On the Lanczos limit formula

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Abstract

In this article we derive a limit representation for a class of analytic functions, which generalizes Lanczos limit formula for Gamma function. This limit representation is valid in the entire complex plane and it is shown to be an analogue of the Euler-Maclaurin summation formula. As examples we provide approximations to Gamma function and Riemann zeta function, and as a corollary we derive a characterization of the nontrivial zeros of the Riemann zeta function.

Keywords: Lanczos limit formula; Euler-Maclaurin summation; Mellin transform

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1 Introduction

In [3] Cornelius Lanczos derived the beautiful limit representation for the gamma function:

$$\begin{aligned}\Gamma(s+1) &= 2\lim_{\gamma\to+\infty}\gamma^s\left[1/2-\exp\left(-1/\gamma\right)\frac{s}{s+1}+\exp\left(-4/\gamma\right)\frac{s(s-1)}{(s+1)(s+2)}-\dots\right]\\ &= 2\lim_{\gamma\to+\infty}\gamma^s{\sum_{k\geq 0}}'(-1)^k\exp\left(-k^2/\gamma\right)\frac{\binom{s}{k}}{\binom{s+k}{k}}. \qquad (1)\end{aligned}$$

Lanczos did not present the proof of this formula, but he stated that it was obtained as a limit of his approximation for Gamma function, which was the main subject of [3]. Lanczos paper [3] was later analyzed in a Ph.D. thesis by G.R.Pugh (see [6]), where the author clarified many subtle points in the Lanczos original paper and proved rigorously the validity of the limit formula for $\mathrm{R}e(s)>0$.

For $\mathrm{R}e(s)>0$ formula (1) can be formally derived as follows: we use the fact that $\frac{\Gamma(n+s/2)}{\Gamma(n+1-s/2)}\sim n^{s-1}$ as $n\to\infty$ to obtain:

$$\begin{aligned}\Gamma\left(s/2\right) &= 2\int\limits_0^\infty \exp(-x^2)x^{s-1}dx = 2\lim_{\gamma\to+\infty}\gamma^{-1}\sum_{n\geq 0}\exp\left(-n^2/\gamma^2\right)\left(n/\gamma\right)^{s-1}\\ &= 2\lim_{\gamma\to+\infty}\gamma^{-s}\sum_{n\geq 0}\exp\left(-n^2/\gamma^2\right)\frac{\Gamma\left(n+s/2\right)}{\Gamma\left(n+1-s/2\right)},\end{aligned}$$

and Lanczos limit formula (1) can be obtained from the above equation. Thus we see that the sum in the equation (1) is an analogue of the Riemann sum discretization for the definite integrals.

In this paper our goal is to generalize (1), to prove its validity in the entire complex plane and to obtain the higher order correction terms (see section 2). In section 3 we apply these results to Gamma function and Riemann zeta function, and as another application we derive an "elementary" characterization of the nontrivial zeros of the Riemann zeta function (which does not involve analytic continuation and is similar to the one derived in [7]).

At last we would like to mention that in this paper we use the following notations:

- The prime in the sum notation indicates that the first term is multiplied by half.
- The Pochhammer symbol is defined as $(a)_n=a(a+1)(a+2)\dots(a+n-1)$.

2 Lanczos limit formula: general setup

We will use the following notation for the Mellin transform: $[\mathcal{M}f](s)=\int\limits_0^\infty f(x)x^{s-1}dx$ (see [4],[5]).

Theorem 1. *Assume that $f(x)$ is an even function, analytic in some region $\mid Im(x)\mid<\epsilon$ and such that all $f^{(i)}(x)$ decrease exponentially as $Re(x)\to\infty$. Then for all $s\in\mathbb{C}$, $(s\neq 0,-2,-4,\dots)$*

as $\gamma \to +\infty$

$$[\mathcal{M}f](s) = \gamma^{-s}{\sum_{n\geq 0}}' f(n/\gamma)\frac{\Gamma(n+s/2)}{\Gamma(n+1-s/2)} + \qquad (2)$$
$$+ \sum_{m=1}^{N-1}(-1)^{m+1}(2\gamma)^{-2m}\frac{p_m(s)}{(2m)!}(1-s)_{2m}[\mathcal{M}f](s-2m) + O\left(\gamma^{-2N}\right).$$

Polynomials $p_m(s)$ *have generating function*

$$\sum_{m\geq 0}\frac{p_m(s)}{(2m)!}u^{2m} = \left(\frac{u}{\sin(u)}\right)^s \qquad (3)$$

and can be computed recursively

$$2n\ p_n(s) = s\sum_{k=1}^{n}\binom{2n}{2k}4^k \mid B_{2k} \mid\ p_{n-k}(s), \quad p_0(s) = 1, \qquad (4)$$

where B_k *are Bernoulli numbers.*

Proof. Define $\tilde{f}(y) = \int\limits_0^\infty f(x)\cos(xy)dx$. Conditions on $f(x)$ guarantee that $\tilde{f}(y)$ is analytic in some region $\mid \mathrm{I}m(s) \mid < \epsilon_1$ and that $\tilde{f}(y)$ decreases exponentially as $y \to \infty$. We start with the integral

$$I(s) = \int\limits_L \tilde{f}(y)y^{-s}dy = \pi\frac{\exp(-\pi i s/2)}{\Gamma(s)}[\mathcal{M}f](s),$$

where the contour of integration is $L = (-\infty, -\epsilon_2] \cup \{\epsilon_2\exp(i\alpha);\ 0 < \alpha < \pi\} \cup [\epsilon_2, \infty)$ for some $\epsilon_2 < \epsilon_1$. Next we choose $\gamma \geq 1$, define $L_1 = (-\infty, -\pi\gamma/2] \cup [\pi\gamma/2, \infty)$ and $L_0 = \gamma^{-1}(L \setminus L_1) = (-\pi/2, -\epsilon_2/\gamma] \cup \{\epsilon_2\exp(i\alpha)/\gamma;\ 0 < \alpha < \pi\} \cup [\epsilon_2/\gamma, \pi/2)$, and separate $I(s)$ into three terms:

$$I(s) = I_1(s,\gamma) + I_2(s,\gamma) + I_3(s,\gamma) \qquad (5)$$
$$= \gamma^{1-s}\int\limits_{L_0}\tilde{f}(\gamma u)\sin(u)^{-s}du + \gamma^{1-s}\int\limits_{L_0}\tilde{f}(\gamma u)u^{-s}\left[1 - \left(\frac{u}{\sin(u)}\right)^s\right]du + \int\limits_{L_1}\tilde{f}(u)u^{-s}du.$$

Our goal is to show that $I_1(s,\gamma)$ can be transformed into the main sum in equation (2), $I_2(s,\gamma)$ gives the asymptotic correction terms and $I_3(s,\gamma)$ decreases exponentially as $\gamma \to \infty$.

First, using the the Poisson summation formula

$$\tilde{f}(\gamma u) = 2\gamma^{-1}{\sum_{n\geq 0}}' f(2n/\gamma)\cos(2nu) - \sum_{n\geq 1}\left[\tilde{f}((\pi n + u)\gamma) + \tilde{f}((\pi n - u)\gamma)\right]$$

and the fact that $\tilde{f}$ decreases exponentially we obtain as $\gamma \to \infty$

$$I_1(s,\gamma) = 2\gamma^{-s}\pi^{3/2}\frac{\exp(-\pi i s/2)}{\Gamma((1+s)/2)}{\sum_{n\geq 0}}' f(2n/\gamma)\frac{(s/2)_n}{\Gamma(n+1-s/2)} + O(\exp(-\epsilon_4\gamma)),$$

where we have also used the following integral (see [2], [6])

$$\int_0^{\pi/2} \cos(2nu)\sin(u)^{-s}dx = \left(\pi^{1/2}/2\right)\Gamma\left((1-s)/2\right)\frac{(s/2)_n}{\Gamma\left(n+1-s/2\right)}.$$

Next we use (3) and obtain that as $\gamma \to \infty$

$$I_2(s,\gamma) = \pi\frac{\exp\left(-\pi is/2\right)}{\Gamma(s)}\sum_{m=1}^{N-1}(-1)^{m+1}\gamma^{-2m}\frac{p_m(s)}{(2m)!}(1-s)_{2m}[\mathcal{M}f](s-2m) + O\left(\gamma^{-2N}\right).$$

And finally we find that $I_3(s,\gamma) = O\left(\exp\left(-\epsilon_3\gamma\right)\right)$ as $\gamma \to \infty$, which follows from the fact that $\tilde{f}(u)$ decreases exponentially as $u \to \infty$.

Now, in order to obtain equation (2) we collect all the terms in (5), use the double argument formula for $\Gamma(s)$ and rescale $\gamma \to 2\gamma$. Recursion (4) for $p_m(s)$ can be derived by differentiating the generating function in equation (3). □

Remark 1: Using recursion relation (4) we compute polynomials $p_1(s), \dots, p_5(s)$:

$$\begin{aligned}
p_1(s) &= \frac{s}{3}\\
p_2(s) &= \frac{s}{15}(5s+2)\\
p_3(s) &= \frac{s}{63}(35s^2+42s+16)\\
p_4(s) &= \frac{s}{135}(175s^3+420s^2+404s+144)\\
p_5(s) &= \frac{s}{99}(385s^4+1540s^3+2684s^2+2288s+768)
\end{aligned}$$

In the following Corollary we include the correction terms into the main sum and thus remove the dependence on $[\mathcal{M}f](s-2m)$ (the proof can be obtained by iterating equation (2)):

Corollary 2. *Assume that $f(x)$ satisfies all the conditions of Theorem 1. Then for all $s \in \mathbb{C}$ ($s \neq 0, -2, -4, \dots$) as $\gamma \to +\infty$*

$$\begin{aligned}
[\mathcal{M}f](s) &= \gamma^{-s}{\sum_{n\geq 0}}' f\left(n/\gamma\right)\frac{\Gamma\left(n+s/2\right)}{\Gamma\left(n+1-s/2\right)} \\
&\times \left[1-\sum_{m=1}^{N-1}\frac{q_m(s)}{2^{2m}(2m)!}\frac{(1-s)_{2m}}{\left(1-s/2+n\right)_m\left(1-s/2-n\right)_m}\right] + O\left(\gamma^{-2N}\right)
\end{aligned} \tag{6}$$

where polynomials $q_m(s)$ are computed recursively: $q_m(s) = p_{m,m-1}(s)$, $p_{k,0}(s) = p_k(s)$ and

$$p_{k,m}(s) = p_{k,m-1}(s) - \binom{2k}{2m}q_{m-1}(s)p_{k-m}(s-2m), \quad m \geq 0, \quad k \geq m+1. \tag{7}$$

Remark 2: Using recursion relation (7) we compute polynomials $q_1(s), \dots, q_5(s)$:

$$\begin{aligned}
q_1(s) &= \frac{s}{3} \\
q_2(s) &= \frac{s}{15}(-5s+22) \\
q_3(s) &= \frac{s}{63}(35s^2-462s+1528) \\
q_4(s) &= \frac{7s}{45}(25s^3-420s^2+2156s-3024) \\
q_5(s) &= \frac{14s}{33}(55s^4-1100s^3+7436s^2-19360s+18624)
\end{aligned}$$

3 Examples: Gamma function and Riemann zeta function

Applying Theorem 1 and Corollary 2 to function $f(x) = \exp(-x^2)$ and rescaling $\gamma \to \gamma^{1/2}$ and $s \to 2-2s$ we obtain a generalization of the Lanczos original formula:

Corollary 3. *For all $s \in \mathbb{C}$ ($s \neq 0, -1, -2 \dots$) as $\gamma \to +\infty$*

$$\begin{aligned}
\Gamma(s) &= 2\left[1+\sum_{m=1}^{N-1}\gamma^{-m}\frac{p_m(2-2s)}{(2m)!}(s-1/2)_m\right]^{-1} \\
&\times \gamma^{s-1}\sum_{n\geq 0}{}'\exp\left(-n^2/\gamma\right)\frac{(1-s)_n}{(s)_n}+O\left(\gamma^{-N}\right)
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\Gamma(s) &= 2\gamma^{s-1}\sum_{n\geq 0}{}'\exp\left(-n^2/\gamma\right)\frac{(1-s)_n}{(s)_n} \\
&\times \left[1-\sum_{m=1}^{N-1}\frac{q_m(2-2s)}{2^{2m}(2m)!}\frac{(2s-1)_{2m}}{(s+n)_m(s-n)_m}\right]+O\left(\gamma^{-N}\right)
\end{aligned} \tag{9}$$

As another example we provide limit representation for the Riemann zeta function. Let us define $\xi(s) = (1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$: it is known that $\xi(s)$ is an entire function which is real for s real and $\xi(s) = \xi(1-s)$ (see [8]). Applying Theorem 1 and Corollary 2 to $f(x) = \frac{x^2}{\sinh(x)^2}$ and rescaling $s \to -s$ we obtain

Corollary 4. *For all $s \in \mathbb{C}$ as $\gamma \to +\infty$*

$$\begin{aligned}
\xi(s) &= \gamma^s\sum_{n\geq 0}{}'\left[\frac{\pi^{1/2}n\gamma^{-1}}{\sinh\left(\pi^{1/2}n\gamma^{-1}\right)}\right]^2\frac{(-s/2)_n}{\Gamma(n+1+s/2)} \\
&- \sum_{m=1}^{N-1}\gamma^{-2m}\frac{p_m(-s)}{(2m)!}((1+s)/2)_m\,\xi(s+2m)+O\left(\gamma^{-2N}\right)
\end{aligned} \tag{10}$$

and

$$\xi(s) = \gamma^s \sum_{n\geq 0}{}' \left[\frac{\pi^{1/2} n\gamma^{-1}}{\sinh\left(\pi^{1/2} n\gamma^{-1}\right)}\right]^2 \frac{(-s/2)_n}{\Gamma\left(n+1+s/2\right)} \times \left[1 - \sum_{m=1}^{N-1} \frac{q_m(-s)}{2^{2m}(2m)!} \frac{(s+1)_{2m}}{(1+s/2+n)_m\,(1+s/2-n)_m}\right] + O\left(\gamma^{-2N}\right) \tag{11}$$

As an application of Corollary 4 we derive an interesting characterization of the nontrivial zeros of the Riemann zeta function. In [7] the author defined the asymptotic convergence degree as

$$\alpha_0(s) = \sum \left\{ a:\ N^{-1} \sum_{n=1}^{N} (n/N)^{-s} = \int_0^1 x^{-s} dx + O\left(N^{-a}\right) \text{ as } N \to +\infty \right\},$$

and it was shown (with the help of the Euler-Maclaurin formula for the zeta function) that $\alpha_0(s)$ is a continuous function on $\mathbb{C} \setminus \{0, 1\}$ except for a jump of $\mathrm{R}e(\rho)$ at each nontrivial zero ρ of the $\zeta(s)$. Following [7] we define the asymptotic convergence degree

$$\alpha(s) = \sup \left\{ a:\ \sum_{n\geq 0}{}' \left[\frac{n\gamma^{-1}}{\sinh\left(n\gamma^{-1}\right)}\right]^2 \frac{(-s/2)_n}{(1+s/2)_n} = O\left(\gamma^{-a}\right) \text{ as } \gamma \to +\infty \right\}.$$

From equation (10) we see that $\alpha(s)$ has jumps of size 2 at the nontrivial zeros of $\zeta(s)$, while it is continuous everywhere else in the complex plane:

$$\alpha(s) = \begin{cases} \mathrm{R}e(s) & \text{if } \xi(s) \neq 0 \\ \mathrm{R}e(s) + 2 & \text{if } \xi(s) = 0 \end{cases}$$

Thus the Riemann Hypothesis is equivalent to each of the following "elementary" statements:

(i) $\alpha(s) < 2$ for $1/2 < \mathrm{R}e(s) < 1$ (or $0 < \mathrm{R}e(s) < 1/2$).

(ii) $\alpha(s)$ is continuous for $1/2 < \mathrm{R}e(s) < 1$ (or $0 < \mathrm{R}e(s) < 1/2$).

Remark 3: Note that using the equation (11) one could define more general functions $\alpha_N(s)$ for every integer $N \geq 1$, which have jumps of the size $2N$ at the nontrivial zeros of $\zeta(s)$.

We would also like to mention that one could derive yet another approximation for the function $\xi(s)$, which is also valid in the entire complex plane:

$$\xi(s) = (1-s)\gamma^s \sum_{k\geq 0}{}' \exp\left(-ik^2/\gamma^2\right) \frac{\pi^{1/2} k\gamma^{-1}}{\sinh\left(\pi^{1/2} k\gamma^{-1}\right)} \frac{(-s/2)_k}{\Gamma\left(1+s/2+k\right)} + s\gamma^{1-s} \sum_{k\geq 0}{}' \exp\left(ik^2/\gamma^2\right) \frac{\pi^{1/2} k\gamma^{-1}}{\sinh\left(\pi^{1/2} k\gamma^{-1}\right)} \frac{((s-1)/2)_k}{\Gamma\left(1+(1-s)/2+k\right)} + O\left(\gamma^{-2}\right). \tag{12}$$

The above equation can be derived with the help of the integral representation (see [8]) $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \Upsilon(s) + \overline{\Upsilon}(1-s)$, where

$$\Upsilon(s) = \exp(-\pi i s/2)\, 2^{-1-s}\pi^{-s/2-1/2}\Gamma((1-s)/2) \int\limits_{\alpha i+\exp(\pi i/4)\mathbb{R}} \exp\left(iw^2/(4\pi)\right) \frac{w^{s-1}}{\sinh(w/2)} dw.$$

Note that the above approximation is also invariant with respect to transformation $s \to 1-s$, and thus satisfies the functional equation.

4 Conclusion

In this article we present an analogue of the Euler-Maclaurin summation: a formula which represents the Mellin transform $[\mathcal{M}f](s)$ as a sum containing $f(n/\gamma)$ (an analogue of the Riemann sum) and higher order asymptotic correction terms. While the original Euler-Maclaurin summation applied to the integrals of Mellin type can give approximation only in some half plane R$e(s) > c$ (see [7] for the Euler-Maclaurin formula for $\zeta(s)$), approximations (2) and (6) converge in the entire complex plane, and thus can serve as analytic continuation for certain meromorphic functions. It is not clear at this point what can be the applications of these formulas: equation 2 can be used for the numerical evaluation of the Mellin transform $[\mathcal{M}f](s)$, but this procedure will be efficient only for s not large (the constant in $O\left(\gamma^{-2N}\right)$ becomes large when s increases) and for which R$e(s)$ is small (otherwise the factor γ^{-s} will become large and will introduce round-off errors). Thus it seems that the Euler-Maclaurin formula, when it can be applied, is better suited for the numerical computations.

At last we would like to mention that results, similar to equations (10) and (12), can also be derived for other Dirichlet L-functions, as well as for some other special functions. One can also generalize Theorem 1 by removing some of the conditions on $f(x)$, but it seems that in this case the approximation will be valid only in some half plane R$e(s) > c$.

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