Integral representations for the Dirichlet L-functions and their expansions in Meixner-Pollaczek polynomials and rising factorials *

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Abstract

In this article we provide integral representations for the Dirichlet beta and Riemann zeta functions, which are obtained by combining Mellin transform with the fractional Fourier transform. As an application of these integral formulas we derive tractable expansions of these L-functions in the series of Meixner-Pollaczek polynomials and rising factorials.

Keywords: Riemann zeta function, Meixner-Pollaczek polynomials, rising factorials, confluent hypergeometric function, Mellin transform, fractional Fourier transform

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1 Introduction

In this article we study the relations between Mellin transform and the fractional powers of Fourier transform and we show how these integral transforms can be used to obtain new results for the Dirichlet L-functions. To illustrate the ideas let us consider the Dirichlet beta function $\beta(s)$, which for $\text{Re}(s) > 0$ is defined as $\beta(s) = \sum\limits_{n\geq 0} \frac{(-1)^n}{(2n+1)^s}$. This function can be obtained as the Mellin transform of $f(x) = \text{sech}(\sqrt{\frac{\pi}{2}}x)$, and the functional equation for $\beta(s)$ follows from the fact that $f(x)$ is invariant under cosine transform $\mathcal{F}_c$ and the integral kernel x^{s-1} of the Mellin transform is also "invariant" under $\mathcal{F}_c$, with s replaced by $1-s$ and some multiplication by functions of s only.

However, more can be said about function $f(x)$: not only it is invariant under $\mathcal{F}_c$, but infinitely many fractional powers $\left(\mathcal{F}_c^{\frac{m}{n}} f\right)(y)$ have quite simple form. But since the integral kernel of the Mellin transform is not invariant under fractional powers of $\mathcal{F}_c$ (in fact we obtain a transform with the confluent hypergeometric function as an integral kernel), we obtain new integral representations for $\beta(s)$.

These integral representations (Theorems 4.1 and 5.1) provide a simple way of obtaining tractable expansions of the Dirichlet L-functions in the series of polynomials, such as Meixner-Pollaczek polynomials or rising factorials. The partial sums of these expansions also satisfy functional equation and the coefficients have a very simple form of the exponential generating function. In a companion paper [9] we study in more detail the expansion of the Riemann Ξ-function in Meixner-Pollaczek polynomials and the zeros of the partial sums of this expansion (see also [2], [5], [6] and [7]).

This article is organized as follows: in section 2 we provide some background material on the fractional Fourier transform and its relation to the Mellin transform. Section 3 is devoted to Mordell integrals, which play the central role in this article since they allow us to compute the fractional Fourier transform of certain hyperbolic functions. In section 4 we study Dirichlet beta function, derive integral representation and expansions in Meixner-Pollaczek polynomials and in rising factorials. Only then in section 5 we study the famous Riemann zeta function $\zeta(s)$: it wasn't given a priority because in this case analysis is somewhat more complicated by the presence of the poles.

At last we would like to mention that everywhere in this article we use the following notation for the rising factorial: $(a)_n = a(a+1)\dots(a+n-1)$.

2 Fractional Fourier and scaled Mellin transforms

In this section we review some background material on the fractional Fourier sine (cosine) transform and Mellin transform (see [12], [17]) which will be used extensively later. The results are presented in the framework of the fractional Hankel transform $\mathcal{H}_\nu^a$, which includes both fractional cosine and sine transforms as the special cases $\nu = \pm\frac{1}{2}$.

Let us define $\mathfrak{A} = L^2([0,\infty), dx)$ to be Hilbert space of the square integrable functions on $[0,\infty)$. We choose a complete orthogonal basis $\phi_n(x) = L_n^\nu(x^2)e^{-\frac{1}{2}x^2}x^{\nu+\frac{1}{2}}$ $(\nu > -1)$, where $L_n^\nu(x)$ are Laguerre polynomials (see [8], [4]). The fractional Hankel transform $\mathcal{H}_\nu^a$ of order a is a unitary operator on $\mathfrak{A}$ defined by $\mathcal{H}_\nu^a \phi_n = e^{-\pi i n a}\phi_n$ (see [17]).

Below we present some of the properties of the fractional Hankel transform which will be used later:

- $\mathcal{H}_\nu^a$ is a unitary operator on $\mathfrak{A}$ such that $\mathcal{H}_\nu^a \mathcal{H}_\nu^b = \mathcal{H}_\nu^{a+b}$ and $(\mathcal{H}_\nu^a)^{-1} = (\mathcal{H}_\nu^a)^* = \mathcal{H}_\nu^{-a}$.
- The integral kernel

$$e^{-\frac{1}{2}(x^2+y^2)}(xy)^{\nu+\frac{1}{2}} \sum_{n\geq 0} \frac{2n!}{\Gamma(n+\nu+1)} e^{-\pi i n a} L_n^\nu(x^2) L_n^\nu(y^2) = \tag{1}$$
$$= \frac{e^{\frac{\pi i}{2}(\nu+1)(a-\hat{a})}}{|\sin\left(\frac{\pi a}{2}\right)|} e^{\frac{i}{2}\cot\left(\frac{\pi a}{2}\right)(x^2+y^2)} \sqrt{xy} J_\nu \left(\frac{xy}{|\sin\left(\frac{\pi a}{2}\right)|}\right),$$

  where $\hat{a} = \operatorname{sign}(\sin(\frac{\pi a}{2}))$ (see [17]).

- When $\nu = -\frac{1}{2}$ we obtain the fractional cosine transform $\mathcal{F}_c^a$ with the integral kernel

$$\sqrt{\frac{2}{\pi}} \frac{e^{\frac{\pi i}{4}(a-\hat{a})}}{|\sin\left(\frac{\pi a}{2}\right)|^{\frac{1}{2}}} e^{\frac{i}{2}\cot\left(\frac{\pi a}{2}\right)(x^2+y^2)} \cos\left(\frac{xy}{|\sin\left(\frac{\pi a}{2}\right)|}\right). \tag{2}$$

- When $\nu = \frac{1}{2}$ we obtain the fractional sine transform $\mathcal{F}_s^a$ with the integral kernel

$$\sqrt{\frac{2}{\pi}} \frac{e^{\frac{3\pi i}{4}(a-\hat{a})}}{|\sin\left(\frac{\pi a}{2}\right)|^{\frac{1}{2}}} e^{\frac{i}{2}\cot\left(\frac{\pi a}{2}\right)(x^2+y^2)} \sin\left(\frac{xy}{|\sin\left(\frac{\pi a}{2}\right)|}\right). \tag{3}$$

Next we define another Hilbert space $\mathfrak{B} = L^2(\mathbb{R}, |\Gamma\left(\lambda + \frac{i}{2}t\right)|^2 dt)$. Here and everywhere else in this article $\lambda = \frac{\nu+1}{2}$ (note that $\lambda > 0$). The scaled Mellin transform is defined as

$$(\mathcal{M}_\lambda f)(s) = \frac{2^{\frac{1}{4}-\frac{s}{2}}}{\Gamma\left(\lambda - \frac{1}{4} + \frac{s}{2}\right)} (\mathcal{M} f)(s),$$

where $(\mathcal{M} f)(s) = \int\limits_0^\infty f(x) x^{s-1} dx$ is the classical Mellin transform.

The following properties of the scaled Mellin transform will be used later:

- $(\mathcal{M}_\lambda \phi_n)(s) = 2^{\lambda-1}(-i)^n P_n^{(\lambda)}\left(\frac{t}{2}\right)$, where $s = \frac{1}{2} + it$ and $P_n^{(\lambda)}\left(\frac{t}{2}\right)$ are the Meixner-Pollaczek polynomials (see [8])
- Parseval identity: if $g(t) = (\mathcal{M}_\lambda f)(s)$, $s = \frac{1}{2} + it$, then $||f||_{\mathfrak{A}}^2 = \frac{1}{2\pi}||g||_{\mathfrak{B}}^2$.
- The action of $\mathcal{M}_\lambda$ on Fourier cosine $\mathcal{F}_c = \mathcal{H}_{-\frac{1}{2}}$ and Fourier sine transforms $\mathcal{F}_c = \mathcal{H}_{\frac{1}{2}}$:

$$(\mathcal{M}_{\frac{1}{4}} \mathcal{F}_c f)(s) = (\mathcal{M}_{\frac{1}{4}} f)(1-s), \quad (\mathcal{M}_{\frac{3}{4}} \mathcal{F}_s f)(s) = (\mathcal{M}_{\frac{3}{4}} f)(1-s). \tag{4}$$

Next we present a result which will be our main tool in the following sections: it describes the action of the scaled Mellin transform $\mathcal{M}_\lambda$ on the fractional Hankel transform $\mathcal{H}_\nu^{-\frac{1}{2}}$ ($\lambda = \frac{\nu+1}{2}$).

Proposition 2.1. *Assume $f(x) \in \mathfrak{A}$ and let $g(t) = (\mathcal{M}_\lambda \mathcal{H}_\nu^{-\frac{1}{2}} f)(t)$. Then $g(t) \in \mathfrak{B}$, $||f||^2_{\mathfrak{A}} = \frac{1}{2\pi}||g||^2_{\mathfrak{B}}$ and $g(t)$ can be represented as*

$$g(t) = \frac{e^{\frac{\pi t}{4}}}{\Gamma(2\lambda)} \int\limits_0^\infty x^{2\lambda-\frac{1}{2}} e^{-\frac{i}{2}x^2} {}_1F_1\left(\lambda + \tfrac{i}{2}t, 2\lambda; ix^2\right) f(x) dx,$$

Furthermore, $f(x) = (\mathcal{H}_\nu^{\frac{1}{2}} \mathcal{M}_\lambda^{-1} g)(x)$ can be expressed as the following integral

$$f(x) = \frac{x^{2\lambda-\frac{1}{2}} e^{-\frac{i}{2}x^2}}{2\pi\Gamma(2\lambda)} \int\limits_{\mathbb{R}} {}_1F_1\left(\lambda + \tfrac{i}{2}t, 2\lambda; ix^2\right) e^{\frac{\pi t}{4}} g(t) |\Gamma\left(\lambda + \tfrac{i}{2}t\right)|^2 dt.$$

Proof: The integral kernel of transformation $\mathcal{M}_\lambda \mathcal{H}_\nu^{-\frac{1}{2}}$ is given by

$$2^{\frac{1}{2}} e^{\frac{\pi i}{2}\lambda} \frac{2^{\frac{1}{4}-\frac{s}{2}}}{\Gamma\left(\lambda - \frac{1}{4} + \frac{s}{2}\right)} \int\limits_0^\infty y^{s-1} e^{-\frac{i}{2}(x^2+y^2)} \sqrt{xy} J_\nu\left(\sqrt{2}xy\right) dy =$$

$$= \frac{1}{\Gamma(2\lambda)} e^{\frac{\pi t}{4}} x^{2\lambda-\frac{1}{2}} e^{-\frac{i}{2}x^2} {}_1F_1\left(\lambda + \tfrac{i}{2}t, 2\lambda; ix^2\right).$$

Similarly, the integral kernel of the inverse transformation $\mathcal{H}_\nu^{\frac{1}{2}} \mathcal{M}_\lambda^{-1}$ is given by

$$\frac{1}{2\pi} 2^{\frac{1}{2}} e^{-\frac{\pi i}{2}\lambda} 2^{-\frac{1}{4}+\frac{s}{2}} \Gamma\left(\lambda - \tfrac{1}{4} + \tfrac{s}{2}\right) \int\limits_0^\infty y^{-s} e^{\frac{i}{2}(x^2+y^2)} \sqrt{xy} J_\nu\left(\sqrt{2}xy\right) dy =$$

$$= \frac{1}{2\pi\Gamma(2\lambda)} e^{\frac{\pi t}{4}} x^{2\lambda-\frac{1}{2}} e^{-\frac{i}{2}x^2} {}_1F_1\left(\lambda + \tfrac{i}{2}t, 2\lambda; ix^2\right) |\Gamma\left(\lambda + \tfrac{i}{2}t\right)|^2.$$

Both integrals were computed with the help of [4].

□

3 Mordell integrals

In this section we review several results about function

$$h(y, \tau) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \frac{e^{\frac{i}{2}\tau x^2} \cos(xy)}{\cosh\left(\sqrt{\frac{\pi}{2}}x\right)} dx, \quad \mathrm{Im}(\tau) \geq 0.$$

Integrals of this type were used by Riemann to obtain functional equation and asymptotic formula for zeta function (see [15]). Later these integrals were studied by Ramanujan ([13]) and by Mordell, who analyzed their behavior with respect to modular transformations (see [10], [11]). An extensive collection of facts about function $h(y,\tau)$ can be found in [16].

In the derivation of the integral representations for Dirichlet L-functions we will need explicit formulas for $h(y,\tau)$, which can be obtained using the following functional equations (see [16] for the proof)

$$h(y,\tau)+h(y+i\sqrt{2\pi},\tau)=\frac{2}{\sqrt{\tau}}e^{\frac{\pi i}{4}-\frac{i}{2\tau}\left(y+i\sqrt{\frac{\pi}{2}}\right)^2}, \tag{5}$$

$$h(y,\tau)+e^{-\sqrt{2\pi}y-\pi i\tau}h(y+i\sqrt{2\pi}\tau,\tau)=2e^{-\sqrt{\frac{\pi}{2}}y-\frac{\pi i\tau}{4}}. \tag{6}$$

If $\tau=\frac{m}{n}$ is an irreducible fraction, then by iterating Eq. (5) m times and Eq. (6) n times we obtain a system of two linear equations in two variables $h_1=h(y,\tau)$ and $h_2=h(y+i\sqrt{2\pi}m,\tau)$. After eliminating h_2 from these equations and simplifying the resulting formula for h_1 we obtain

$$h(y,\tau)=\frac{1}{\cosh\left(n\sqrt{\frac{\pi}{2}}y\right)}\left[G_{\frac{1}{2},-}\left(\sqrt{\tfrac{\pi}{2}}y,\tau,n\right)+\frac{1}{\sqrt{\tau}}e^{\frac{\pi i}{4}-\frac{i}{2\tau}y^2}G_{\frac{1}{2},-}\left(\sqrt{\tfrac{\pi}{2}}\tfrac{y}{\tau},-\tfrac{1}{\tau},m\right)\right], \tag{7}$$

where the quadratic Gauss sum is defined as $G_{a,\pm}(y,\tau,n)=\sum\limits_{k=0}^{n-1}(\pm1)^k e^{-\pi i\tau(k+a)^2+(n-2k-2a)y}$. The following two integrals can also be expressed in terms of $h(y,\tau)$ and thus computed explicitly for rational τ:

$$\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \frac{e^{\frac{i}{2}\tau x^2}\sin(xy)}{e^{\sqrt{2\pi}x}+1}dx=-\frac{1}{2\sqrt{\tau}}e^{-\frac{\pi i}{4}-\frac{i}{2\tau}y^2}\left(1+\Phi\left(e^{-\frac{\pi i}{4}}\frac{y}{\sqrt{2\tau}}\right)\right)- \tag{8}$$
$$-\quad\frac{1}{e^{n\sqrt{\frac{\pi}{2}}y}+(-1)^n e^{-n\sqrt{\frac{\pi}{2}}y}}\left[G_{\frac{1}{2},+}\left(\sqrt{\tfrac{\pi}{2}}y,\tau,n\right)-\frac{1}{\sqrt{\tau}}e^{-\frac{\pi i}{4}-\frac{i}{2\tau}y^2}G_{0,-}\left(\sqrt{\tfrac{\pi}{2}}\tfrac{y}{\tau},-\tfrac{1}{\tau},m\right)\right]$$

$$\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \frac{e^{\frac{i}{2}\tau x^2}\sin(xy)}{e^{\sqrt{2\pi}x}-1}dx=\frac{1}{2\sqrt{\tau}}e^{-\frac{\pi i}{4}-\frac{i}{2\tau}y^2}\left(1+\Phi\left(e^{-\frac{\pi i}{4}}\frac{y}{\sqrt{2\tau}}\right)\right)+ \tag{9}$$
$$+\quad\frac{1}{e^{n\sqrt{\frac{\pi}{2}}y}+(-1)^{n+m} e^{-n\sqrt{\frac{\pi}{2}}y}}\left[G_{0,+}\left(\sqrt{\tfrac{\pi}{2}}y,\tau,n\right)-\frac{1}{\sqrt{\tau}}e^{-\frac{\pi i}{4}-\frac{i}{2\tau}y^2}G_{0,+}\left(\sqrt{\tfrac{\pi}{2}}\tfrac{y}{\tau},-\tfrac{1}{\tau},m\right)\right]$$

4 Dirichlet beta function

In this section we illustrate the interplay between Mellin transform and the fractional cosine transform on the example of the Dirichlet beta function $\beta(s)$, which for $\mathrm{Re}(s)>0$ can be defined as $\beta(s)=L(s,\chi_4)=\sum\limits_{n\geq0}\frac{(-1)^n}{(2n+1)^s}$. We also define $\xi(s,\chi_4)=2^s\pi^{-\frac{s}{2}}\Gamma\left(\frac{1+s}{2}\right)\beta(s)$ and $\Xi(t,\chi_4)=\xi\left(\frac{1}{2}+it,\chi_4\right)$.

Our main result in this section is the following integral representation for $\Xi(t,\chi_4)$:

Theorem 4.1. *For all $t \in \mathbb{C}$*

$$\Xi(t, \chi_4) e^{-\frac{\pi t}{4}} = 2 \int_0^\infty e^{-\frac{i}{2}y^2} {}_1F_1\left(\tfrac{1}{4} + \tfrac{i}{2}t, \tfrac{1}{2}; iy^2\right) \frac{\sin\left(\frac{y^2}{2} + \frac{\pi}{8}\right)}{\cosh(\sqrt{\pi}y)} dy. \tag{10}$$

Proof: We start with the function $f(x) = \operatorname{sech}\left(\sqrt{\frac{\pi}{2}}x\right)$ and find that

$$(\mathcal{M}_{\frac{1}{4}} f)(s) = \frac{2^{\frac{1}{4}}}{\sqrt{\pi}} \xi(s, \chi_4), \quad \operatorname{Re}(s) > 0. \tag{11}$$

Note that the functional equation $\xi(s, \chi_4) = \xi(1-s, \chi_4)$ follows at once from Eq. (4) and the fact that $f(x)$ is invariant under Fourier cosine transform $\mathcal{F}_c$ (see [4]).

Now we use Eq. (7) and compute $(\mathcal{F}_c^{\frac{1}{2}} f)(y)$:

$$F(y) = (\mathcal{F}_c^{\frac{1}{2}} f)(y) = \sqrt{\frac{2}{\pi}} C\left(\tfrac{\pi}{4}, -\tfrac{1}{2}\right) \int_0^\infty e^{\frac{i}{2}(x^2+y^2)} \frac{\cos(\sqrt{2}xy)}{\cosh\left(\sqrt{\frac{\pi}{2}}x\right)} dx = 2^{\frac{5}{4}} \frac{\sin\left(\frac{y^2}{2} + \frac{\pi}{8}\right)}{\cosh(\sqrt{\pi}y)},$$

Next we use the identity $f(x) = (\mathcal{F}_c^{-\frac{1}{2}} F)(x)$, Eq. (11) and Proposition 2.1 to obtain

$$\frac{2^{\frac{1}{4}}}{\sqrt{\pi}} \xi(s, \chi_4) = (\mathcal{M}_{\frac{1}{4}} \mathcal{F}_c^{-\frac{1}{2}} F)(s) = \frac{e^{\frac{\pi t}{4}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty e^{-\frac{i}{2}y^2} {}_1F_1\left(\tfrac{1}{4} + \tfrac{i}{2}t, \tfrac{1}{2}; iy^2\right) F(y) dy,$$

which ends the proof.

□

Corollary 4.2. *For all $t \in \mathbb{C}$*

$$\Xi(t, \chi_4) e^{-\frac{\pi t}{4}} = \sum_{n \geq 0} a_n P_n^{\left(\frac{1}{4}\right)}\left(\frac{t}{2}\right) \tag{12}$$

where the generating function for coefficients $\{a_n\}$ is

$$\sum_{n \geq 0} \frac{a_n}{n!} x^n = \sqrt{2\pi} \frac{\sin\left(x + \frac{\pi}{8}\right)}{\cosh(\sqrt{2\pi x})}. \tag{13}$$

Proof: A rigorous proof (and the integral representation for a_n) can be obtained by expanding the confluent hypergeometric function in (10) in Meixner-Pollaczek polynomials (see [8]):

$$e^{-\frac{iy}{2}} {}_1F_1(\lambda + it, 2\lambda; iy) = \sum_{n=0}^\infty \frac{(-1)^n}{2^n (2\lambda)_n} P_n^{(\lambda)}(t) y^n. \tag{14}$$

However we decided to present here a more intuitive argument, which shows why the generating function for the coefficients a_n is necessarily the same as the function in the integral representation (10) (up to a simple change of variables).

First we assume that function $\Xi(t,\chi_4)e^{\frac{\pi t}{4}}$ lies in the Hilbert space $\mathfrak{B}$ and we expand it in the orthogonal basis given by Meixner-Pollaczek polynomials: $\Xi(t,\chi_4)e^{\frac{\pi t}{4}} = \sum\limits_{n\geq 0}(-1)^n a_n P_n^{\left(\frac{1}{4}\right)}\left(\frac{t}{2}\right)$. Using the orthogonality relation for the Meixner-Pollaczek polynomials (see [8]) we find that the coefficients a_n are given by

$$a_n = \frac{(-1)^n\sqrt{2}n!}{4\pi\Gamma(\frac{1}{2}+n)}\int\limits_{\mathbb{R}}\Xi(t,\chi_4)e^{\frac{\pi t}{4}}P_n^{\left(\frac{1}{4}\right)}\left(\frac{t}{2}\right)\left|\Gamma\left(\tfrac{1}{4}+\tfrac{i}{2}t\right)\right|^2 dt.$$

The generating function for $\{a_n\}$ is computed using Eq. (14):

$$\sum_{n\geq 0}\frac{a_n}{n!}x^n = \frac{\sqrt{2}e^{-ix}}{4\pi^{\frac{3}{2}}}\int\limits_{\mathbb{R}}{}_1F_1\left(\tfrac{1}{4}+\tfrac{i}{2}t,\tfrac{1}{2};2ix\right)\Xi(t,\chi_4)e^{\frac{\pi t}{4}}\left|\Gamma\left(\tfrac{1}{4}+\tfrac{i}{2}t\right)\right|^2 dt,$$

and using Proposition 2.1 and the integral representation (10) we find that the above integral must be equal to $\sqrt{2\pi}\,\frac{\sin\left(x+\frac{\pi}{8}\right)}{\cosh(\sqrt{2\pi x})}$.

□

Corollary 4.3. *For all $t\in\mathbb{C}$*

$$\Xi(t,\chi_4)\,e^{-\frac{\pi t}{4}} = \sum_{n\geq 0}\left[\frac{b_n\,(-i)^n}{n!}\left(\tfrac{1}{4}+\tfrac{i}{2}t\right)_n + \frac{\overline{b}_n\,i^n}{n!}\left(\tfrac{1}{4}-\tfrac{i}{2}t\right)_n\right], \qquad (15)$$

where the generating functions for coefficients $\{b_n\}$ is

$$\sum_{n\geq 0}\frac{b_n}{n!}x^n = \sqrt{\pi}e^{\frac{\pi i}{4}+i\frac{x}{2}}\frac{\sin\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\cosh(\sqrt{\pi x})}.$$

Proof: Again we start with the integral representation (10) and rewrite it as $\Xi(t,\chi_4)\,e^{-\frac{\pi t}{4}} = \Psi(t)+\overline{\Psi(t)}$, where

$$\Psi(t) = e^{\frac{3\pi i}{8}}\int\limits_0^\infty\frac{e^{-iy^2}{}_1F_1\left(\frac{1}{4}+\frac{it}{2},\frac{1}{2};iy^2\right)}{\cosh(\sqrt{\pi}y)}dy.$$

Next we use the definition of the confluent hypergeometric function and expand it in the power series in y (see [4]). Integrating term by term we find that the coefficients b_n have the following integral representation:

$$b_n = \frac{e^{\frac{3\pi i}{8}}(-1)^n}{\left(\frac{1}{2}\right)_n}\int\limits_0^\infty\frac{e^{-iy^2}y^{2n}}{\cosh(\sqrt{\pi}y)}dy. \qquad (16)$$

One can find using the above formula that for n large $|b_n|\sim n^\alpha e^{-\sqrt{\frac{\pi}{2}n}}$ for some α, thus the series (15) converges for all complex t. The exponential generating function for the coefficients $\{b_n\}$ is computed using Eq. 7.

□

5 Riemann zeta function

We adopt the following standard definitions (see [14],[15]): $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ and $\Xi(t) = \xi\left(\frac{1}{2}+it\right)$. Our main result in this section is the following integral representation for $\Xi(t)$:

Theorem 5.1. *For all $t \in \mathbb{C}$*

$$\Xi(t)e^{-\frac{\pi t}{4}} = \frac{1}{2}\cos\left(\frac{\pi}{8}\right) - t\sin\left(\frac{\pi}{8}\right) + (1+4t^2)\int_0^\infty ye^{-\frac{i}{2}y^2}{}_1F_1\left(\tfrac{3}{4}+\tfrac{i}{2}t, \tfrac{3}{2}; iy^2\right)\frac{\sin\left(\frac{y^2}{2}+\frac{\pi}{8}\right)}{e^{2\sqrt{\pi}y}+1}dy. \tag{17}$$

Proof: Define function $f(x) = \frac{1}{e^{\sqrt{2\pi x}}-1} - \frac{1}{\sqrt{2\pi x}}$. The first step is to find that

$$(\mathcal{M}_{\frac{3}{4}}f)(s) = \frac{2^{\frac{1}{4}}}{\sqrt{\pi}}\frac{\xi(s)}{s(s-1)}, \quad \text{Re}(s) \in (0,1). \tag{18}$$

Again, the functional equation $\xi(s) = \xi(1-s)$ follows from the fact that f is invariant under Fourier sine transform $\mathcal{F}_s$ (see [4]) and Eq. (4).

Using Eq. (9) we find the fractional sine transform $\mathcal{F}_s^{\frac{1}{2}}$ of function $f(x)$:

$$F(y) = (\mathcal{F}_s^{\frac{1}{2}}f)(y) = 2^{\frac{1}{4}}\left[-2\frac{\sin\left(\frac{y^2}{2}+\frac{\pi}{8}\right)}{e^{2\sqrt{\pi}y}+1} + \frac{1}{2i}\left(\phi(y) - \overline{\phi(y)}\right)\right], \tag{19}$$

where $\phi(y) = e^{i\frac{y^2}{2}+\frac{\pi i}{8}}\left(1-\Phi\left(e^{\frac{\pi i}{4}}y\right)\right)$. Note that function $\phi(y)$ is analytic, and as $y \to +\infty$ we have (see [4]) $\phi(y) \sim \frac{1}{\sqrt{\pi}}\frac{e^{-i\frac{y^2}{2}-\frac{\pi i}{8}}}{y} + O\left(\frac{1}{y^2}\right)$, thus $\phi(y)$ is in the Hilbert space $\mathfrak{A}$. Next we find that

$$(\mathcal{M}_{\frac{3}{4}}\mathcal{F}_s^{-\frac{1}{2}}\phi)(s) = \frac{i}{\sqrt{\pi}}\frac{e^{\frac{\pi i}{4}(1-s)}}{s-1}, \quad (\mathcal{M}_{\frac{3}{4}}\mathcal{F}_s^{-\frac{1}{2}}\overline{\phi})(s) = \frac{i}{\sqrt{\pi}}\frac{e^{-\frac{\pi i}{4}s}}{s}, \tag{20}$$

and to finish the proof we only need to combine Eqs. (18), (19), (20) and Proposition 2.1.

□

It is interesting to note that Eq. (17) is essentially equivalent to the integral representation $\Xi(t) = \left(\frac{1}{4}+t^2\right)\left[\Upsilon(\frac{1}{2}+it) + \overline{\Upsilon(\frac{1}{2}+it)}\right]$, where $\Upsilon(s)$ is defined by

$$\Upsilon(s) = -\frac{1}{4}e^{\frac{\pi i}{2}(s-1)}2^{s-1}\pi^{\frac{s}{2}-1}\Gamma\left(\frac{s}{2}\right)\int_L \frac{e^{\frac{ix^2}{4\pi}}}{\sinh(\frac{x}{2})}x^{-s}dx. \tag{21}$$

and the integral is taken along the line $L = e^{\frac{\pi i}{4}}\mathbb{R} + \pi i$ (see [15]). One can obtain formula (17) by applying Plancherel theorem for sine transform to the functions inside the integral in Eq. (21).

It is also of interest to compare integral representation (17) with the well-known Riemann formula (see [15]):

$$2\xi(s) = 1 + s(s-1)\int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\frac{1}{x}\sum_{n\geq 1} e^{-\pi n^2 x}dx = \tag{22}$$

$$= 1 + s(s-1)\sum_{n\geq 1}\left[\frac{\Gamma\left(\frac{s}{2},\pi n^2\right)}{(\sqrt{\pi}n)^s} + \frac{\Gamma\left(\frac{1-s}{2},\pi n^2\right)}{(\sqrt{\pi}n)^{1-s}}\right].$$

The following proposition shows that Eqs. (22) and (17) are just "extreme" cases ($\alpha = \frac{\pi}{2}$ and $\alpha = 0$ correspondingly) of the more general result:

Proposition 5.2. *For all* $\alpha \in [0,\pi]$

$$2\xi(s) = se^{\frac{i}{2}(\frac{\pi}{2}-\alpha)(1-s)} + (1-s)e^{-\frac{i}{2}(\frac{\pi}{2}-\alpha)s} + s(s-1)\sum_{n\geq 1}\left[\frac{\Gamma\left(\frac{s}{2}, e^{-i(\frac{\pi}{2}-\alpha)}\pi n^2\right)}{(\sqrt{\pi}n)^s} + \frac{\Gamma\left(\frac{1-s}{2}, e^{i(\frac{\pi}{2}-\alpha)}\pi n^2\right)}{(\sqrt{\pi}n)^{1-s}}\right] \tag{23}$$

and

$$\Xi(t)e^{-\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)t} = \frac{1}{2}\cos\left(\frac{\pi}{8}-\frac{\alpha}{4}\right) - t\sin\left(\frac{\pi}{8}-\frac{\alpha}{4}\right) + (1+4t^2)\int_0^\infty ye^{-\frac{i}{2}y^2}{}_1F_1\left(\tfrac{3}{4}+\tfrac{i}{2}t, \tfrac{3}{2}; iy^2\right)\vartheta(y)dy, \tag{24}$$

where $\vartheta(y) = Re\left[e^{\frac{3\pi i}{8}+\frac{\alpha i}{4}-\frac{i}{2}y^2}\sum_{n\geq 1}\exp\left(\pi i n^2 e^{i\alpha} - 2\sqrt{\pi}nye^{\frac{i\alpha}{2}}\right)\right]$.

Proof: To derive (23) one should start with the function $\psi(y) = \sum_{n\geq 1} e^{\pi i n^2 y} = \frac{1}{2}\left(\theta_3(0,y)-1\right)$ and follow the lines of Riemann's proof (see [15]), but take Mellin transform along the line $y \in e^{i\alpha}\mathbb{R}_+$, $\alpha \in [0,\pi]$. Formula (24) is obtained from (23) with the help of expression for the incomplete Gamma function as the Laplace transform of the confluent hypergeometric function (see [4]):

$$\frac{\Gamma\left(\frac{s}{2}, e^{-i(\frac{\pi}{2}-\alpha)}\pi n^2\right)}{(\sqrt{\pi}n)^s} = 4ie^{-i\left(\frac{\pi}{2}-\alpha\right)\frac{s}{2}}\int_0^\infty ye^{-\frac{i}{2}y^2}{}_1F_1\left(\tfrac{3}{4}+\tfrac{i}{2}t, \tfrac{3}{2}; iy^2\right)\exp\left(\pi i n^2 e^{i\alpha} - 2\sqrt{\pi}nye^{\frac{i\alpha}{2}}\right)dy.$$

□

Next we derive an expansion of the Riemann Xi function in the Meixner-Pollaczek polynomials (see [9] for the detailed analysis of the coefficients of this expansion and zeros of its partial sums).

Corollary 5.3. *For all* $t \in \mathbb{C}$

$$\Xi(t)e^{-\frac{\pi t}{4}} = \frac{1}{2}\cos\left(\frac{\pi}{8}\right) - t\sin\left(\frac{\pi}{8}\right) + (1+4t^2)\sum_{n\geq 0} a_n P_n^{\left(\frac{3}{4}\right)}\left(\frac{t}{2}\right), \tag{25}$$

where the generating function for coefficients $\{a_n\}$ *is*

$$\sum_{n\geq 0}\frac{a_n}{n!}x^n = \frac{1}{4}\sqrt{\frac{\pi}{x}}\left[-\sin\left(x+\frac{\pi}{8}\right)\tanh\left(\sqrt{2\pi x}\right) + \frac{1}{2i}\left(\Phi\left(e^{\frac{\pi i}{4}}\sqrt{x}\right)e^{ix+\frac{\pi i}{8}} - \Phi\left(e^{-\frac{\pi i}{4}}\sqrt{x}\right)e^{-ix-\frac{\pi i}{8}}\right)\right].$$

Proof: Again we start with Eq. (17), expand the confluent hypergeometric function in the series of Meixner-Pollaczek polynomials (see Eq. (14)) and integrate term by term to find that the coefficients are given by

$$a_n = \frac{(-1)^n}{(2n+1)!!}\int_0^\infty \frac{\sin\left(\frac{y^2}{2}+\frac{\pi}{8}\right)}{e^{2\sqrt{\pi}y}+1}y^{2n+1}dy.$$

The exponential generating function for $\{a_n\}$ is computed using Eq. (8).

□

Following the lines of the proof of Corollary 4.3 we obtain the following expansion of $\Xi(t)e^{-\frac{\pi t}{4}}$ in rising factorials:

Corollary 5.4. *For all* $t \in \mathbb{C}$

$$\Xi(t)e^{-\frac{\pi t}{4}} = \frac{1}{2}\cos\left(\frac{\pi}{8}\right) - t\sin\left(\frac{\pi}{8}\right) + (1+4t^2)\sum_{n\geq 0}\left[\frac{b_n\,(-i)^n}{n!}\left(\tfrac{3}{4}+\tfrac{i}{2}t\right)_n + \frac{\bar{b}_n\, i^n}{n!}\left(\tfrac{3}{4}-\tfrac{i}{2}t\right)_n\right],$$

where the generating function for the coefficients $\{b_n\}$ *is*

$$\sum_{n\geq 0}\frac{b_n}{n!}x^n = \frac{e^{-\frac{3\pi i}{8}}}{16}\sqrt{\frac{\pi}{x}}\left[e^{ix}\Phi\left(e^{\frac{\pi i}{4}}\sqrt{x}\right) + \frac{1-e^{ix}\cosh(\sqrt{\pi x})}{\sinh(\sqrt{\pi x})}\right].$$

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