{1}/n) = \begin{cases} -1 + \alpha - \frac{\alpha}{n_v - 1}, & \text{if } i = j, \\ -\frac{\alpha}{2(n_v - 1)}, & \text{if } i \neq j, \text{ and } i, j \text{ are both incident} \\ & \text{to some vertex } x, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$(47) \quad \mathbf{D} = \left(-1 + \alpha - \alpha \frac{1}{n_v - 1} \right) \mathbf{I} - \frac{\alpha}{2(n_v - 1)} \mathbf{A},$$

where \mathbf{A} is the adjacency matrix of L . According to [5], Corollary 1.4.2, the matrix \mathbf{A} has an eigenvalue -2 of degree $n - n_v$. This shows that the matrix \mathbf{D} has an eigenvalue

$$-1 + \alpha - \alpha \frac{1}{n_v - 1} - \frac{\alpha}{2(n_v - 1)} \times (-2) = -1 + \alpha > 0$$

of multiplicity $n - n_v$, and therefore $\vec{1}/n$ is a linearly unstable equilibrium. \square

3. Detailed analysis of star, triangle and whisker-graph WARMs. The goal of this section is to examine the star-graph WARM, triangle-graph WARM and the whisker-graph WARM in detail, and then to prove Theorem 4. In the first two cases we also present a complete descriptions of the set of equilibria and the set of linearly stable equilibria. Throughout this section, we use the notation $(u)_m$ to denote the vector $(u, \dots, u) \in \mathbb{R}^m$.

3.1. *Star graph WARM.* Throughout this section, we consider a G -WARM where G is a star graph on n edges, and write our vectors up to permutations, ordered in decreasing order. The next theorem is the main result of this section.

THEOREM 6. *Fix $n \geq 2$ and let $k \in [n]$. The following are the equilibria for the star graph-WARM:*

- (i) $(1/n)_n$ for $\alpha > 1$;
- (ii) $((v)_k, (u)_{n-k})$ for $n/2 \leq k \leq n - 1$ and $\alpha > n + 1$, where $v > u$ and $v = v(\alpha)$ is (strictly) increasing in α to $v(+\infty) = (k + 1)/(k(n + 1))$.

Also, for $k < n/2$ there exists $\tilde{\alpha} = \tilde{\alpha}(k, n) \in (1, n + 1)$ and the following equilibria for the star-graph WARM:

- (iii) $((v)_k, (u)_{n-k})$ for $\alpha > \tilde{\alpha}$, where $v > u$ and $v(\alpha)$ is (strictly) increasing in α to $v(+\infty) = (k + 1)/(k(n + 1))$;
- (iv) $((v)_k, (u)_{n-k})$ for $\alpha \in [\tilde{\alpha}, n + 1)$, where $v > u$ and $v(\alpha)$ is (strictly) decreasing in α to $v(n + 1-) = 1/n$.

These are the only equilibria in the star-graph WARM (up to permutations). Moreover, $(1/n)_n$ is a linearly stable equilibrium if and only if $\alpha < n + 1$ (it is critical when $\alpha = n + 1$). The equilibrium described in (ii) and its permutations are linearly stable if and only if $k = 1$ and $n = 2$. The equilibrium described in (iii) and its permutations are linearly stable if and only if $k = 1$ and $\alpha > \tilde{\alpha}(1, n)$. All other equilibria are not linearly stable.

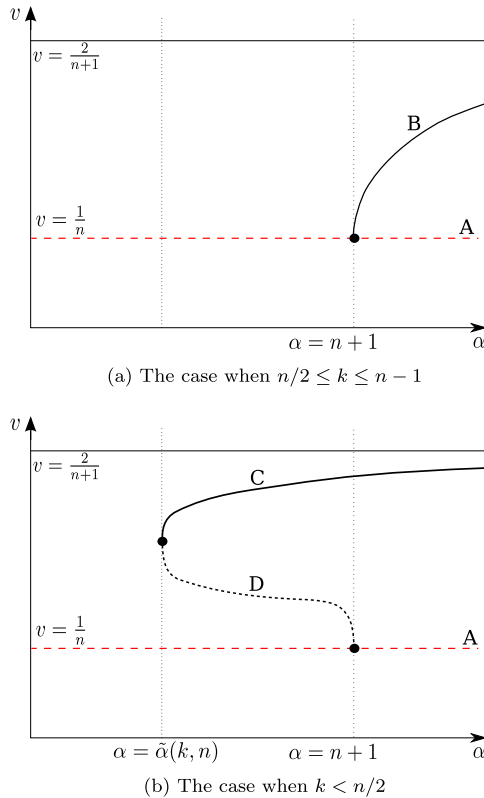


FIG. 1. Equilibria of the form $((v)_k, (u)_{n-k})$ in the star-graph WARM, where $v \geq u$. The red line A corresponds to the equilibrium $(1/n)_n$ where $v = u = 1/n$. The lines B, C and D are the graphs of $v = v(\alpha)$ and correspond to equilibria described in items (ii), (iii) and (iv) in Theorem 6.

REMARK 6. We would like to point out that the values of v and u which appear in different items of Theorem 6 are not the same. These values are characterised by the fact that $v > u$ and by the increase/decrease properties of v as a function of α . Figure 1 gives a visual interpretation of different kinds of equilibria described in Theorem 6 and the relation between them.

The star-graph WARM with $n = 2$ edges will be important for us later, and we state our result for this case explicitly in the next corollary. Note that the star-graph WARM with two edges is the same as the simplest line graph, and it also corresponds (after a time-change) to Example 2 with $n = 2$ and $p = 1/2$.

COROLLARY 1. For the star graph with two edges, the following is true: For $\alpha = 3$, $\mathcal{E} = \{(1/2, 1/2)\}$ and this equilibrium is critical, while for every $\alpha \neq 3$ there exists a unique (up to permutations) $(v, u) \in \mathcal{S}$, where $v = v(\alpha) \geq 1/2$. Moreover,

$v(\alpha)$ is a continuous function of α , that is (strictly) increasing for $\alpha > 3$ from $v(3) = 1/2$ to $v(+\infty) = 2/3$, and such that $v(\alpha) = 1/2$ for $\alpha < 3$.

The proof of Theorem 6 will be given via a sequence of lemmas. Recall that for the star graph on n edges, any equilibrium $\vec{v} \in \mathcal{E}$ must satisfy

$$(48) \quad v_i = \frac{1}{n+1} + \frac{1}{n+1} \cdot \frac{v_i^\alpha}{v_1^\alpha + \dots + v_n^\alpha}, \quad 1 \leq i \leq n.$$

Then clearly $\vec{v} \in \mathcal{E}$ must satisfy $1/(n+1) < v_i < 2/(n+1)$ for each edge $i \in [n]$, therefore all equilibria are internal, and $v_i/v_j \in [1/2, 2]$.

LEMMA 5. Assume that $\vec{v} \in \mathcal{E}$ for the star graph-WARM with n edges. Then $\vec{v} = (1/n)_n$ or there exist $v > u$ and $k \in [n-1]$ such that (up to permutations) $\vec{v} = ((v)_k, (u)_{n-k})$.

PROOF. Fix $\alpha > 1$. Assume that \vec{v} is an equilibrium and let $\delta = \delta(\vec{v}) = \sum_{i=1}^n v_i^\alpha \in (0, 1)$. Define a function $f_\delta: (0, 1) \mapsto \mathbb{R}$ by

$$(49) \quad f_\delta(x) = x^{-1}(1 + \delta^{-1}x^\alpha),$$

and notice that (48) is equivalent to $f_\delta(v_i) = n+1$ for each i . We check that $f'_\delta(x) = x^{-2}((\alpha-1)\delta^{-1}x^\alpha - 1)$, therefore, the function $f'_\delta(x)$ has at most one zero in the interval $1/(n+1) < x < 2/(n+1)$. According to Rolle's theorem, there are at most two solutions to $f_\delta(x) = n+1$ in the interval $1/(n+1) \leq x \leq 2/(n+1)$, whence any equilibrium \vec{v} has at most 2 distinct components. \square

LEMMA 6. The set of equilibria for the star graph-WARM is as claimed in Theorem 6(i)–(iv).

PROOF. The fact that $(1/n)_n$ is an equilibrium for all $\alpha > 1$ is immediate from Proposition 1. Otherwise, for any $\vec{v} \in \mathcal{E}$ by Lemma 5 there exist $v > u$ and $k \in [n-1]$ such that (up to permutations) $\vec{v} = ((v)_k, (u)_{n-k})$.

Any $\vec{v} \in \mathcal{E}$ if and only if it satisfies (48). For \vec{v} of the form $\vec{v} = ((v)_k, (u)_{n-k})$, (48) is equivalent to a single equation

$$(50) \quad u = \frac{1}{n+1} + \frac{1}{n+1} \cdot \frac{u^\alpha}{kv^\alpha + (n-k)u^\alpha},$$

plus the balance equation $kv + (n-k)u = 1$. We introduce a new variable

$$t = \ln(v/u) = \ln((1/u - n + k)/k).$$

From the above formula, it follows that $u = (n + k(e^t - 1))^{-1}$, and then equation (50) gives us

$$(51) \quad \frac{1}{n + k(e^t - 1)} = \frac{1}{n+1} + \frac{1}{n+1} \cdot \frac{1}{n - k + ke^{\alpha t}}.$$

Solving the above equation for $e^{\alpha t}$, we obtain

$$(52) \quad e^{\alpha t} = \frac{n + 1 - k}{k} \cdot \frac{e^t - a}{b - e^t},$$

where we have denoted $a := (n - k)/(n - k + 1)$ and $b := (1 + k)/k$. Let us define

$$(53) \quad f_{k,n}(t) := \ln\left(\frac{n + 1 - k}{k} \cdot \frac{e^t - a}{b - e^t}\right), \quad \ln(a) < t < \ln(b), 1 \leq k \leq n - 1.$$

Then formula (52) is equivalent to the equation

$$(54) \quad \alpha t = f_{k,n}(t),$$

and this will be the starting point for our analysis.

Let us investigate the function $t \mapsto f_{k,n}(t)$ in more detail. We check that

$$f'_{k,n}(t) = \frac{a}{e^t - a} + \frac{b}{b - e^t}, \quad f''_{k,n}(t) = \frac{(b - a)e^t(e^{2t} - ab)}{(e^t - a)^2(b - e^t)^2}.$$

From (53) and the above two formulas, we easily obtain the following facts:

- (I) $f_{k,n}(t)$ is a (strictly) increasing function and $f_{k,n}(t) \rightarrow +\infty$ as $t \uparrow \ln(b)$;
- (II) $f_{k,n}(0) = 0$ and $f'_{k,n}(0) = n + 1$;
- (III) $f_{k,n}(t)$ is (strictly) concave for $t \in (\ln(a), \tilde{t})$ and (strictly) convex for $t \in (\tilde{t}, \ln(b))$, where $\tilde{t} := \ln(ab)/2$;
- (IV) $f''_{k,n}(0) = (2k - n)(n + 1)$;
- (V) The inflection point \tilde{t} satisfies $\tilde{t} \leq 0$ if $k \geq n/2$ and $\tilde{t} > 0$ if $k < n/2$;
- (VI) For all $t \in (\ln(a), \ln(b))$ we have $f'_{k,n}(t) \geq f'_{k,n}(\tilde{t}) = \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}} > 1$.

Let us first consider the case when $k < n/2$. Then the function $t \mapsto f_{k,n}(t)$ is (strictly) concave on $(0, \tilde{t})$ and (strictly) convex on $(\tilde{t}, \ln(b))$. The graph of $f_{k,n}(t)$ is shown in Figure 2(a). Note that there exists a unique $\tilde{\alpha}(n, k)$ such that the straight line $y = \tilde{\alpha}t$ is tangent to $y = f_{k,n}(t)$ at the point \tilde{t} (this is the blue line in Figure 2). Convexity properties of $f_{k,n}(t)$ described in item (III) above and the fact that $f'_{k,n}(0) = n + 1$ imply $\tilde{\alpha} < n + 1$ and item (VI) shows that $\tilde{\alpha} > 1$. We see that for $\alpha > \tilde{\alpha}$ there is a solution $t_2(\alpha)$ to (54) that is a strictly increasing function of α ; for $\alpha \in (\tilde{\alpha}, n + 1)$ there is another solution $t_1(\alpha)$ that is strictly decreasing in α and satisfies $t_1(\alpha) < t_2(\alpha)$. When $\alpha \rightarrow \tilde{\alpha}^+$, these two solutions become equal. There are no other solutions to (54), as otherwise we would have additional inflection points. Recall that we have $u = (n + k(e^t - 1))^{-1}$ and $v = ue^t$. Let us denote by $v_1(\alpha)$ and $v_2(\alpha)$ the values corresponding to $t_1(\alpha)$ and $t_2(\alpha)$. It is clear from items (I)–(III) [see also Figure 2(b)] that when $\alpha \uparrow +\infty$ ($\alpha \uparrow n + 1$), we have $t_2(\alpha) \uparrow \ln(b)$ [resp., $t_1(\alpha) \downarrow 0$], which corresponds to $v_2(\alpha) \uparrow (k + 1)/(k(n + 1))$ [resp., $v_1(\alpha) \downarrow 1/n$]. This demonstrates both the existence and uniqueness of equilibria satisfying (iii) and (iv) of Theorem 6, respectively.

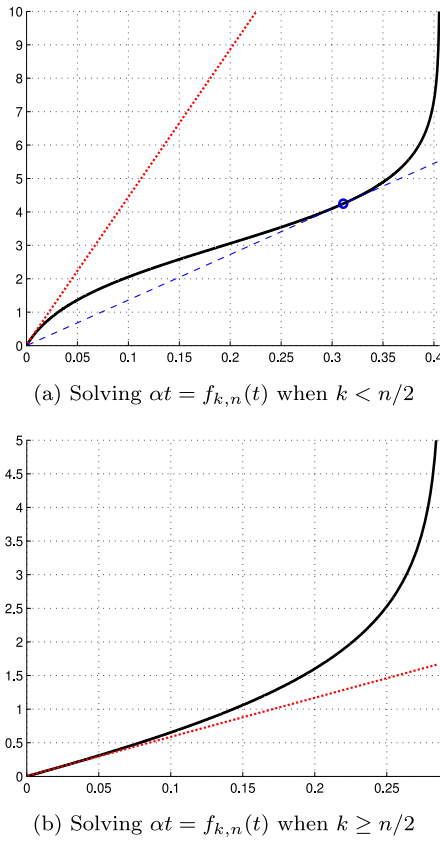


FIG. 2. The black solid curve is the graph of $y = f_{k,n}(t)$ and the red dotted line is $y = (n + 1)t$. In Figure 2(a), the blue dashed line is $y = \tilde{\alpha}t$ where $\tilde{\alpha} = \alpha(k, n)$.

When $k \geq n/2$ the situation is simpler, as the function $t \mapsto f_{k,n}(t)$ is (strictly) convex on $(0, \ln(b))$. Since $f'_{k,n}(0) = n + 1$, we see that for every $\alpha > n + 1$ there exists a unique positive solution $t = t_3(\alpha)$ to (54), and that this solution is (strictly) increasing in α to $t_3(+\infty) = \ln(b)$, which corresponds to $v_3(\alpha) \uparrow (k + 1)/(k(n + 1))$. For $\alpha \leq n + 1$, there are no positive solutions to (54). See Figure 2(b). \square

For $\vec{v} \in \mathbb{R}^n$ and $a > 0$, write $\vec{v}^a = (v_1^a, \dots, v_n^a)$, so that, for example, $((v)_k, (u)_{n-k})^a = ((v^a)_k, (u^a)_{n-k})$.

LEMMA 7. Assume $\vec{v} = ((v)_k, (u)_{n-k}) \in \mathcal{E}$ for some $1 \leq k \leq n - 1$ and $v > u$. Let $\eta = kv^\alpha + (n - k)u^\alpha$ and $\xi = \alpha(n + 1)^{-1}\eta^{-2}$. Then $\vec{v} \in \mathcal{S}$ (critical if equality holds below) if and only if

$$k = 1 \quad \text{and} \quad \xi(uv)^{\alpha-1} < 1, \quad \text{or}$$

$$k \geq 2 \quad \text{and} \quad v < \frac{\alpha}{(\alpha - 1)(n + 1)}.$$

PROOF. The matrix \mathbf{D} of partial derivatives has entries

$$D_{ii} = -1 + \xi \times \begin{cases} v^{\alpha-1}\eta - v^{2\alpha-1}, & \text{if } i \leq k, \\ u^{\alpha-1}\eta - u^{2\alpha-1}, & \text{if } i > k, \end{cases}$$

$$D_{ij} = -\xi \times \begin{cases} v^{2\alpha-1}, & \text{if } i, j \leq k, \\ v^\alpha u^{\alpha-1}, & \text{if } i \leq k < j, \\ v^{\alpha-1} u^\alpha, & \text{if } j \leq k < i, \\ u^{2\alpha-1}, & \text{if } i, j > k. \end{cases}$$

Let

$$(55) \quad \vec{x} = \vec{v}^\alpha \quad \text{and} \quad \vec{w} = -\xi \vec{v}^{\alpha-1}.$$

Let \mathbf{Z} be a diagonal matrix with $Z_{ii} = D_{ii} + z_i$, where $\vec{z} = -\lambda \vec{1} + \xi \vec{v}^{2\alpha-1}$. Then

$$Z_{ii} = -(1 + \lambda) + \xi \eta \begin{cases} v^{\alpha-1}, & \text{if } i \leq k, \\ u^{\alpha-1}, & \text{if } i > k, \end{cases}$$

and $\mathbf{D} - \lambda \mathbf{I} = \mathbf{Z} + \vec{x} \vec{w}^T$. It follows from Lemma 4 that

$$\begin{aligned} \det(\mathbf{D} - \lambda \mathbf{I}) &= \det(\mathbf{Z}) + \vec{w}^T \text{adj}(\mathbf{Z}) \vec{x} \\ &= Z_{1,1}^k Z_{n,n}^{n-k} - \xi v^{2\alpha-1} k Z_{1,1}^{k-1} Z_{n,n}^{n-k} - \xi u^{2\alpha-1} (n - k) Z_{1,1}^k Z_{n,n}^{n-k-1} \\ &= Z_{1,1}^{k-1} Z_{n,n}^{n-k-1} (Z_{1,1} Z_{n,n} - \xi v^{2\alpha-1} k Z_{n,n} - \xi u^{2\alpha-1} (n - k) Z_{1,1}). \end{aligned}$$

After a lot of simplifying, using the definition of η and that $kv + (n - k)u = 1$ we get that the term in brackets is zero if and only if

$$(1 + \lambda)^2 - (1 + \lambda)\xi(uv)^{\alpha-1} = 0,$$

that is, if and only if $\lambda = -1$ or $\lambda = -1 + \xi(uv)^{\alpha-1}$. The latter is negative precisely when $\xi(uv)^{\alpha-1} < 1$.

When $n - k - 1 > 0$ we also have an eigenvalue $\lambda = -1 + \xi \eta u^{\alpha-1}$ at which $Z_{n,n} = 0$. This eigenvalue is negative when $\xi \eta u^{\alpha-1} < 1$. Note that

$$(56) \quad \eta = v^{\alpha-1}(kv + (n - k)u(u/v)^{\alpha-1}) < v^{\alpha-1}(kv + (n - k)u) = v^{\alpha-1},$$

thus the condition $\xi(uv)^{\alpha-1} < 1$ implies $\xi \eta u^{\alpha-1} < 1$. This proves Lemma 7 in the case $k = 1$.

When $k - 1 > 0$, we also have an eigenvalue $\lambda = -1 + \xi \eta v^{\alpha-1}$ at which $Z_{1,1} = 0$. This eigenvalue is negative when $\xi \eta v^{\alpha-1} < 1$. Since $u < v$, we have that $\xi \eta u^{\alpha-1} < \xi \eta v^{\alpha-1}$. Thus in the case $k \geq 2$ the equilibrium $((u)_k, (v)_{n-k})$ is linearly stable if and only if the following two conditions are satisfied:

$$(i) \xi(uv)^{\alpha-1} < 1 \quad \text{and} \quad (ii) \xi \eta v^{\alpha-1} < 1.$$

However, performing a similar computation to (56) we can show that $\eta > u^{\alpha-1}$, thus condition (ii) above implies condition (i). Since v satisfies

$$v = \frac{1}{n+1} + \frac{1}{n+1} \times \frac{v^\alpha}{kv^\alpha + (n-k)u^\alpha} = \frac{1}{n+1} + \frac{\xi\eta v^\alpha}{\alpha},$$

the condition $\xi\eta v^{\alpha-1} < 1$ is equivalent to $v < \frac{\alpha}{(\alpha-1)(n+1)}$, and we have proved Lemma 7 when $k \geq 2$. \square

REMARK 7. The proof of Lemma 7 shows that if $k \geq 2$ and $((v)_k, (u)_{n-k}) \in \mathcal{S}$ then $\xi(uv)^{\alpha-1} < 1$. This observation will be useful when proving Theorem 6 below.

LEMMA 8. Assume that $((v)_k, (u)_{n-k}) \in \mathcal{E}$ with $v > u$ and ξ and η are defined as in Lemma 7. Then the condition $\xi(uv)^{\alpha-1} < 1$ is equivalent to $\partial v/\partial \alpha > 0$.

PROOF. We use the same notation as in the proof of Lemma 6, that is, $t = \ln(v/u)$. Taking the derivative $\partial/\partial \alpha$ of both sides of equation (51), we obtain, with $t' = \frac{dt}{d\alpha}$,

$$\frac{e^t t'}{(n+k(e^t-1))^2} = \frac{1}{n+1} \cdot \frac{e^{\alpha t}(t+\alpha t')}{(n-k+ke^{\alpha t})^2}.$$

Rewriting this equation in terms of u and v (recall that $u = 1/(n+k(e^t-1))$ and $e^t = v/u$) we obtain

$$uv t' = \frac{t+\alpha t'}{n+1} \cdot \frac{(uv)^\alpha}{(kv^\alpha + (n-k)u^\alpha)^2},$$

which is equivalent to

$$(57) \quad t' = \frac{t}{\alpha} \left(\frac{1}{\xi(uv)^{\alpha-1}} - 1 \right)^{-1}.$$

Since $t > 0$, we see that $\xi(uv)^{\alpha-1} < 1$ if and only if $t' > 0$, the latter statement being equivalent to $\partial v/\partial \alpha > 0$ (since $t = \log(v(n-k)/(1-kv))$ is an increasing function of v). \square

PROOF OF THEOREM 6. By Lemma 6, the set of equilibria is as claimed in Theorem 6(i)–(iv), and it remains to verify the claims about stability.

The fact that $(1/n)_n \in \mathcal{S}$ if and only if $\alpha < n+1$ follows from part (iii) of Proposition 2 (as does the statement about criticality). By Lemma 5, all other equilibria are of the form $((v)_k, (u)_{n-k})$ for some $v > u$, $1 \leq k \leq n-1$ (up to permutations).

If $n = 2$, then $k = 1 \geq n/2$, and as in Lemma 6 (or just claim (ii) of the theorem) there exists a unique equilibrium of the form (v, u) with $v > u$ if and only if $\alpha > n+1$, and moreover $\alpha \mapsto v(\alpha)$ is increasing to $2/3$ as $\alpha \rightarrow +\infty$. This result combined with Lemmas 8 and 7 proves Theorem 6 when $n = 2$.

For $n > 2$, if $k = 1$ and $\alpha \in (\tilde{\alpha}(1, n), n + 1)$, then as in Lemma 6 there exist two equilibria of the form $(v, (u)_{n-1})$, one of which has $\partial v/\partial \alpha < 0$ and the other $\partial v/\partial \alpha > 0$. Lemmas 7 and 8 tell us that linear stability is equivalent to $\partial v/\partial \alpha > 0$, so this shows that only the latter equilibria is linearly stable. When $\alpha \geq n + 1$ we have a unique equilibrium of the form $(v, (u)_{n-1})$ with $v > u$, and since $\partial v/\partial \alpha > 0$ it is linearly stable.

For $n > 2$ and $k > 1$, if $k \geq n/2$ or $k < n/2$ and $\alpha > n + 1$ then we have only one equilibrium of the form $((v)_k, (u)_{n-k})$ with $v > u$, which exists for $\alpha > n + 1$. However, if $\alpha > n + 1$ then $\alpha/((\alpha - 1)(n + 1)) < 1/n$, and Lemma 7 tells us that such an equilibrium cannot be linearly stable (since $v > u$ implies $v > 1/n$). If instead $1 < k < n/2$ and $\alpha \in (\tilde{\alpha}(k, n), n + 1)$ then we have two equilibria of the form $((v)_k, (u)_{n-k})$ with $v > u$, corresponding to the two solutions of the equation $\alpha t = f_{k,n}(t)$ in the proof of Lemma 6 [see also Figures 1(b) and 2(a)]. Let us denote these equilibria

$$\vec{v}^{(1)} = \vec{v}^{(1)}(\alpha) = ((v^{(1)})_k, (u^{(1)})_{n-k})$$

and similarly for $\vec{v}^{(2)} = \vec{v}^{(2)}(\alpha)$. We assume that $v^{(1)} < v^{(2)}$. From the proof of Lemma 6, we know that $v^{(1)}$ is a decreasing function of α while $v^{(2)}$ is an increasing function of α .

From Remark 7 and Lemma 8, $\vec{v}^{(1)}$ cannot be linearly stable since $v^{(1)}(\alpha)$ is decreasing in α .

Let us consider the equilibrium $\vec{v}^{(2)}$. If this equilibrium is stable, then from Lemma 7, we find that $v^{(2)} < \alpha/((\alpha - 1)(n + 1))$. Since $v^{(1)} < v^{(2)}$, we see that $v^{(1)}$ also satisfies the condition $v^{(1)} < \alpha/((\alpha - 1)(n + 1))$, therefore, $\vec{v}^{(1)}$ must be a stable equilibrium due to Lemma 7. Thus, we have arrived at a contradiction (since we know that $\vec{v}^{(1)}$ cannot be linearly stable), and we conclude that $\vec{v}^{(2)}$ is not linearly stable.

Finally, when $n > 2$, $1 \leq k \leq n/2$ and $\alpha = \tilde{\alpha}(k, n)$, the two equilibria $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ coincide [see Figure 1(b)] and they are critical. To see this, recall that $t = \ln(v/u)$ and check that $dt/d\alpha \rightarrow \infty$ as $\alpha \rightarrow \tilde{\alpha}^+$ [see Figures 1(b) and 2(a)]. Formula (57) then implies that $\xi(uv)^{\alpha-1} \rightarrow 1$ as $\alpha \rightarrow \tilde{\alpha}^+$, therefore, one of the eigenvalues is zero and the equilibrium is either critical (when $k = 1$) or linearly unstable (when $k > 1$). \square

3.2. *Triangle graph WARM.* In this section, we consider a G -WARM, where G is the triangle graph. Again we will consider the equilibria up to permutations and will list (v_1, v_2, v_3) in the decreasing order $v_1 \geq v_2 \geq v_3$. The following theorem is our main result in this section.

THEOREM 7. *The equilibria for the triangle-graph WARM are:*

- (i) $(1/3, 1/3, 1/3)$, for all $\alpha > 1$;
- (ii) $(1/2, 1/2, 0)$, for all $\alpha > 1$;

(iii) $(v, u, 0)$ for $\alpha > 3$, where $v > u$ and $v(\alpha)$ increases from $v(3+) = 1/2$ to $v(+\infty) = 2/3$;

(iv) (v, v, u) , for $\alpha \in (1, 4/3)$, where $v > u$ and $v(\alpha)$ decreases from $v(1+) = 1/2$ to $v(4/3-) = 1/3$;

(v) (v, u, u) , for $\alpha > 4/3$, where $v > u$ and $v(\alpha)$ increases from $v(4/3+) = 1/3$ to $v(+\infty) = 2/3$.

Their stability properties are as follows:

Equilibrium (i) is linearly stable if and only if $\alpha < 4/3$.

Equilibrium (ii) is linearly stable if and only if $\alpha < 3$.

Equilibrium (iii) is linearly stable for all $\alpha > 3$.

The equilibria (i), (ii), (iii) are critical if and only if equality holds above.

Equilibria (iv) and (v) are linearly unstable.

The proof of Theorem 7 will be completed via a sequence of lemmas.

LEMMA 9. *Equilibria as described in items (iv) and (v) in Theorem 7 exist, and are the unique equilibria of the form (v, v, u) and (v, u, u) with $v > u$.*

PROOF. In the case of a triangle graph-WARM, equations (3) give us the following system:

$$\begin{aligned}
 (58) \quad v_1 &= \frac{1}{3} \frac{v_1^\alpha}{v_1^\alpha + v_2^\alpha} + \frac{1}{3} \frac{v_1^\alpha}{v_1^\alpha + v_3^\alpha}, \\
 v_2 &= \frac{1}{3} \frac{v_2^\alpha}{v_2^\alpha + v_3^\alpha} + \frac{1}{3} \frac{v_2^\alpha}{v_1^\alpha + v_2^\alpha}, \\
 v_3 &= \frac{1}{3} \frac{v_3^\alpha}{v_1^\alpha + v_3^\alpha} + \frac{1}{3} \frac{v_3^\alpha}{v_2^\alpha + v_3^\alpha}.
 \end{aligned}$$

Suppose that (v, u, u) is an equilibrium with $v > u$. Write $e^t = v/u$, and note that $t > 0$. From the condition $v + 2u = 1$, we find that $u = (2 + e^t)^{-1}$ and $v = (1 + 2e^{-t})^{-1}$. Then the second equation in (58) gives us

$$(59) \quad \frac{1}{2 + e^t} = \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{1 + e^{\alpha t}},$$

which can be rewritten in the form

$$e^{\alpha t} = \frac{3e^t}{4 - e^t},$$

which is equivalent to

$$(60) \quad h(t) := \ln\left(\frac{3}{4 - e^t}\right) = (\alpha - 1)t.$$

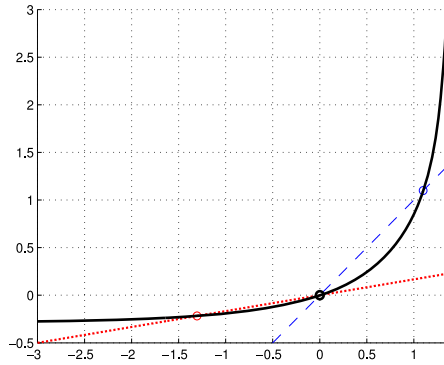


FIG. 3. Finding equilibria of the form (v, u, u) and (v, v, u) . The black solid curve is the graph of the function $y = h(t) = \ln(3/(4 - \exp(t)))$, the straight lines correspond to graphs of the functions $y = (\alpha - 1)t$ for $\alpha = 2$ (blue dashed) and $\alpha = 7/6$ (red dotted).

The graph of the function $t \mapsto h(t)$ is given in Figure 3. One can check that the function $h(t)$ is (strictly) convex on $t \in (0, \log(4))$ and satisfies $h(0) = 0$ and $h'(0) = 1/3$, therefore, (60) has a positive solution $t = t(\alpha)$ if and only if $\alpha > 4/3$ (and this solution is necessarily unique). Moreover, $t(\alpha)$ is increasing which implies that $v(\alpha) = 1/(2 + e^{-t(\alpha)})$ is an increasing function. Finally, $t(4/3) = 0$ and $t(+\infty) = \ln(4)$, which gives us $v(4/3+) = 1/3$ and $v(+\infty) = 2/3$. This verifies part (v) in Theorem 7.

Let us now consider an equilibrium of the form (v, v, u) with $v > u$. This case is equivalent to the previous one, except that now we have $u/v = e^t$ and $t < 0$. One can check that t also must satisfy (60), and that (60) has a negative solution if and only if $\alpha \in (1, 4/3)$. This solution $t = t(\alpha)$ is unique, and is increasing in α , which translates into the property that $v(\alpha) = 1/(2 + e^{t(\alpha)})$ is a decreasing function. Since $t(4/3) = 0$ and $t(1) = -\infty$, we see that $v(1+) = 1/2$ and $v(4/3-) = 1/3$. \square

LEMMA 10. For $\alpha \in (1, 4/3)$, there are no equilibria other than the ones described in items (i), (ii), (iv) of Theorem 7.

PROOF. First, assume that $(v, u, 0)$ is an equilibrium. Then (v, u) is an equilibrium for the line graph-WARM with two edges, and Corollary 1 shows that for $\alpha \in (1, 4/3)$ the only such equilibrium is $(1/2, 1/2)$. This shows that there do not exist any other equilibria of the form $(v, 1 - v, 0)$. Let us consider (v_1, v_2, v_3) , where $v_1 \geq v_2 \geq v_3 > 0$. We will show that if $\alpha \in (1, 4/3)$ and (v_1, v_2, v_3) is an equilibrium, then necessarily $v_1 = v_2$. Assume $v_1 > v_2$. We introduce the new variables $s \geq 0$ and $a \geq 1$

$$(v_2/v_1)^\alpha = e^{-s}, \quad (v_2/v_3)^\alpha = a.$$

Dividing the second equation in (58) by the first one, we get

$$\frac{v_2}{v_1} = \frac{(1 + (v_3/v_2)^\alpha)^{-1} + (1 + (v_1/v_2)^\alpha)^{-1}}{(1 + (v_2/v_1)^\alpha)^{-1} + (1 + (v_3/v_1)^\alpha)^{-1}}.$$

In our new notation, this is equivalent to

$$e^{-s/\alpha} = \frac{(1 + a^{-1})^{-1} + (1 + e^s)^{-1}}{(1 + e^{-s})^{-1} + (1 + a^{-1}e^{-s})^{-1}}.$$

After some simple algebra, we rewrite the above equation in the form

$$(61) \quad e^{(1-1/\alpha)s} = \frac{1 + 2a + ae^s}{1 + a + 2ae^s} \times \frac{1 + ae^s}{1 + a}.$$

Finally, let us define the function

$$(62) \quad f_a(s) := \ln(1 + 2a + ae^s) + \ln(1 + ae^s) - \ln(1 + a + 2ae^s) - \ln(1 + a),$$

and from equation (61) we obtain

$$(63) \quad (1 - 1/\alpha)s = f_a(s).$$

We will show that for all $a \geq 1$ and for all $\beta := (1 - 1/\alpha) \in (0, 1/4)$, the equation $f_a(s) = \beta s$, $s \geq 0$ has a unique solution $s = 0$, which implies that $v_1 = v_2$. From (62), we see that

$$f'_a(s) = 1 - \frac{1 + 2a}{1 + 2a + ae^s} - \frac{1}{1 + ae^s} + \frac{1 + a}{1 + a + 2ae^s}$$

and after some tedious calculations we obtain

$$4f'_a(s) - 1 = \frac{6a^3e^{3s} + 3a^2(a + 1)e^{2s} + (6a^3 - 8a^2 - 4a)e^s - 2a^2 - 3a - 1}{(1 + 2a + ae^s)(1 + ae^s)(1 + a + 2ae^s)}.$$

Note that, for all $s > 0$, and all $a \geq 1$

$$\begin{aligned} 6a^3e^{3s} + (6a^3 - 8a^2 - 4a)e^s &> 6a^3e^s + (6a^3 - 8a^2 - 4a)e^s \\ &= 4ae^s(3a^2 - 2a - 1) \geq 0, \end{aligned}$$

and

$$\begin{aligned} 3a^2(a + 1)e^{2s} - 2a^2 - 3a - 1 &> 3a^3 + a^2 - 3a - 1 \\ &= (3a + 1)(a^2 - 1) \geq 0. \end{aligned}$$

Therefore, we have proved that $f'_a(s) > 1/4$ for all $a \geq 1$ and all $s > 0$. As a result, for all $\beta \in (0, 1/4)$ it is true that the function $s \mapsto f_a(s) - \beta s$ is strictly increasing, and since $f_a(0) = 0$ it shows that the only nonnegative solution to $f_a(s) = \beta s$ is $s = 0$. \square

LEMMA 11. *For $\alpha \geq 4/3$, there are no equilibria other than the ones described in items (i), (ii), (iii), (v) of Theorem 7.*

PROOF. We assume that $\alpha \geq 4/3$ and $v_2 > v_3 > 0$, with the aim of obtaining a contradiction. We start by rewriting the second and the third equations in (58) as follows:

$$\frac{3}{v_2^{\alpha-1}} = \frac{a + 2b + c}{(a + b)(b + c)},$$

$$\frac{3}{v_3^{\alpha-1}} = \frac{a + b + 2c}{(a + c)(b + c)},$$

where we have denoted $a = v_1^\alpha$, $b = v_2^\alpha$ and $c = v_3^\alpha$. Dividing the second equation by the first one, we obtain

$$\left(\frac{v_2}{v_3}\right)^{\alpha-1} = \frac{(a + b + 2c)(a + b)}{(a + 2b + c)(a + c)}.$$

Some simple algebra shows that the above equation is equivalent to

$$\left(\frac{v_2}{v_3}\right)^{\alpha-1} - 1 = \frac{b^2 - c^2}{(a + 2b + c)(a + c)}.$$

Since $b^2 - c^2 = (b - c)(b + c) = (b/c - 1)(b + c)c$, the previous equation can be rewritten as

$$(64) \quad \frac{v_2}{v_3} \times \frac{(v_2/v_3)^{\alpha-1} - 1}{(v_2/v_3)^\alpha - 1} = \frac{v_2}{v_3} \times \frac{(b + c)c}{(a + 2b + c)(a + c)}.$$

Let us denote the expression in the left-hand side (resp., in the right-hand side) as L (resp., R). Our first goal is to prove that $L > 1/4$. Let us denote $w = v_2/v_3 > 1$. Then

$$(65) \quad L := w \frac{w^{\alpha-1} - 1}{w^\alpha - 1} = 1 - \frac{w - 1}{w^\alpha - 1}.$$

It is easy to check that for all $\alpha > 1$ the function $z \mapsto (z^\alpha - 1)/(z - 1)$ is strictly increasing for $z \in (1, \infty)$, therefore, we have

$$\frac{w^\alpha - 1}{w - 1} > \lim_{z \rightarrow 1^+} \frac{z^\alpha - 1}{z - 1} = \alpha.$$

This implies $(w - 1)/(w^\alpha - 1) < 1/\alpha$ and

$$(66) \quad L = 1 - \frac{w - 1}{w^\alpha - 1} > 1 - \frac{1}{\alpha} \geq 1/4.$$

Our second goal is to prove that $R \leq 1/4$. Let us denote $x = v_2/v_1$ and $y = v_3/v_2$, so that $v_2 = xv_1$ and $v_3 = xyv_1$. Note that the inequality $v_1 \geq v_2 > v_3 > 0$ implies $0 < x \leq 1$ and $0 < y < 1$. We rewrite the right-hand side in (64) as

$$(67) \quad R := \frac{v_2}{v_3} \times \frac{(b + c)c}{(a + 2b + c)(a + c)} = \frac{v_2(v_2^\alpha + v_3^\alpha)v_3^{\alpha-1}}{(v_1^\alpha + 2v_2^\alpha + v_3^\alpha)(v_1^\alpha + v_3^\alpha)}$$

$$= \frac{x^{2\alpha}y^{\alpha-1}(1 + y^\alpha)}{(1 + x^\alpha(2 + y^\alpha))(1 + x^\alpha y^\alpha)} =: f(x, y).$$

First, we check that for all $q > 0$ the function $z \mapsto z^2/((1+z(2+q))(1+zq))$ is increasing for $z > 0$, thus

$$\sup_{0 < z \leq 1} \frac{z^2}{(1+z(2+q))(1+zq)} = \frac{z^2}{(1+z(2+q))(1+zq)} \Big|_{z=1} = \frac{1}{(3+q)(1+q)}.$$

Therefore, from the above identity and (67) we obtain

$$\begin{aligned} R &\leq \sup_{0 < t < 1} \left[\sup_{0 < s \leq 1} f(s, t) \right] \\ (68) \quad &= \sup_{0 < t < 1} t^{\alpha-1} (1+t^\alpha) \left[\sup_{0 < s \leq 1} \frac{s^{2\alpha}}{(1+s^\alpha(2+t^\alpha))(1+s^\alpha t^\alpha)} \right] \\ &= \sup_{0 < t < 1} \frac{t^{\alpha-1}}{3+t^\alpha}. \end{aligned}$$

Consider the function $g(t) := t^{\alpha-1}/(3+t^\alpha)$. We compute

$$\frac{dg(t)}{dt} = \frac{t^{\alpha-2}(3(\alpha-1)-t^\alpha)}{(3+t^\alpha)^2}.$$

Since $3(\alpha-1) \geq 1$ for $\alpha \geq 4/3$, we see that $dg(t)/dt > 0$ for $0 < t < 1$, thus $g(t)$ is increasing for $t \in (0, 1)$ and

$$\sup_{\substack{0 < s \leq 1 \\ 0 < t < 1}} f(s, t) = \sup_{0 < t < 1} \frac{t^{\alpha-1}}{3+t^\alpha} = \frac{t^{\alpha-1}}{3+t^\alpha} \Big|_{t=1} = \frac{1}{4}.$$

The above equation combined with (64), (66) and (68) imply $1/4 < L = R \leq 1/4$. This shows that our initial assumption $v_2 > v_3 > 0$ cannot be true, therefore, $v_3 = 0$ or $v_2 = v_3$. \square

LEMMA 12. *Let*

$$\eta := \frac{\alpha(uv)^\alpha}{3(u^\alpha + v^\alpha)^2}.$$

An equilibrium of the form (v, u, u) or (v, v, u) for $v > u$ is linearly stable if and only if both $\eta < uv$ and $\eta < u - \frac{\alpha}{6}$.

PROOF. Assume that $(v_1, v_2, v_3) = (v, u, u)$ and $v \neq u$. The Jacobian matrix is of the form

$$(69) \quad \mathbf{D} = \begin{pmatrix} -1 + \frac{2\eta}{v} & -\frac{\eta}{u} & -\frac{\eta}{u} \\ -\frac{\eta}{v} & -1 + \frac{\alpha}{12u} + \frac{\eta}{u} & -\frac{\alpha}{12u} \\ -\frac{\eta}{v} & -\frac{\alpha}{12u} & -1 + \frac{\alpha}{12u} + \frac{\eta}{u} \end{pmatrix}.$$

One can check that

$$\det(\mathbf{D} - \lambda \mathbf{I}) = -(\lambda + 1) \left(\lambda + 1 - \frac{\eta}{uv}(v + 2u) \right) \left(\lambda + 1 - \frac{\alpha + 6\eta}{6u} \right).$$

Since $v + 2u = 1$, we see that the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = -1 + \frac{\eta}{uv}, \quad \lambda_3 = -1 + \frac{\alpha + 6\eta}{6u}. \quad \square$$

LEMMA 13. *The equilibrium of Theorem 7(v) is not linearly stable.*

PROOF. Assume that (v, u, u) is an equilibrium, such that $v > u$ and $\alpha > 4/3$. In order to show that (v, u, u) is not a linearly stable equilibrium it is enough to prove that $\eta > u - \alpha/6$ (see Lemma 12). Define $r = v/u$. The condition $\eta > u - \alpha/6$ is equivalent to

$$\frac{1}{2} + \frac{r^\alpha}{(1 + r^\alpha)^2} > \frac{3}{\alpha(2 + r)}.$$

This inequality is obvious if $\alpha > 2$, so we only need to consider $\alpha \in (4/3, 2]$. Let us introduce the new variable $z = r^{\alpha/2} - 1$, so that $r = (1 + z)^{2/\alpha}$. With this notation, we need to prove that for all $\alpha \in (4/3, 2]$ and all $z > 0$

$$\frac{1}{2} + \frac{(1 + z)^2}{(1 + (1 + z)^2)^2} > \frac{3}{\alpha(2 + (1 + z)^{2/\alpha})}.$$

For all $\alpha \in (4/3, 2]$ and all $z > 0$, we have $(1 + z)^{2/\alpha} \geq 1 + z$, therefore,

$$\frac{3}{\alpha(2 + (1 + z)^{2/\alpha})} \leq \frac{3}{\alpha(3 + z)} < \frac{9}{4(3 + z)}.$$

So it is enough to show that for all $z > 0$

$$\frac{1}{2} + \frac{(1 + z)^2}{(1 + (1 + z)^2)^2} > \frac{9}{4(3 + z)}.$$

Multiplying both sides by $(1 + (1 + z)^2)^2(3 + z)$ and simplifying the resulting expressions, we obtain that the above inequality is equivalent to

$$2z^5 + 5z^4 + 8z^3 + 12z^2 + 12z > 0 \quad \text{for all } z > 0,$$

which is obviously true. \square

LEMMA 14. *The equilibrium of Theorem 7(iv) is not linearly stable.*

PROOF. We will show that the first condition of Lemma 12 is not satisfied, that is $\eta > uv$ for all $\alpha \in (1, 4/3)$.

Assume that (v, v, u) is an equilibrium. We use a similar parameterization as in the proof of Lemma 9 (though with reversed roles for u and v): $u/v = e^t$, $v =$

$(2 + e^t)^{-1}$. Since $v > u$, we have $t < 0$. From the proof of Lemma 9, we know that $\frac{dt}{d\alpha} > 0$. We consider t as a function of α . Equation (59) gives us

$$\frac{d}{d\alpha} \left[\frac{1}{2 + e^t} \right] = \frac{d}{d\alpha} \left[\frac{1}{6} + \frac{1}{3} \cdot \frac{1}{1 + e^{\alpha t}} \right],$$

which is equivalent to

$$\frac{e^t t'}{(2 + e^t)^2} = \frac{1}{3} \cdot \frac{e^{\alpha t} (t + \alpha t')}{(1 + e^{\alpha t})^2},$$

where $t' := \frac{dt}{d\alpha}$. Since $t < 0$ and $t' > 0$,

$$\frac{e^t}{(2 + e^t)^2} < \frac{1}{3} \cdot \frac{e^{\alpha t} \alpha}{(1 + e^{\alpha t})^2}.$$

Since $e^t = u/v$ and $(2 + e^t)^{-1} = v$, the above inequality gives us

$$uv < \frac{\alpha (uv)^\alpha}{3(u^\alpha + v^\alpha)^2}.$$

Applying Lemma 12, we conclude that (v, v, u) is not a linearly stable equilibrium. □

PROOF OF THEOREM 7. The equilibrium (i) and its stability properties are given by Propositions 1 and 2. The equilibria (ii) and (iii) and their stability properties are immediate from Theorem 3 and Corollary 1.

By Lemma 9, the equilibria in (iv) and (v) exist, and by Lemmas 9–11, there are no equilibria other than (i)–(v) as claimed in the theorem. The claimed stability properties of the equilibria in (iv) and (v) are given by Lemmas 12–13. □

3.3. Whisker graph WARM. In this section, we consider G -WARMs where G is a whisker graph. Since we already understand the star-graph setting, let us in this section restrict our attention to whisker graphs that are not star graphs. The following theorem is the main result of this section.

THEOREM 8. *On the symmetric whisker graph, with $r \geq 1$ there exists $\alpha(r) > 1$ such that for any $\alpha > \alpha(r)$ there exist two equilibria of the form $((v)_r, u, (v)_r)$, both with $v < u$, exactly one of which is linearly stable. For the linearly stable equilibrium, the function $u(\alpha)$ increases to $u(+\infty) = (r + 1)^{-1}$. When $\alpha = \alpha(r)$, there exists a unique critical equilibrium of the form $((v)_r, u, (v)_r)$ with $u > 0$, and there do not exist any equilibria of the form $((v)_r, u, (v)_r)$ for $\alpha < \alpha(r)$.*

The proof of Theorem 8 is similar to the proof of our results for the star-graph WARMs given in Section 3.1 (though the matrix computations are more complicated). First, we need to establish several auxiliary results.

LEMMA 15. For all $\alpha > 1$, all equilibria for a whisker graph are of the form (70) $((v)_{k_r}, (u)_{r-k_r}, v_{r+1}, (v')_{k_s}, (u')_{s-k_s})$.

PROOF. For the (r, s) -whisker graph (with $r + 1 + s = n$), $\vec{v} \in \mathcal{E}$ if and only if \vec{v} satisfies (for all $i = 1, \dots, n$)

$$(71) \quad 0 = F(\vec{v})_i = -v_i + \frac{1}{n+1} \begin{cases} 1 + \frac{v_i^\alpha}{\delta_r}, & i \leq r, \\ v_{r+1}^\alpha \left[\frac{1}{\delta_r} + \frac{1}{\delta_s} \right], & i = r + 1, \\ 1 + \frac{v_i^\alpha}{\delta_s}, & r + 2 \leq i \leq n, \end{cases}$$

where $\delta_r = \sum_{i=1}^{r+1} v_i^\alpha$ and $\delta_s = \sum_{i=r+1}^n v_i^\alpha$. Fixing δ_r and repeating the proof of Lemma 5 with f given by (49) (with δ_r instead of δ), we have that for any equilibrium \vec{v} on a whisker graph, $\{v_1, \dots, v_r\}$ has at most 2 distinct elements [only one element when $\delta_r \notin \frac{(\alpha-1)}{(n+1)^\alpha}(1, 2^\alpha)$]. Similarly, $\{v_{r+2}, \dots, v_n\}$ has at most 2 distinct elements [only one element when $\delta_s \notin \frac{(\alpha-1)}{(n+1)^\alpha}(1, 2^\alpha)$]. \square

Note that $v_{r+1} \geq 0$ and all other entries are bounded above and below by $2/(n+1)$ and $1/(n+1)$, respectively. For such \vec{v} , we have that $\delta_r = k_r v^\alpha + (r - k_r) u^\alpha + v_{r+1}^\alpha$, and similarly $\delta_s = k_s (v')^\alpha + (s - k_s) (u')^\alpha + v_{r+1}^\alpha$.

Next, we investigate the eigenvalues of the Jacobian \mathbf{D} via the determinant of $\mathbf{M} \equiv \mathbf{D} - \lambda \mathbf{I}$. The matrix \mathbf{M} is complicated, but admits a certain block structure that can be exploited to give an expression for its determinant (see Lemmas 16 and 17 below).

Letting $\xi_r = \frac{\alpha}{(n+1)\delta_r^2}$ and $\xi_s = \frac{\alpha}{(n+1)\delta_s^2}$ and recalling (12) and (13), we have that

$$(72) \quad D_{i,i} = -1 + \frac{\alpha}{n+1} \begin{cases} v_i^{\alpha-1} \left[\frac{\delta_r - v_i^\alpha}{\delta_r^2} \right], & i \leq r, \\ v_{r+1}^{\alpha-1} \left[\frac{(\delta_r - v_{r+1}^\alpha)}{\delta_r^2} + \frac{(\delta_s - v_{r+1}^\alpha)}{\delta_s^2} \right], & i = r + 1, \\ v_i^{\alpha-1} \left[\frac{\delta_s - v_i^\alpha}{\delta_s^2} \right], & r + 2 \leq i \leq n \end{cases}$$

$$= -1 + \begin{cases} \xi_r v^{\alpha-1} (\delta_r - v^\alpha), & i \leq k_r, \\ \xi_r u^{\alpha-1} (\delta_r - u^\alpha), & k_r + 1 \leq i \leq r, \\ \xi_r v_{r+1}^{\alpha-1} (\delta_r - v_{r+1}^\alpha) \\ \quad + \xi_s v_{r+1}^{\alpha-1} (\delta_s - v_{r+1}^\alpha), & i = r + 1, \\ \xi_s (v')^{\alpha-1} (\delta_s - (v')^\alpha), & r + 2 \leq i \leq r + 2 + k_s, \\ \xi_s (u')^{\alpha-1} (\delta_s - (u')^\alpha), & r + 2 + k_s \leq i \leq n. \end{cases}$$

Moreover, $D_{i,\ell} = 0$ if $i \leq r$ and $\ell \geq r + 2$ (or vice versa) and otherwise

$$D_{i,\ell} = -v_i^\alpha v_\ell^{\alpha-1} \begin{cases} \xi_r, & i, \ell \leq r + 1, i \neq \ell, \\ \xi_s, & i, \ell \geq r + 1, i \neq \ell. \end{cases}$$

Now $\mathbf{M} \equiv \mathbf{D} - \lambda \mathbf{I}$ is of the form

$$\mathbf{M} = \left(\begin{array}{c|c|c} \mathbf{A} & \vec{g} & 0 \\ \hline \vec{h}^T & a & \vec{t}^T \\ \hline 0 & \vec{z} & \mathbf{B} \end{array} \right),$$

where $\mathbf{A} \in \mathbb{R}^{r \times r}$ has the same form as the matrix $\mathbf{D} - \lambda \mathbf{I}$ in the case of the star-graph on r edges,

$$\vec{g}^T = -\frac{\alpha v_{r+1}^{\alpha-1}}{(n+1)\delta_r^2} ((v^\alpha)_{k_r}, (u^\alpha)_{r-k_r}) \in \mathbb{R}^r,$$

$$\vec{h}^T = -\frac{\alpha v_{r+1}^\alpha}{(n+1)\delta_r^2} ((v^{\alpha-1})_{k_r}, (u^{\alpha-1})_{r-k_r}) \in \mathbb{R}^r$$

and $a = D_{r+1,r+1} - \lambda$ etc. We have that

$$\vec{g} = -\xi_r v_{r+1}^{\alpha-1} \vec{x}_r, \quad \vec{h} = v_{r+1}^\alpha \vec{w}_r,$$

where \vec{x}_r and \vec{w}_r are defined as in (55) (but with ξ_r instead of ξ), that is,

$$\vec{x}_r^T = ((v^\alpha)_{k_r}, (u^\alpha)_{r-k_r}), \quad \text{and}$$

$$\vec{w}_r^T = -\xi_r ((v^{\alpha-1})_{k_r}, (u^{\alpha-1})_{r-k_r}).$$

Similarly, $\mathbf{B} \in \mathbb{R}^{s \times s}$ has the same form as the matrix $\mathbf{D} - \lambda \mathbf{I}$ in the case of the star-graph on s edges and

$$\vec{z} = -\xi_s v_{r+1}^{\alpha-1} \vec{x}'_s, \quad \vec{t} = v_{r+1}^\alpha \vec{w}'_s.$$

The following lemma expresses the determinant of \mathbf{M} in terms of its block components \mathbf{A} and \mathbf{B} .

LEMMA 16. *The determinant of \mathbf{M} is given by*

$$(73) \quad \det(\mathbf{M}) = a \det(\mathbf{A}) \det(\mathbf{B}) - (\det(\mathbf{B}) \vec{h}^T \text{adj}(\mathbf{A}) \vec{g} + \det(\mathbf{A}) \vec{t}^T \text{adj}(\mathbf{B}) \vec{z}).$$

PROOF. First, note that $\det(\mathbf{M}) = \det(\mathbf{H})$, where

$$\mathbf{H} = \left(\begin{array}{cc|c} \mathbf{A} & \mathbf{0} & \vec{g} \\ \mathbf{0} & \mathbf{B} & \vec{z} \\ \hline \vec{h}^T & \vec{t}^T & a \end{array} \right).$$

Let $\mathbf{R} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$. Then using the block matrix form of \mathbf{H} ,

$$\det(\mathbf{H}) = (a + 1) \det(\mathbf{R}) - \det\left(\mathbf{R} + \begin{pmatrix} \vec{g} \\ \vec{z} \end{pmatrix} (\vec{h}^T, \vec{t}^T)\right).$$

Now by definition of adj, we have that $\mathbf{R} \operatorname{adj}(\mathbf{R}) = \det(\mathbf{R})\mathbf{I}$, from which it follows easily that for \mathbf{R} of the form $\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$

$$\operatorname{adj}(\mathbf{R}) = \begin{pmatrix} \det(\mathbf{B}) \operatorname{adj}(\mathbf{A}) & \mathbf{0} \\ \mathbf{0} & \det(\mathbf{A}) \operatorname{adj}(\mathbf{B}) \end{pmatrix}.$$

Combining this with Lemma 4, we arrive at

$$\begin{aligned} \det(\mathbf{H}) &= (a + 1) \det(\mathbf{R}) - \left(\det(\mathbf{R}) + (\vec{h}^T, \vec{t}^T) \operatorname{adj}(\mathbf{R}) \begin{pmatrix} \vec{g} \\ \vec{z} \end{pmatrix} \right) \\ &= a \det(\mathbf{R}) - (\det(\mathbf{B}) \vec{h}^T \operatorname{adj}(\mathbf{A}) \vec{g} + \det(\mathbf{A}) \vec{t}^T \operatorname{adj}(\mathbf{B}) \vec{z}). \end{aligned}$$

But $\det(\mathbf{R}) = \det(\mathbf{A}) \det(\mathbf{B})$, yielding (73). \square

Now from the proof of Lemma 7 we know that \mathbf{A} and \mathbf{B} can be written in the form $\mathbf{A} = \mathbf{Z} + \vec{x}_r \vec{w}_r^T$ and $\mathbf{B} = \mathbf{Z}' + \vec{x}'_s (\vec{w}'_s)^T$ and where \mathbf{Z} and \mathbf{Z}' are diagonal matrices with

$$\begin{aligned} Z_{ii} &= -(1 + \lambda) + \delta_r \xi_r \begin{cases} v^{\alpha-1}, & i \leq k_r, \\ u^{\alpha-1}, & k_r < i \leq r, \end{cases} \\ Z'_{ii} &= -(1 + \lambda) + \delta_s \xi_s \begin{cases} (v')^{\alpha-1}, & i \leq k_s, \\ (u')^{\alpha-1}, & k_s < i \leq n - r - 1. \end{cases} \end{aligned}$$

Using Lemma 4, the various components in (73) can be expressed in terms of \mathbf{Z} and \mathbf{Z}' , as shown in the following result.

LEMMA 17. *The determinant of \mathbf{M} satisfies*

$$(74) \quad \det(\mathbf{M}) = a[\det(\mathbf{Z}) + \vec{w}_r^T \operatorname{adj}(\mathbf{Z}) \vec{x}_r][\det(\mathbf{Z}') + (\vec{w}'_s)^T \operatorname{adj}(\mathbf{Z}') \vec{x}'_s]$$

$$(75) \quad + [\det(\mathbf{Z}') + (\vec{w}'_s)^T \operatorname{adj}(\mathbf{Z}') \vec{x}'_s][\xi_r v_{r+1}^{2\alpha-1} \vec{w}_r^T \operatorname{adj}(\mathbf{Z}) \vec{x}_r]$$

$$(76) \quad + [\det(\mathbf{Z}) + \vec{w}_r^T \operatorname{adj}(\mathbf{Z}) \vec{x}_r][\xi_s v_{r+1}^{2\alpha-1} (\vec{w}'_s)^T \operatorname{adj}(\mathbf{Z}') \vec{x}'_s].$$

PROOF. Recall that $\mathbf{A} = \mathbf{Z} + \vec{x}_r \vec{w}_r^T$ and $\mathbf{B} = \mathbf{Z}' + \vec{x}'_s (\vec{w}'_s)^T$. Therefore, by Lemma 4,

$$\det \mathbf{A} = \det(\mathbf{Z}) + \vec{w}_r^T \operatorname{adj}(\mathbf{Z}) \vec{x}_r, \quad \text{and}$$

$$\det \mathbf{B} = \det(\mathbf{Z}') + (\vec{w}'_s)^T \operatorname{adj}(\mathbf{Z}') \vec{x}'_s.$$

Since $\mathbf{A} = \mathbf{Z} + \vec{x}_r \vec{w}_r$ and $\vec{g} = -\xi_r v_{r+1}^{\alpha-1} \vec{x}_r$ and $\vec{h} = v_{r+1}^\alpha \vec{w}_r$,

$$(77) \quad \mathbf{A} + \vec{g} \vec{h}^T = \mathbf{Z} + \vec{x}_r \vec{w}_r^T - \xi_r v_{r+1}^{2\alpha-1} \vec{x}_r \vec{w}_r^T = \mathbf{Z} + (1 - \xi_r v_{r+1}^{2\alpha-1}) \vec{x}_r \vec{w}_r^T.$$

Applying Lemma 4 to the right-hand side of (77) gives

$$\det(\mathbf{A} + \vec{g} \vec{h}^T) = \det(\mathbf{Z}) + (1 - \xi_r v_{r+1}^{2\alpha-1}) \vec{w}_r^T \operatorname{adj}(\mathbf{Z}) \vec{x}_r.$$

On the other hand, applying Lemma 4 to the left-hand side of (77) gives

$$\vec{h}^T \text{adj}(\mathbf{A})\vec{g} = \det(\mathbf{A} + \vec{g}\vec{h}^T) - \det(\mathbf{A}).$$

Thus

$$\begin{aligned} \vec{h}^T \text{adj}(\mathbf{A})\vec{g} &= \det(\mathbf{A} + \vec{g}\vec{h}^T) - \det(\mathbf{A}) \\ &= \det(\mathbf{Z}) + (1 - \xi_r v_{r+1}^{2\alpha-1})\vec{w}_r^T \text{adj}(\mathbf{Z})\vec{x}_r - \det(\mathbf{A}) \\ &= -\xi_r v_{r+1}^{2\alpha-1} \vec{w}_r^T \text{adj}(\mathbf{Z})\vec{x}_r, \end{aligned}$$

since $\det(\mathbf{A}) = \det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z})\vec{x}_r$. Handling the \mathbf{B} terms similarly, we can rewrite (73) in the form of (74)–(76) as claimed. \square

Note that if $v_{r+1} = 0$ then $a = -(1 + \lambda)$, the two terms (75) and (76) vanish and we recover the fact (see Theorem 3) that the case $v_{r+1} = 0$ is linearly stable if and only if each of the remaining star graphs is linearly stable.

Lemma 17 is useful since we can give explicit (albeit complicated) expressions for every term appearing in (74)–(76) as follows. First,

$$\det(\mathbf{Z}) = (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{k_r} (-(1 + \lambda) + \delta_r \xi_r u^{\alpha-1})^{r-k_r},$$

and

$$\begin{aligned} &\vec{w}_r^T \text{adj}(\mathbf{Z})\vec{x}_r \\ &= -\xi_r \left(\sum_{i=1}^{k_r} v^{2\alpha-1} [(-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{k_r-1} (-(1 + \lambda) + \delta_r \xi_r u^{\alpha-1})^{r-k_r}] \right. \\ &\quad + \sum_{i=k_r+1}^r u^{2\alpha-1} [(-(1 + \lambda) + \delta_r \xi_r u^{\alpha-1})^{k_r} \\ &\quad \left. \times (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-k_r-1} \right], \end{aligned}$$

and if both $k_r \geq 1$ and $r - k_r \geq 1$ this becomes

$$\begin{aligned} &\vec{w}_r^T \text{adj}(\mathbf{Z})\vec{x}_r \\ &= -\xi_r (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{k_r-1} (-(1 + \lambda) + \delta_r \xi_r u^{\alpha-1})^{r-k_r-1} \\ &\quad \times (k_r v^{2\alpha-1} (-(1 + \lambda) + \delta_r \xi_r u^{\alpha-1}) \\ &\quad + (r - k_r) u^{2\alpha-1} (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})). \end{aligned}$$

Similarly,

$$\det(\mathbf{Z}') = (-(1 + \lambda) + \delta_s \xi_s v^{\alpha-1})^{k_s} (-(1 + \lambda) + \delta_s \xi_s (u')^{\alpha-1})^{s-k_s},$$

and

$$\begin{aligned}
 & (\vec{w}'_s)^T \text{adj}(\mathbf{Z}') \vec{x}'_s \\
 &= -\xi_s \left(\sum_{i=1}^{k_s} (v')^{2\alpha-1} [-(1+\lambda) + \delta_s \xi_s (v')^{\alpha-1}]^{k_s-1} \right. \\
 &\quad \times [-(1+\lambda) + \delta_s \xi_s (u')^{\alpha-1}]^{s-k_s} \\
 &\quad \left. + \sum_{i=k_s+1}^s (u')^{\alpha-1} [-(1+\lambda) + \delta_s \xi_s (u')^{\alpha-1}]^{k_s} \right. \\
 &\quad \left. \times [-(1+\lambda) + \delta_s \xi_s (v')^{\alpha-1}]^{s-k_s-1} \right).
 \end{aligned}$$

We now apply Lemma 17 to the completely symmetric case $r = s = k_r = k_s$, $v = v'$ to establish the following “whisker” analogue of the star-graph criterion, Lemma 7.

LEMMA 18. *For the symmetric whisker graph with $r = s = k_r = k_s$, $\vec{v} = ((v)_r, v_{r+1}, (v)_r)$ is a linearly stable equilibrium if and only if*

$$\xi_r v_{r+1}^{\alpha-1} v^{\alpha-1} < 1, \quad \text{and in the case } r > 1 \text{ also } \delta_r \xi_r v^{\alpha-1} < 1.$$

PROOF. We have that $\mathbf{Z} = \mathbf{Z}'$, etc. in Lemma 17, and thus

$$\begin{aligned}
 \det(\mathbf{M}) &= a[\det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r]^2 \\
 &\quad + 2[\xi_r v_{r+1}^{2\alpha-1} \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r][\det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r] \\
 &= [\det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r] (a[\det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r] \\
 &\quad + 2[\xi_r v_{r+1}^{2\alpha-1} \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r]).
 \end{aligned}$$

Here $\det(\mathbf{Z}) = -(1 + \lambda) + \delta_r \xi_r v^{\alpha-1}$ and

$$\begin{aligned}
 & \det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r \\
 &= (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^r - r \xi_r v^{2\alpha-1} (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} \\
 &= (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} [-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1} - r \xi_r v^{2\alpha-1}] \\
 &= (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} [-(1 + \lambda) + \xi_r v^{\alpha-1} v_{r+1}^\alpha],
 \end{aligned}$$

so $\lambda = \delta_r \xi_r v^{\alpha-1} - 1$ and $\lambda = \xi_r v^{\alpha-1} v_{r+1}^\alpha - 1$ are eigenvalues, with the first of multiplicity $r - 1$ (vanishing when $r = 1$).

Recall from (72) that

$$\begin{aligned} a &= D_{r+1,r+1} - \lambda = -(1 + \lambda) + \frac{\alpha v_{r+1}^{\alpha-1}}{n + 1} \left[\frac{(\delta_r - v_{r+1}^\alpha)}{\delta_r^2} + \frac{(\delta_s - v_{r+1}^\alpha)}{\delta_s^2} \right] \\ &= -(1 + \lambda) + v_{r+1}^{\alpha-1} (\xi_r (\delta_r - v_{r+1}^\alpha) + \xi_s (\delta_s - v_{r+1}^\alpha)) \\ &= -(1 + \lambda) + 2r v_{r+1}^{\alpha-1} \xi_r v^\alpha, \end{aligned}$$

where $\delta_r - v_{r+1}^\alpha = \sum_{i=1}^r v_i^\alpha$ and $\delta_s - v_{r+1}^\alpha = \sum_{i=r+2}^n v_i^\alpha$. Thus,

$$\begin{aligned} &\frac{\det(\mathbf{M})}{\det(\mathbf{Z}) + \vec{w}_r^T \text{adj}(\mathbf{Z}) \vec{x}_r} \\ &= (-(1 + \lambda) + 2r v_{r+1}^{\alpha-1} \xi_r v^\alpha) (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} \\ &\quad \times [-(1 + \lambda) + \xi_r v^{\alpha-1} v_{r+1}^\alpha] \\ &\quad - 2r \xi_r v^{2\alpha-1} (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} \xi_r v_{r+1}^{2\alpha-1} \\ &= (-(1 + \lambda) + \delta_r \xi_r v^{\alpha-1})^{r-1} (1 + \lambda) [(1 + \lambda) - \xi_r v_{r+1}^{\alpha-1} v^{\alpha-1}], \end{aligned}$$

where we have used $2rv + v_{r+1} = 1$. The corresponding eigenvalues are

$$\lambda = \delta_r \xi_r v^{\alpha-1} - 1, \quad \lambda = -1, \quad \text{and} \quad \lambda = \xi_r v_{r+1}^{\alpha-1} v^{\alpha-1} - 1,$$

with the former not being present when $r = 1$. \square

PROOF OF THEOREM 8. To establish the existence of equilibria of the form $((v)_r, u, (v)_r)$ we need to show that the equation

$$(78) \quad u = \frac{1}{r + 1} \frac{u^\alpha}{u^\alpha + rv^\alpha},$$

has a solution $u > 0, v > 0$, satisfying $u + 2rv = 1$. We define $u/v = e^t$, then $v = 1/(2r + e^t), u = e^t/(2r + e^t)$ and we check that (78) is equivalent

$$(79) \quad e^{\alpha t} = \frac{(r + 1)e^t}{2 - e^t}$$

which we rewrite in the form

$$(80) \quad (\alpha - 1)t = \ln\left(\frac{r + 1}{2 - e^t}\right).$$

The function $h_r(t) := \ln((r + 1)/(2 - e^t))$ is convex, increasing and strictly positive on $t \in (-\infty, \ln(2))$. The graph of this function is shown in Figure 4. Since the function is convex, increasing and $h_r(0) > 0$ it is clear that there exists a unique $t(r) \in \mathbb{R}$ such that the tangent line to the graph of $y = h_r(t)$ at point $t(r)$ passes through the origin. Let us denote the slope of this tangent line by m and

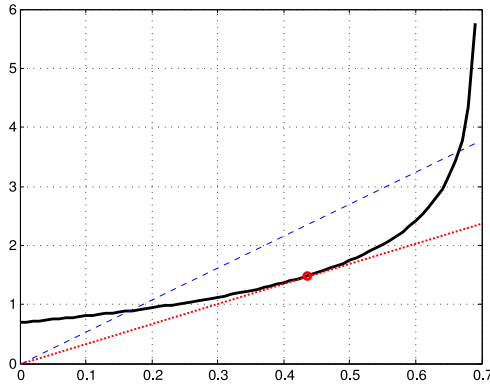


FIG. 4. The graph of functions $y = h_r(t)$ (black solid line), $y = (\alpha(r) - 1)t$ (red dotted line) and $y = (\alpha - 1)t$ for $\alpha > \alpha(r)$ (blue dashed line). The point $(t(r), h_r(t(r)))$ is marked by a red circle.

define $\alpha(r) = m + 1$. An equivalent characterization of $\alpha(r)$ is that the equation $h_r(t) = (\alpha - 1)t$ will have two solutions for $\alpha > \alpha(r)$ and no solutions for $\alpha < \alpha(r)$. See Figure 4.

Thus, we have now proved that: (i) For $\alpha < \alpha(r)$, there do not exist equilibria of the form $((v)_r, u, (v)_r)$; (ii) For all $\alpha > \alpha(r)$, equation (80) has two solutions, $0 < t_1(\alpha) < t_2(\alpha) < \ln(2)$, such that $t_1(\alpha)$ is decreasing in α and $t_2(\alpha)$ is increasing in α . As $\alpha \rightarrow \alpha(r)^+$ we have $t_1(\alpha) \rightarrow t_2(\alpha)$. These two solutions give us two equilibria of the form $((v)_r, u, (v)_r)$ [recall that $v = 1/(2r + e^t)$ and $u = e^t/(2r + e^t)$].

Next, let us investigate stability of these equilibria. According to Lemma 18, the equilibrium is linearly stable if and only if

$$(81) \quad \frac{\alpha(uv)^{\alpha-1}}{2(r+1)(u^\alpha + rv^\alpha)^2} < 1 \iff e^{(\alpha-1)t_2(\alpha)} > \alpha(r+1)/2,$$

and in the case $r > 1$ also

$$(82) \quad \frac{\alpha v^{\alpha-1}}{2(r+1)(u^\alpha + rv^\alpha)} < 1 \iff e^{(\alpha-1)t_2(\alpha)} > \alpha/2.$$

The fact that the two inequalities in (81) are equivalent follows easily from (78), and the same applies to the two inequalities in (82). It is clear that the inequality in (81) implies the one in (82). Taking the derivative $\partial/\partial\alpha$ of equation (80), we check that

$$\frac{dt}{d\alpha} = \frac{(r+1)t}{2e^{(\alpha-1)t} - \alpha(r+1)},$$

thus the inequality (81) is satisfied if and only if $dt/d\alpha > 0$. One of the two equilibria that we have found [the one corresponding to the solution $t_1(\alpha)$] is decreasing in α , therefore, it cannot possibly be a stable equilibrium. At the same time,

the second solution $t_2(\alpha)$ is increasing in α , therefore, the condition (81) is satisfied. Thus, we have proved that the second equilibrium (the one with $v/u = e^{t_2(\alpha)}$) is linearly stable. As $\alpha \rightarrow \alpha(r)^+$, we have $dt/d\alpha \rightarrow \infty$ (see Figure 4), thus at $\alpha = \alpha(r)$ we have $2e^{(\alpha-1)t} - \alpha(r+1) = 0$ and the equilibrium corresponding to $t_1(\alpha(r)) = t_2(\alpha(r))$ is critical. \square

3.4. Proof of Theorem 4. The proof of Theorem 4 follows from Theorems 6, 7 and 8, together with Theorem 3. Indeed, in Theorem 6 we established that for every $\alpha > 3$ the star-graph WARM has a linearly stable equilibrium. Similarly, Theorem 8 states that any symmetric whisker-graph WARM has a linearly stable equilibrium if α is large enough. Together with Theorem 3, these imply Theorem 4(i). Claim (ii) of Theorem 4 is an immediate consequence of Theorem 7.

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