Using q-calculus to study LDL^T factorization of a certain Vandermonde matrix

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May 2, 2017

Abstract

We use tools from q-calculus to study LDL^T decomposition of the Vandermonde matrix V_q with entries $v_{i,j} = q^{ij}$. We prove that the matrix L is given as a product of diagonal matrices and the lower triangular Toeplitz matrix T_q with elements $t_{i,j} = 1/(q;q)_{i-j}$, where $(z;q)_k$ is the q-Pochhammer symbol. We investigate some properties of the matrix T_q, in particular, we compute explicitly the inverse of this matrix.

Keywords: Vandermonde matrix, LDL^T decomposition, Toeplitz matrix, q-Binomial Theorem, q-Pochhammer symbol, discrete Fourier transform
2010 Mathematics Subject Classification : Primary 15A23, Secondary 15B05

1 Introduction and main results

Let us consider a Vandermonde matrix

$$V_q := \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ 1 & q^2 & q^4 & q^6 & \dots \\ 1 & q^3 & q^6 & q^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

of size $n \times n$. In the special case when $q = e^{-2\pi \mathrm{i}/n}$, this matrix is called *the discrete Fourier transform matrix*. Explicit matrix factorizations of the discrete Fourier transform matrix are very important, since they are often used in various versions of the Fast Fourier Transform algorithm [5]. Motivated by this connection, in this note we plan to study the LDL^T factorization of the matrix V_q and to investigate the properties of the factors appearing in such decomposition. The tools and techniques, which are used to prove our results, come from q-calculus.

First, let us present several definitions and notation. In what follows, we assume that $n \in \mathbb{N}$ and $q \in \mathbb{C}$. We define the q-Pochhammer symbol

$$(z;q)_n := (1-z)(1-zq)\cdots(1-zq^{n-1}), \quad n \ge 1, \tag{1}$$

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Research supported by the Natural Sciences and Engineering Research Council of Canada.

and $(z;q)_0 := 1$. We will denote by I the $n \times n$ identity matrix. The following matrices of size $n \times n$ will be used frequently in this paper: a lower-triangular Toeplitz matrix $T_q = \{t_{i,j}\}_{0 \le i,j \le n-1}$ defined by $t_{i,j} = 1/(q;q)_{i-j}$ if $i \ge j$, and a diagonal matrix $P_q = \{p_{i,j}\}_{0 \le i,j \le n-1}$ having elements $p_{i,i} = (q;q)_i$, or, more explicitly,

$$T_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{(q;q)_1} & 1 & 0 & 0 & \dots \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \dots \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad P_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & (q;q)_1 & 0 & 0 & \dots \\ 0 & 0 & (q;q)_2 & 0 & \dots \\ 0 & 0 & 0 & (q;q)_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the matrices T_q and $T_{q^{-1}}$ are well-defined for all $q \in \mathbb{C} \setminus \mathcal{A}_n$, where the set $\mathcal{A}_n$ is given by

$$\mathcal{A}_n := \{q \in \mathbb{C} : q^j = 1 \text{ for some } j = 1, 2, \dots, n-1\}.$$

In our first result we identify explicitly the matrices appearing in the LDL^T factorization of the Vandermonde matrix V_q.

Theorem 1. *Assume that $q \in \mathbb{C} \setminus \mathcal{A}_n$. Then $V_q = LDL^T$, where $L = P_q T_q (P_q)^{-1}$ and $D = \{d_{i,j}\}_{0 \le i,j \le n-1}$ is a diagonal matrix having elements $d_{i,i} = (-1)^i q^{i(i-1)/2} (q;q)_i$.*

In section 2 we give a very simple proof of Theorem 1 (our proof is based on the q-Binomial Theorem). Alternatively, one could derive this result starting from formulas (2.4) and (2.5) in the paper [4] by Oruc and Phillips, who use symmetric functions to study LU decomposition of general Vandermonde matrices.

Remark 1. It is easy to see that the entries of the matrix $L = P_q T_q (P_q)^{-1}$ are given by

$$l_{i,j} = \frac{(q;q)_i}{(q;q)_j (q;q)_{i-j}}, \quad i \ge j. \tag{2}$$

This matrix is known in the literature as *the q-Pascal matrix* and it has appeared in [2, 3].

In our second result we present some properties of the Toeplitz matrix T_q, including an explicit formula for its inverse. First we define the following two matrices of size $n \times n$:

$$S := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad D_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & q & 0 & 0 & \dots \\ 0 & 0 & q^2 & 0 & \dots \\ 0 & 0 & 0 & q^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3}$$

Theorem 2. *Assume that $q \in \mathbb{C} \setminus \mathcal{A}_n$. Then*

(i) $(T_q)^{-1} = T_{q^{-1}}(I - S) = D_{q^{-1}} T_{q^{-1}} D_q$;

(ii) for $m \in \mathbb{N}$ we have

$$T_q D_{q^{-m}} T_{q^{-1}} D_{q^m} = I + \sum_{j=1}^{m-1} \frac{(q^{1-m};q)_j}{(q;q)_j} S^j. \tag{4}$$

Remark 2. Note that the matrix $H := (I - S)^{-1}$, which appears in item (i), is a lower triangular Toeplitz matrix having elements $h_{i,j} = 1$ if $i \ge j$ and $h_{i,j} = 0$ otherwise. Similarly, the matrix in the right-hand side of (4) is a lower-triangular Toeplitz matrix, having m non-zero diagonals: this matrix has coefficient 1 on the main diagonal and the coefficient $(q^{1-m};q)_j/(q;q)_j$ on the sub-diagonal number j, for $1 \le j \le m-1$.

2 Proofs

The only tool that will be needed for proving Theorems 1 and 2 is the q-Binomial Theorem (see [1][Theorem 10.2.1]), which states that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{j\geq 0} \frac{(a;q)_j}{(q;q)_j} z^j, \quad |q|<1, \ |z|<1. \tag{5}$$

Here $(z;q)_\infty := \prod_{l\geq 0}(1-zq^l)$ and it is clear that this infinite product converges for all $z \in \mathbb{C}$ and $|q|<1$. We also record here the following two corollaries of the q-Binomial Theorem, which will be needed later:

$$\frac{1}{(z;q)_\infty} = \sum_{j\geq 0} \frac{z^j}{(q;q)_j}, \quad |q|<1, \ |z|<1, \tag{6}$$

$$(z;q)_\infty = \sum_{j\geq 0} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j} z^j, \quad |q|<1, \ z\in\mathbb{C}. \tag{7}$$

Proof of Theorem 1: Using formula (2) and considering an element (i,j) of the matrix LDL^T we see that formula $V_q = LDL^T$ is equivalent to the following identity: for any integers $i,j\geq 0$

$$q^{ij} = \sum_{k=0}^{\min(i,j)} \frac{(-1)^k q^{k(k-1)/2}(q;q)_i(q;q)_j}{(q;q)_k(q;q)_{i-k}(q;q)_{j-k}}. \tag{8}$$

We will prove the above identity by writing the Taylor series of the function

$$g(u,v) := \frac{(uv;q)_\infty}{(u;q)_\infty(v;q)_\infty}, \quad |u|<1, \ |v|<1, \ |q|<1,$$

in two different ways. First of all, from formula (5) we obtain

$$g(u,v) = \frac{1}{(v;q)_\infty} \times \frac{(uv;q)_\infty}{(u;q)_\infty} = \frac{1}{(v;q)_\infty} \sum_{i\geq 0} \frac{(v;q)_i}{(q;q)_i} u^i.$$

Using the fact that $(v;q)_i/(v;q)_\infty = 1/(q^i v;q)_\infty$ and expanding this expression in Taylor series in v via (6) we conclude that

$$g(u,v) = \sum_{i\geq 0}\sum_{j\geq 0} \frac{q^{ij}u^i v^j}{(q;q)_i(q;q)_j}. \tag{9}$$

On the other hand, we can obtain the series expansion of $g(u,v)$ by applying formulas (6) and (7) in the form

$$(uv;q)_\infty = \sum_{k\geq 0} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} u^k v^k,$$

$$\frac{1}{(u;q)_\infty} = \sum_{l\geq 0} \frac{u^l}{(q;q)_l},$$

$$\frac{1}{(v;q)_\infty} = \sum_{m\geq 0} \frac{v^m}{(q;q)_m}.$$

We multiply the above three series expansions and obtain a Taylor series representation in the form

$$g(u,v)=\sum_{k\geq 0}\sum_{l\geq 0}\sum_{m\geq 0}\frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k(q;q)_l(q;q)_m}u^{k+l}v^{k+m}. \tag{10}$$

Comparing the coefficients in front of the term $u^i v^j$ in both formulas (9) and (10) gives us the desired result (8). □

Proof of Theorem 2: Let us prove the identity $T_q T_{q^{-1}} = (I-S)^{-1}$, which is equivalent to the first equality in item (i) (the second equality in (i) follows from formula (4) with $m=1$). The main idea of the proof is that the Toeplitz matrix T_q can be expressed in the following form

$$T_q = I + \sum_{j\geq 1}\frac{S^j}{(q;q)_j}, \tag{11}$$

where S is the matrix defined in (3). The above formula is easy to derive, given that for $1\leq j\leq n-1$ the entries of the matrix S^j have value 1 on the sub-diagonal number j and value zero everywhere else. In particular, S^j is a zero matrix for $j\geq n$, thus the series in (11) terminates at $j=n-1$. Similarly, using the identity

$$(1/q;1/q)_j = (-1)^j q^{-j(j+1)/2}(q;q)_j, \tag{12}$$

and formula (11) we obtain

$$T_{q^{-1}} = I + \sum_{j\geq 1}\frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j}(qS)^j. \tag{13}$$

Now, assume that $|q|<1$. Then formulas (6) and (11) give us

$$T_q = [(S;q)_\infty]^{-1} = (I-S)^{-1}\times(I-qS)^{-1}\times(I-q^2S)^{-1}\times\cdots. \tag{14}$$

Similarly, formulas (7) and (13) give us

$$T_{q^{-1}} = (qS;q)_\infty = (I-qS)\times(I-q^2S)\times(I-q^3S)\times\cdots. \tag{15}$$

From the above two identities we see that all the terms $(I-q^iS)$ in the product $T_qT_{q^{-1}}$ are cancelled, except for the first term $(I-S)^{-1}$, thus we obtain $T_qT_{q^{-1}}=(I-S)^{-1}$ for $|q|<1$. We extend this result from $|q|<1$ to the general case $q\in\mathbb{C}\setminus\mathcal{A}_n$ by analytical continuation in q.

The proof of formula (4) uses the same ideas. Again, first we assume that $|q|<1$. From (12) we check that $D_{q^{-m}}T_{q^{-1}}D_{q^m}$ is a Toeplitz matrix of the form

$$D_{q^{-m}}T_{q^{-1}}D_{q^m} = I + \sum_{j\geq 1}\frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j}(q^{1-m}S)^j = (q^{1-m}S;q)_\infty.$$

Using the above result and formula (14) we obtain

$$T_qD_{q^{-m}}T_{q^{-1}}D_{q^m} = [(S;q)_\infty]^{-1}\times(q^{1-m}S;q)_\infty = (q^{1-m}S;q)_{m-1}.$$

The desired desired result (4) follows by applying (5) and analytical continuation in q. □

Acknowledgements

The author would like to thank an anonymous referee for careful reading of the paper and for suggesting several improvements. The research is supported by the Natural Sciences and Engineering Research Council of Canada.

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