Transformations of Markov processes and classification scheme for solvable driftless diffusions

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Abstract

Analytically tractable driftless diffusions, such as geometric Brownian motion, constant elasticity of variance (CEV) process and processes with quadratic volatility, are the main building blocks in various derivative pricing models in Mathematical Finance. All these processes share an important analytical property: the corresponding backward Komogorov equation can be solved explicitly by a reduction to the hypergeometric equation. In this article we classify all driftless diffusions which satisfy the above property. This classification scheme includes all known one-dimensional stationary driftless diffusion processes as well as new rich families of analytically tractable diffusions.

Keywords: driftless diffusions, Doob's h-transform, scale transformation, Liouville transformation, option pricing models.

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1 Introduction

The applicability of option pricing models in Mathematical Finance hinges on the computability of prices for various liquidly traded derivatives and the flexibility of the model to reflect econometric evidence and to be calibrated to market data. Analytic closed form solvability is the traditional way of facilitating computability by reducing to the evaluation of hypergeometric functions. Within the class of solvable models, the more realistic a model is, the smaller is the chance of obtaining any explicit result. Therefore it is important to study "solvable" diffusion processes and to try to find as many of them as possible. In this paper, we present a classification scheme of solvable one-dimensional stationary driftless diffusions, where we define a process as being "solvable", if we can transform an eigenvalue problem corresponding to the Kolmogorov equation into a (generalized) hypergeometric equation.

There has been a lot of work in the last several decades both on analytically tractable diffusions and on reduction of diffusions to simpler processes. For example, [9] and [4] study the question of transforming the Kolmogorov equation for a given diffusion process into the heat equation. In [5] the author uses the theory of Lie groups to study the more general question of mapping a two-dimensional partial differential equation (PDE) into a PDE with constant coefficients. In [2] the authors introduce a general method to transform diffusion processes by means of diffeomorphisms and measure changes into a driftless diffusion while preserving solvability. If the original process is solvable, as for instance would be the Bessel process, the method in [2] allows one to express the transition probability density function (pdf) of the new process in terms of that of the original process (see also [1]). This technique was generalized in [3] and [12] and translated into probabilistic language of Doob's h-transform and scale transformation, which in turn allowed to apply these methods to birth-and-death processes. The paper [8] studies solvable time-dependent driftless diffusions, while in [10] the author uses supersymmetric methods to generate new analytical solutions to the Kolmogorov and Black-Scholes equations. We would also like to mention an interesting result proved in [13] on the classification of diffusion processes whose Markov generator admits orthogonal polynomials as eigenfunctions.

On the other hand, there have appeared several important papers in Theoretical Physics, which addressed the following, closely related, question: for which potential terms $U(x)$ the corresponding Schrödinger equation $-f''(x)+U(x)f(x) = Ef(x)$ can be solved in terms of hypergeometric functions? It seems that the first results in this area were obtained in [7] and [15], and were subsequently generalized in [14], which in turn became the main inspiration for our work in this area.

In this paper, we work with a stationary one-dimensional Markov diffusion processes X_t defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathrm{P})$, taking values in an interval $D_X \subseteq \mathbb{R}$ and started in the interior of D_X. Local dynamics of X_t is defined by the Markov generator

$$\mathcal{L}_X = b(x)\partial_x + \frac{1}{2}\sigma(x)^2\partial_x^2. \tag{1}$$

We also assume that coefficients $b(x)$ and $\sigma(x)$ are smooth and bounded in the D_X, and that $\sigma(x)$ is uniformly bounded away from zero. We will also use notation (X_t, P) when it is necessary to specify the reference measure.

Remark 1. A full specification of the dynamics of X_t would require a discussion of the boundary conditions. However we intentionally do not address this subject, as we are only interested in the behavior of X_t up to the first time it hits the boundary. In particular, by shrinking domain D_X if necessary, we can always assume that both boundaries are regular absorbing.

The following class of transformation was introduced in [2] and was later studied in [3] and [12].

Definition 2. A *stochastic transformation* of (X_t, P) is defined by a triple $\{\rho, h, Y\}$ where

- $h(x)$ is a solution to $\mathcal{L}_X h = \rho h$, satisfying $h(X_0) > 0$

- $Y(x) : D_X \mapsto D_Y \subseteq \mathbb{R}$ is a diffeomorphism

such that the process $Y_t = Y(X_t)$ has zero drift under the new measure P^h which is defined as the Doob's h-transform of the measure P:

$$\frac{\mathrm{d}\mathrm{P}_t^h}{\mathrm{d}\mathrm{P}_t} = e^{-\rho t} h(X_t). \tag{2}$$

Remark 3. Notice that we do not require the function $h(x)$ to be positive everywhere in D_X. The condition $h(X_0) > 0$ guarantees that there exists an interval $\tilde{D}_X \subseteq D_X$ such that $X_0 \in \tilde{D}_X$ and $h(x) > 0$ for all $x \in \tilde{D}_X$. Thus we can just restrict process X_t to $\tilde{D}_X$ by stopping it at the first time it touches the boundary of $\tilde{D}_X$.

The following theorem, proved in [3] and [12], completely characterizes stochastic transformations in terms of solutions of an eigenvalue equation $\mathcal{L}_X f = \rho f$.

Theorem 1. *$\{\rho, h, Y\}$ is a stochastic transformation if and only if*

$$\begin{cases} h(x) = c_1 f_1(x) + c_2 f_2(x) \\ h(x)Y(x) = c_3 f_1(x) + c_4 f_2(x) \end{cases} \tag{3}$$

where $f_1(x)$ and $f_2(x)$ are two linearly independent solutions to a second order linear ODE $\mathcal{L}_X f = \rho f$ and $c_i \in \mathbb{R}$ are such that $c_1 c_4 - c_2 c_3 \neq 0$ and $h(X_0) > 0$.

The proof of Theorem 1 is rather simple (see [3], [12]) and is based on the following two well-known facts: the Doob's h-transform transforms the generator $\mathcal{L}_X$ to

$$\mathcal{L}_X^h = \frac{1}{h(x)} \mathcal{L}_X h(x) - \rho$$

and the condition that $Y(X_t)$ is driftless under P^h is equivalent to $\mathcal{L}_X^h Y(x) = 0$, which can be rewritten as

$$\mathcal{L}_X \left[h(x) Y(x) \right] = \rho \left[h(x) Y(x) \right],$$

which in turn implies the second equality in equation (3). The fact that $Y(x)$ is a diffeomorphism can be easily established by showing that $Y'(x) \neq 0$ with the help of the Wronskian of equation $\mathcal{L}_X f = \rho f$ (see [2],[3], [12] for all the details).

Definition 4. Let (X_t, P) and (Y_t, Q) be two driftless diffusion processes. We say that they are *equivalent*

$$(X_t, \mathrm{P}) \sim (Y_t, \mathrm{Q})$$

if there exists a stochastic transformation $\{\rho, h, Y\}$ of (X_t, P) which maps it into (Y_t, Q). An equivalence class of (X_t, P) will be denoted as

$$\mathfrak{M}(X_t) = \mathfrak{M}(X_t, \mathrm{P}) = \{(Y_t, \mathrm{Q}) : \quad (X_t, \mathrm{P}) \sim (Y_t, \mathrm{Q})\}.$$

It remains to check that definition 4 correctly defines an equivalence relation. Reflexivity is obvious (take $\rho = 0$, $h(x) \equiv 1$ and $Y(x) = x$), the proof of transitivity is rather simple and the following lemma states that symmetry holds.

Lemma 2. *Assume (X_t, P) is driftless and $\{\rho, h, Y\}$ maps (X_t, P) to (Y_t, Q). Define $X(y) : D_Y \mapsto D_X$ to be an inverse of $Y(x)$. Then $\{-\rho, 1/h(X(y)), X(y)\}$ is a stochastic transformation of (Y_t, Q) which maps (Y_t, Q) to (X_t, P).*

Remark 5. An equivalence class $\mathfrak{M}(X_t)$ can be similarly defined even when process X_t has nonzero drift: in this case it would be a class of all driftless processes Y_t that can be obtained from X_t by a stochastic transformation. Note that in this case $X_t \notin \mathfrak{M}(X_t)$.

Next we present the central definition in this paper:

Definition 6. The diffusion process X_t is called *solvable*, if there exists a ρ-independent change of variables $z = Z(x)$ and a function $h(z,\rho)$ such that all solutions to $\mathcal{L}_X f(x) = \rho f(x)$ are of the form $h(Z(x),\rho)F(Z(x),\rho)$, where $F(z,\rho)$ is a solution to the (generalized) hypergeometric equation

$$A(z)f''(z) + B(z)f'(z) + Cf(z) = 0. \tag{4}$$

Here $A(z) \in P_2$, $B(z) \in P_1$ and $C \in \mathbb{C}$, where P_i denotes the set of all polynomials over $\mathbb{C}$ of degree less than or equal to i.

The above terminology is justified by the fact that if we can compute all solutions to an eigenvalue equation $\mathcal{L}_X f = \rho f$, then we know the Green's function of a diffusion process in closed form (see [6]), which by itself can provide a rich amount of important information about the process, and which can be also used for expressing the transitional probability function as an inverse Laplace transform. Another convincing argument for the importance of this definition is the fact that all known one-dimensional stationary driftless processes which have any degree of analytical tractability seem to belong to this class.

Remark 7. The requirement that the change of variables be independent of ρ cannot be dropped since otherwise every diffusion process would be solvable, see [14] for the discussion of a similar situation in the case of solvable Schrödinger equations.

2 Invariance under stochastic transformations

In this section we study the following question: under what conditions can a driftless process Y_t be obtained from a diffusion process X_t by a stochastic transformation? Theorem 3 provides a simple answer to this question: we just need to compute a certain function for both processes X_t and Y_t which is defined in terms of the drift and diffusion coefficients of the Markov generator, and which is invariant under stochastic transformations. By comparing these invariant functions, we can easily decide whether the two processes are related.

Firstly, let us introduce the following two types of transformations of a second order linear ODE: $\mathcal{L}_X f = \rho f$ which correspond to change of variable (Ito's formula) and Doob's h-transform. A third type of transformation, which does not have an immediate probabilistic interpretation is introduced in the next section and will be central in our derivation of classification results.

In this section, we assume that the Markov generator of X_t is given by (1).

(i) *Transformation of type I:* change of variables $y = Y(x)$

$$\mathcal{L}_X f = \rho f \mapsto \left[\mathcal{L}_X Y(x)\right]\partial_y g + \frac{1}{2}\left[\sigma(x)Y'(x)\right]^2 \partial_y^2 g = \rho g. \tag{5}$$

The solutions to the transformed equation are given $g(y) = f(X(y))$, where $x = X(y)$ is the inverse of $Y(x)$ and $f(x)$ is a solution of the original equation.

(ii) *Transformation of type II:* gauge transformation (a combination of left and right multiplication)

$$\begin{aligned} \mathcal{L}_X f = \rho f \quad &\mapsto \quad h^{-1}\mathcal{L}_X h g = \rho g \\ &\leftrightarrow \quad \left[b(x) + \sigma(x)^2\frac{h'(x)}{h(x)}\right]\partial_x g(x) + \frac{1}{2}\sigma(x)^2\partial_x^2 g(x) + \frac{\mathcal{L}h(x)}{h(x)}g(x) = \rho g(x). \end{aligned} \tag{6}$$

The solutions to the transformed equation are given $g(x) = f(x)h(x)^{-1}$, where $f(x)$ is a solution of the original equation.

The next theorem is our main result in this section. It provides an invariant with respect to stochastic transformations. The proof is based on the reduction of an eigenfunction equation $\mathcal{L}_X f = \rho f$ to a certain canonical form using the above two types of transformations. The next step is to verify that the potential term in this canonical form is actually an invariant.

Theorem 3. *For a diffusion process X_t defined by a generator (1) we define:*

$$J_X(z) = \frac{1}{4}\left(\sigma(x)\sigma''(x) - \frac{1}{2}(\sigma'(x))^2 + 2\left[\frac{2b(x)\sigma'(x)}{\sigma(x)} - b'(x) - \frac{b^2(x)}{\sigma^2(x)}\right]\right), \tag{7}$$

where the change of variables $x = X(z)$ is defined by $dx = \sigma(x)dz$. Similarly we define $J_Y(z)$ for a driftless process Y_t. Then Y_t can be obtained as a stochastic transformation $\{\rho, h, Y\}$ of X_t if and only if there exists $z_0 \in \mathbb{R}$ such that $J_Y(z) \equiv J_X(z - z_0) - \rho$.

Proof. The main idea is to transform operator $\mathcal{L}_X$ into the following canonical form:

$$\mathcal{L}_X = b(x)\partial_x + \frac{1}{2}\sigma(x)^2\partial_x^2 \mapsto \mathcal{L}_c = \frac{1}{2}\partial_z^2 + I(z). \tag{8}$$

It is easy to see that the above form cannot be changed by using only gauge transformations and changes of variables: any nontrivial change of variables would necessarily introduce a "volatility" term as in (5), while any gauge transformation would introduce a "drift" term, as in (6). This means that function $I(z)$ is an invariant for all processes in $\mathfrak{M}(X_t)$.

Thus all we need to do is to find this function $I(z)$. We achieve this in two steps: first change of variables $z = Z(x)$ so that $dz = dx/\sigma(x)$, this transforms $\mathcal{L}_X$ into

$$\mathcal{L}_X \mapsto \tilde{\mathcal{L}} = \frac{1}{2}\partial_z^2 + \left[\frac{b(x)}{\sigma(x)} - \frac{\sigma'(x)}{2}\right]\partial_z$$

Next, to remove the "drift" term we perform a gauge transformation with factor $h(z)$ defined by

$$\frac{dh}{dz} = -h\left[\frac{b(x)}{\sigma(x)} - \frac{\sigma'(x)}{2}\right].$$

Using formula (6) we find that the potential term is given by $I(z) = h^{-1}(z)\tilde{\mathcal{L}}h(z)$, which after some algebraic manipulations can be transformed into (7), thus completing the proof. □

Below we present two examples of how one can use the above theorem.

Example 8. Consider a process Y_t from the quadratic volatility family

$$\sigma_Y(y) = a_2y^2 + a_1y + a_0$$

and let us verify how this reduces to a Brownian motion $X_t = W_t$ by a stochastic transformation. Using theorem 3 this becomes a trivial exercise, as one can easily check that $J_W(z) \equiv 0$ and $J_Y(z) \equiv \text{const}$.

Example 9. A less trivial example is to show that a Bessel process $dX_t = adt + \sqrt{X_t}dW_t$ is related to the constant elasticity of variance (CEV) process $dY_t = Y_t^\theta dW_t$. Again, we use (7) to find that

$$J_X(z) = z^{-2}(-2a^2 + 2a - 3/8), \quad J_Y(z) = \frac{1}{4}z^{-2}\frac{\frac{\theta^2}{2} - \theta}{(1-\theta)^2},$$

and when $\theta = 1 - \frac{1}{2(2a-1)}$ we have $J_Y(z) = J_X(z)$, thus processes Y_t should be related to the Bessel process by a stochastic transformation $\{\rho, h, Y\}$ with $\rho = 0$.

3 Classification Results

In this section we prove two classification results, which completely characterize stationary one-dimensional driftless diffusions which are solvable according to definition 6. The first theorem provides an explicit form of the volatility function of a solvable process Y_t, while the second theorem describes all solvable processes as those which belong to an equivalence class $\mathfrak{M}(X_t)$ of some rather simple diffusion process X_t. The techniques that we'll use to prove these results are similar to those used in [14] and [15].

As we have discussed in the previous section, stochastic transformations preserve the potential term in the canonical form (8) of the eigenvalue equation $\mathcal{L}_X f = \rho f$. The central tool for our classification results is the third type of transformation, which preserves solvability and at the same time allows one to change the potential term as follows:

(iii) *Transformation of type III:* left multiplication of both sides of an equation by a function $\gamma(x)$

$$\begin{aligned} \mathcal{L}_X f = \rho f \quad &\mapsto \quad \gamma \mathcal{L}_X f = \gamma \rho f \\ &\leftrightarrow \quad \gamma(x) b(x) \partial_x f(x) + \frac{1}{2}\gamma(x)\sigma(x)^2 \partial_x^2 f(x) = \rho \gamma(x) g(x). \end{aligned} \tag{9}$$

This transformation does not alter the solutions of the eigenvalue equation.

To prove our first classification result, we start with an eigenvalue equation $\mathcal{L}_X f = \rho f$ for a diffusion process and reduce it to canonical form. Then we do the same for the hypergeometric equation (4). By equating potential terms of these canonical forms, we ensure that the eigenvalue equation can be mapped into an hypergeometric equation (4), and thus the process is solvable in the sense of definition 6. It turns out that the condition that potential terms are equal for all z and ρ can provide an explicit characterization of a solvable process.

Theorem 4. *A driftless process Y_t is solvable in the sense of definition 6 if and only if its volatility function is of the following form:*

$$\sigma_Y(y) = \sigma_Y(Y(z)) = D\frac{W(z)}{(c_1F_1(z) + c_2F_2(z))^2}\frac{A(z)}{\sqrt{R(z)}}, \tag{10}$$

where $D \in \mathbb{R}$, $A(z)$ and $R(z)$ are polynomials of degree less than or equal than two, the change of variables is given by

$$y = Y(z) = \frac{c_3F_1(z) + c_4F_2(z)}{c_1F_1(z) + c_2F_2(z)}, \quad c_1c_4 - c_3c_2 \neq 0, \tag{11}$$

where $F_i(z)$ are linearly independent solutions to (4) and $W(z)$ is the Wronskian of equation (4) given by

$$W(z) = \exp\left(-\int^z \frac{B(t)}{A(t)}dt\right). \tag{12}$$

Proof. We start with a driftless process Y_t defined by a volatility function $\sigma_Y(y)$. Let us consider an eigenvalue equation $\mathcal{L}_Y f = \rho f$:

$$\frac{1}{2}\sigma_Y^2(y) f''(y) = \rho f(y) \tag{13}$$

To reduce this equation to the hypergeometric equation (4), we multiply both sides of equation (14) by a ρ-independent function $\gamma(y)$ to be specified later

$$\sigma_Y^2(y)\gamma(y) f''(y) = 2\rho\gamma(y) f(y)$$

Next, we reduce the above equation into canonical form

$$f''(z) + I_1(z)f(z) = 0,$$

using only a change of variables and a gauge transformation. This can be achieved in the following way: we apply the change of variables

$$\frac{dy}{dz} = \sigma_Y(y)\sqrt{\gamma(y)} \tag{14}$$

followed by a gauge transformation with function $h(z) = \sqrt{dy/dz}$. One can check that this leads to an equation in canonical form with a potential term

$$I_1(z) = \frac{1}{2}\{y, z\} - 2\rho\gamma(y(z)),$$

where $\{y, z\}$ is the *Schwarzian derivative* of y with respect to z (see [11]):

$$\{y, z\} = \left(\frac{y''(z)}{y'(z)}\right)' - \frac{1}{2}\left(\frac{y''(z)}{y'(z)}\right)^2.$$

The next step is to reduce the hypergeometric equation (4) to canonical form. The simplest way of achieving this is to apply a gauge transformation with $h(z) = \sqrt{W(z)}$, where $W(z)$ is the Wronskian given by equation (12) and divide both sides of the equation by $A(z)$. One can check that this yields a potential term

$$I_2(z) = \frac{4A(z)C - 2A(z)B(z)' + 2B(z)A(z)' - B(z)^2}{4A(z)^2}.$$

Notice that, by varying $A(z) \in P_2$, $B(z) \in P_1$ and $C \in \mathbb{R}$, we obtain an arbitrary polynomial of degree two or less in the numerator of $I_2(z)$. This can be easily verified by checking three cases corresponding to the degree of polynomial $A(z)$ being equal to zero, one or two. We can thus write $I_2(z)$ as

$$I_2(z) = \frac{Q(z)}{A(z)}. \tag{15}$$

In order to ensure that the eigenvalue equation $\mathcal{L}_X f = \rho f$ can be transformed into a hypergeometric equation, we want the potential term $I_1(z)$ to be equal to the potential term of the generalized hypergeometric equation $I_2(z)$:

$$\frac{1}{2}\{y, z\} - 2\rho\gamma(y(z)) = \frac{Q(z)}{A(z)^2} \quad \text{for all} \quad z, \rho \in \mathbb{C} \tag{16}$$

where $Q(z)$ and $A(z)$ are polynomials in P_2 with coefficients possibly dependent on ρ. Note that $\{y, z\}$ and $\gamma(y(z))$ are both independent of ρ, therefore by setting first $\rho = 0$ and then $\rho = 1$ we find that both functions $\{y, z\}$ and $\gamma(y(z))$ must be rational functions in z, and then by considering all the different cases, depending on the degrees of these rational functions, one can prove (see [14] for all the details) that equation (16) implies that there exist two polynomials $T(z), R(z) \in P_2$, independent of ρ, such that $Q(z) = T(z) - 2\rho R(z)$. Thus we arrive at the following system of equations

$$\begin{cases} \frac{1}{2}\{y, z\} = \frac{T(z)}{A^2(z)} \\ \gamma(y(z)) = \frac{R(z)}{A^2(z)} \end{cases} \tag{17}$$

According to Theorem 10.1.1 in [11], all the solutions to the equation $\frac{1}{2}\{y,z\} = \frac{T(z)}{A^2(z)}$ are given by

$$y(z) = \frac{c_3 f_1(z) + c_4 f_2(z)}{c_1 f_1(z) + c_2 f_2(z)}, \tag{18}$$

where f_1 and f_2 are linearly independent solutions to the equation

$$f''(z) + \frac{T(z)}{A^2(z)} f(z) = 0. \tag{19}$$

The above equation is already in canonical form, with potential term of the same class as the potential term (15) for hypergeometric equation, thus we readily identify all solutions to (19) as $f(z) = F(z)/\sqrt{W(z)}$, where $F(z)$ is a solution to the hypergeometric equation (4). From this fact and equation (18) we deduce that

$$y(z) = \frac{c_3 F_1(z) + c_4 F_2(z)}{c_1 F_1(z) + c_2 F_2(z)}$$

where $F_1(z)$ and $F_2(z)$ are two linearly independent solutions to the hypergeometric equation (4).

Now we can find the volatility function $\sigma_Y(y)$. We use equation (14) to find that

$$\sigma_Y(Y(z)) = y'(z) \frac{1}{\sqrt{\gamma(y(z))}},$$

and $\gamma(y(z))$ is given by the second equation in (17). Using the properties of the Wronskian we find that the derivative $y'(z)$ can be computed as follows:

$$y'(z) = \frac{DW(z)}{(c_1 F_1(z) + c_2 F_2(z))^2},$$

which gives us formula (10) and ends the proof. □

The next theorem provides a different, more constructive, point of view on classification of solvable diffusions. Here we start with a class of rather simple diffusion processes X_t, which contains as a particular case Ornstein-Uhlenbeck, CIR and Jacobi diffusions, and then we recover all solvable diffusions as elements in the corresponding equivalence class $\mathfrak{M}(X_t)$.

Theorem 5. *A driftless process Y_t is solvable in the sense of definition 6 if and only if it can be obtained by a stochastic transformation of a process X_t with a Markov generator given by:*

$$\mathcal{L}_X = (a + bx)\frac{A(x)}{R(x)}\partial_x + \frac{1}{2}\frac{A(x)^2}{R(x)}\partial_x^2, \tag{20}$$

where $A(x), R(x) \in P_2$.

Proof. First we will prove that all the processes in the class $\mathfrak{M}(X_t)$ are solvable. As we already discussed, this statement is equivalent to establishing that the volatility function is of the form (10)) and, conversely, that any solvable process belongs to $\mathfrak{M}(X_t)$ for some process X_t with generator given by (20).

Firstly, let us characterize the processes in the class $\mathfrak{M}(X_t)$. We need to find two linearly independent solutions to the eigenvalue equation $\mathcal{L}_X f = \rho f$ with $\mathcal{L}_X$ given by (20). By dividing both sides of this equation by $A(x)$ and performing a gauge transformation to remove the drift term we arrive at the equation in the canonical form:

$$F''(x) + \frac{Q(x) - 2\rho R(x)}{A^2(x)} F(x) = 0,$$

where $Q(x) \in P_2$. The potential term in the above equation is of the same form as (15), thus we use the same argument as in the proof of theorem 4, and find that all solutions to this equation are given by $f(x) = F(x)/\sqrt{W(x)}$, where F is a solution to a hypergeometric equation (4). Using theorem 1 which characterizes all stochastic transformations, we find that the change of variables should be given by

$$Y(x) = \frac{c_1 F_1(x) + c_2 F_2(x)}{c_3 F_1(x) + c_4 F_2(x)},$$

and then the volatility function $\sigma_Y(y)$ of any process $Y_t \in \mathfrak{M}(X_t)$ is computed as

$$\sigma_Y(y) = Y'(x)\frac{A(x)}{\sqrt{R(x)}} = D\sqrt{A(x)}\frac{W(x)}{(c_1F_1(x) + c_2F_2(x))^2}\sqrt{\frac{A(x)}{R(x)}},$$

which coincides with equation (10). This means that every process in $\mathfrak{M}(X_t)$ is solvable, and it is clear that any solvable process with volatility function of the form (10) belongs to some class $\mathfrak{M}(X_t)$ with X_t defined by (20). $\square$

The process by which we have transformed an eigenvalue equation of a driftless process Y_t into a canonical form in the proof of theorem 4 is called *Liouville transformation* (see [16], [17] and [11]). We have three types of transformations, which act on a second order linear ODE. One can easily see that it is possible to transform any given second order linear ODE into a canonical form using any two of these three transformations, and the canonical form would be invariant with respect to any two transformations. The crucial step is that it is possible to use all three transformations to change the potential term in the canonical form, which explains why transformations of type III, even though they do not change the solutions of the ODE, play the most important role the proof of theorem 4.

We can illustrate the meaning of a Liouville transformation, and in particular transformation of type III, as follows. Assume that we start with a diffusion process $X_t^{(1)}$. By applying a stochastic transformations we construct an equivalence class $\mathfrak{M}(X_t^{(1)})$. By a gauge transformation and by a change of variables we transform the eigenvalue equation $\mathcal{L}_{X^{(1)}} f = \rho f$ into the corresponding canonical form, with the potential term $J_{X^{(1)}}(z)$, given by (7). This function is invariant under stochastic transformations, but can be changed if we use transformation of type III first, followed by transformations of type I and II. Thus we arrive at a new canonical form, which can be mapped into an eigenvalue equation of a new diffusion process $X_t^{(2)}$. Next, as previously, we construct a family of driftless processes $\mathfrak{M}(X_t^{(2)})$. The important fact is that processes in $\mathfrak{M}(X_t^{(2)})$ and $\mathfrak{M}(X_t^{(1)})$ are not related by a stochastic transformation, otherwise they would have the same potential terms $J_{X^{(1)}}$ and $J_{X^{(2)}}$. Thus transformations of type III allow one to obtain new processes, which would be precluded had we restrict ourselves only to diffeomorphisms and gauge transformations. The following diagram illustrates these ideas and also the significance of the Liouville transformation:

$$\begin{array}{ccccc}
\mathfrak{M}(X_t^{(1)}) & & \mathfrak{M}(X_t^{(2)}) & & \mathfrak{M}(X_t^{(3)}) \\
\wr & & \wr & & \wr \\
\text{stochastic} & & \text{stochastic} & & \text{stochastic} \\
\text{transformations} & & \text{transformations} & & \text{transformations} \\
\wr & & \wr & & \wr \\
X_t^{(1)} & \not\sim & X_t^{(2)} & \not\sim & X_t^{(3)} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
J_{X^{(1)}}(z) & \Longleftrightarrow & J_{X^{(2)}}(z) & \Longleftrightarrow & J_{X^{(3)}}(z) \\
 & \text{Liouville} & & \text{Liouville} & \\
 & \text{transformation} & & \text{transformation} &
\end{array}$$

References

[1] C. Albanese and G. Campolieti, *Advanced derivatives pricing and risk management*, Academic Press, 2005.

[2] C. Albanese, G. Campolieti, P. Carr, and A. Lipton, *Black-Scholes goes hypergeometric*, Risk, 14 (2001), pp. 99–103.

[3] C. Albanese and A. Kuznetsov, *Unifying volatility models*, Risk, 17 (2004), pp. 94–98.

[4] G. Bluman, *On the transformation of diffusion processes into the wiener process*, SIAM J. Appl. Math., 39 (1980), pp. 238–247.

[5] ——, *On mapping linear partial differential equations to constant coefficient equations*, SIAM J. Appl. Math., 43 (1983), pp. 1259–1273.

[6] A. Borodin and P. Salminen, *Handbook of Brownian motion - facts and formulae*, Birkhauser, 1996.

[7] A. Bose, Nuovo Cimento, 32 (1964), p. 679.

[8] P. Carr, P. Laurence, and T. Wang, *Generating integrable one dimensional driftless diffusions*, Comptes Rendus Mathematique, 343 (2006), pp. 393–398.

[9] I. D. Cherkasov, *On the transformation of the diffusion process to a Wiener process*, Theor. Probability Appl., 2 (1957), pp. 373–377.

[10] P. Henry-Labordère, *Solvable local and stochastic volatility models: supersymmetric methods in option pricing*, Quantitative Finance, 7 (2007), pp. 525–535.

[11] E. Hille, *Ordinary differential equations in the complex domain*, Wiley, 1976. reprinted by Dover, 1997.

[12] A. Kuznetsov, *Solvable Markov processes*, PhD dissertation, University of Toronto, 2004. ISBN 0612944069.

[13] O. Mazet, *Classification des semi-groupes de diffusion sur $\mathbb{R}$ associés à une famille de polynômes orthogonaux.*, Azéma, J. (ed.) et al., Séminaire de probabilités XXXI.Berlin: Springer. Lect. Notes Math., 1655 (1997), pp. 40–53.

[14] R. Milson, *Liouville transformation and exactly solvable Schrödinger equations*, International Journal of Theoretical Physics, 37 (1998), pp. 1735–1752.

[15] G. Natanzon, *Study of the one-dimensional Schrödinger equation generated from the hypergeometric equation*, Vestnik Leningradskogo Universiteta, (1971), pp. 22–28.

[16] F. Olver, *Asymptotics and special functions*, Academic Press, 1974.

[17] D. Zwillinger, *Handbook of differential equations*, Academic Press, 3 ed., 1997.