

Computing the finite-time expected discounted penalty function for a family of Lévy risk processes

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Abstract

Ever since the first introduction of the Expected Discounted Penalty Function (EDPF) [14], it has been widely acknowledged that it contains information that is relevant from a risk management perspective. Expressions for the EDPF are now available for a wide range of models, in particular for a general class of Lévy risk processes [4, 5]. Yet, in order to capitalize on this potential for applications, these expressions must be computationally tractable enough as to allow for the evaluation of associated risk measures such as VaR or CVaR. Most of the models studied so far offer few interesting examples for which computation of the associated EDPF can be carried out to the last instances where evaluation of risk measures is possible. Another drawback of existing examples is that the expressions are available for an infinite-time horizon EDPF only. Yet, realistic applications would require the computation of an EDPF over a finite-time horizon. In this paper we address these two issues by studying examples of risk processes for which numerical evaluation of the EDPF can be readily implemented. These examples are based on the recently introduced meromorphic processes [20], including the beta and theta families of Lévy processes [18, 19], whose construction is tailor-made for computational ease. We provide expressions for the EDPF associated with these processes and we discuss in detail how a finite-time horizon EDPF can be computed for these families. We also provide numerical examples for different choices of parameters in order to illustrate how ruin-based risk measures can be computed for these families of Lévy risk processes.

1 Introduction

In a now classical paper [14] Gerber and Shiu introduced the concept of Expected Discounted Penalty Function (EDPF). This so-called Gerber-Shiu function is a functional of the ruin time (i.e., the first time the reserve level of a firm becomes negative), the surplus prior to ruin, and the deficit at ruin. It has always been acknowledged that the EDPF is an object containing relevant information from a risk management point of view and thereby lies the ultimate motivation for its analysis. The EDPF has been

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extensively studied and generalized to various scenarios and there is now a wide range of models for which expressions of the EDPF are available. All of these models incorporate different levels of complexity into the picture. In this paper, we focus on the so-called Lévy risk models. In the last decade, there have been several research articles studying the suitability of Lévy processes in a risk theory context. It is generally accepted that these so-called Lévy risk models are a natural generalization of the Cramer-Lundberg model and that their study brings new insight to the classical field of ruin theory. Several families of Lévy processes have been put forward as risk models and we now have a well-established literature on the subject. For a thorough discussion on the suitability of these processes as risk models we refer the reader to [25], [12] and references therein.

As it turns out, the first-passage problem for Lévy processes is well understood and recent results in this area can be applied to the ruin problem in order to gain interesting insight (see for instance [4, 5]). These are quite general results given in terms of expressions that can be evaluated, in theory, using Fourier inversion techniques. But if one is interested in employing in risk management applications of all these tools recently developed in ruin theory, then it is of uttermost importance to exhibit relevant examples for which ruin probabilities and VaR for the deficit at ruin can be computed with ease. Yet, in the literature there are no empirical studies for which these quantities have been evaluated and this is mainly due to the computational challenges that lie behind these expressions. Another drawback of existing models is that the expressions available are for an infinite-time horizon EDPF only. However, realistic applications would require the computation of an EDPF over a finite-time horizon.

The aim of this paper is two-fold. We first discuss a family of meromorphic Lévy risk processes, recently introduced in [20], which include the beta and theta families of Lévy processes introduced in [18] and [19]. We derive expressions for their associated infinite-time EDPF and we discuss in detail how to compute the finite-time EDPF and its particular cases. This is of particular interest from a practical point of view since we provide expressions and methods to compute ruin-based risk measures for these processes. It turns out that these processes share a key feature with other families of processes that have already been used as risk models and therefore a case can be made for their adequacy as insurance models. Here we discuss some of appealing features of these processes, in particular those related to the computational tractability of their associated EDPF. A second objective of this paper is to provide a detailed numerical study of the EDPF, and most importantly, of the finite-time horizon EDPF under this particular family of Lévy risk process. Surprisingly, although we find studies of the finite-time ruin probability and their applications [3, 23], it turns out that the natural related concept of finite-time EDPF has not been formally discussed in the literature and we give a novel definition for this object in this paper. In fact, as we write this, we are aware of at least one work in progress where a different definition for a finite-time EDPF is discussed [13]. Here, we show how a finite-time horizon EDPF can be easily computed for these families of risk processes. We also explain in detail how to carry out numerical computation of ruin related quantities that can be used as tools in risk management applications. In particular we compute ruin probabilities and VaR for the deficit at ruin in both infinite-time and finite-time horizons. This tractability is a key argument motivating the study of these processes as valid models of insurance reserves.

This paper is organized as follows. In Section 2 we introduce the general model based on a Lévy risk process and the definition of the EDPF focusing on those particular cases that are relevant in risk management applications. In particular, we introduce the definition of a finite-time EDPF. In Section 3 we introduce meromorphic processes, including the beta and theta families of Lévy risk processes. We discuss some of their features that make them suitable for insurance applications at, both, theoretical and practical levels. In Section 4 and 5, we present the main contribution of the paper in the form of expressions that allow us to compute, respectively, the infinite-time and finite-time EDPF for these

families of Lévy risk models. In Section 6, we present the second contribution of the paper in the form of a detailed numerical analysis explaining how the main results of the paper can be implemented. We present a comparative analysis and we provide results of several numerical experiments for computing both finite and infinite time EDPF. In particular, we turn our attention to the evaluation of quantities that are of interest from a risk management perspective.

2 Lévy Risk Models, Expected Discounted Penalty Functions and Risk Management Applications

We consider a very general setup that generalizes the standard Cramer-Lundberg model. The model discussed in this paper is

$$X_t := x + Y_t, \quad t \geq 0, \quad (2.1)$$

where $x \geq 0$ is the initial surplus and Y is a spectrally negative Lévy process ($Y_0 = 0$), representing the net aggregate cash inflow of an insurance company. As is customary, the symbols \mathbb{E}_x and \mathbb{P}_x will denote the expectation and the probability measure related to the process started at x , and if the process is started from zero we will use simple notations \mathbb{E} and \mathbb{P} .

Notice that since Y is a general spectrally negative Lévy process, the model in (2.1) contains all elements of a traditional risk model. Indeed, the constant rate premium is included as the drift of Y , the so-called perturbation comes in as the Brownian component of Y and the pure aggregate claims is present as the jump part of Y , which could be set as a compound Poisson or an infinite activity process. With this in mind, we assume the process Y to have a positive drift such that $\mathbb{E}[Y_1] > 0$ in order to avoid the possibility that X becomes negative almost surely. This condition is often expressed in terms of a safety loading. Indeed, it is standard to write the drift component within Y as a loaded premium. For instance, notice that we can recuperate the classical Cramer-Lundberg model if $Y_t = ct - S_t$ where $c := (1 + \theta)\mathbb{E}[S_1]$ and S is a compound Poisson process modeling aggregate claims. The drift c , with a positive safety loading $\theta > 0$, is the collected premium rate. We do not use the notion of safety loading in this paper in order to simplify notation but we stress the fact that this concept is implicitly considered within the drift of Y when we impose the condition $\mathbb{E}[Y_1] > 0$.

One of the advantages of considering a general Lévy risk model like (2.1) is that we can use the tools and methods of the fluctuation theory of Lévy processes to bring new insight into the ruin problem. Indeed, the model in (2.1) contains the classical Cramer-Lundberg model as a particular case allowing for a more comprehensive understanding of the ruin problem. Moreover, the process in (2.1) models the aggregate claims as one single object unlike the Cramer-Lundberg case where the aggregate claims is seen as having two sources of randomness. We recall that, unlike a Lévy risk process, the classical risk process models claims severities and arrivals separately. This difference is interesting since we can work directly with tractable laws for the aggregate claims like the gamma or inverse Gaussian when using a model like in (2.1) whereas in the classical case we have to work with a compound law. For a more extensive discussion on Lévy risk models we refer to [12]. We also remark that the surplus process defined in (2.1) encompasses, among others, the risk models studied in [5, 11, 16, 25].

One of the main objects of interest in ruin theory is the *ruin time*, τ , representing the first passage time of X below zero when $X_0 = x$, i.e.

$$\tau := \inf\{t > 0 : X_t < 0\},$$

where we set $\tau = +\infty$ if $X_t \geq 0$ for all $t \geq 0$.

A thorough understanding of this downward passage problem requires the following concepts. We define the running infimum and the running supremum of a given Lévy process X by

$$\underline{X}_t := \inf_{0 \leq s \leq t} X_s \quad \text{and} \quad \overline{X}_t := \sup_{0 \leq s \leq t} X_s .$$

Now, let us consider the risk process X in (2.1). The following quantities characterize the first downward passage of X below zero and contain information on the ruin event that could be relevant in risk management applications:

- the ruin time, τ ;
- the deficit at ruin, $-X_\tau$;
- the surplus immediately prior to ruin, $X_{\tau-}$;
- the last minimum surplus level before ruin, $\underline{X}_{\tau-}$.

The joint law of the above random variables has been studied in [7] as part of the so-called quintuple law at first passage. We refer to [21] for additional details on this result and related definitions.

All of these quantities encapsulate relevant knowledge about the ruin event. Clearly, a risk manager would be interested in gaining information regarding the probability of ruin $\mathbb{P}_x(\tau < \infty)$ which gives information on how likely the reserve is to face all claims. Even more valuable information can be found in the distribution of the deficit at ruin (depleting the risk reserve by a few dollars rather than by a couple of millions certainly has different implications for an insurance company). Information about the distribution of the reserve level just prior to ruin and of the last minimum surplus level sheds light on the structure of the paths leading to ruin.

In [14], the authors studied the ruin event in the compound Poisson case by analyzing the joint law of τ , the deficit at ruin, $-X_\tau$, and the surplus before ruin, $X_{\tau-}$ in one single object, the EDPF. Following the same order of ideas, we study the EDPF under the general model (2.1) but with a view towards potential risk management applications. This means that the standard infinite-time horizon definition of the EDPF needs to be accompanied by an alternative finite-time horizon definition of the EDPF. In this paper we set out to study the following two generalized EDPF's for the model (2.1).

Definition 1. The Generalized EDPF ϕ associated with the risk process (2.1) is given by

$$\phi(x; q) := \mathbb{E}_x \left[e^{-q\tau} w \left(-X_\tau, X_{\tau-}, \underline{X}_{\tau-} \right) \mathbb{I}_{\{\tau < \infty\}} \right] , \quad (2.2)$$

where $q \geq 0$ and w is a bounded measurable function on \mathbb{R}_+^3 satisfying $w(0, 0, 0) = w_0 > 0$.

Definition 2. The Generalized finite-time EDPF ϕ_t associated with the risk process (2.1) is given by

$$\phi_t(x; q) := \mathbb{E}_x \left[e^{-q\tau} w \left(-X_\tau, X_{\tau-}, \underline{X}_{\tau-} \right) \mathbb{I}_{\{\tau < t\}} \right] , \quad (2.3)$$

where $q \geq 0$ and w is a bounded measurable function on \mathbb{R}_+^3 satisfying $w(0, 0, 0) = w_0 > 0$.

We will write $\phi(x) = \phi(x; q)$ and $\phi_t(x) = \phi_t(x; q)$ in case we do not need to stress the dependence on q . Notice that, as a function of t , the finite-time EDPF $\phi_t(x; q)$ is monotonic increasing and it converges as $t \rightarrow \infty$ to $\phi(x; q)$.

Definition 1 was first introduced in [5] whereas our definition 2 seems to be new. In fact, we would like to remark at this point that there does not seem to be previous discussions in the literature on

finite-time EDPF. There is however at least one work in progress that discusses an alternative definition in different settings (see [13]).

Notice that the EDPF's in (2.2) and (2.3) are, by definition, functions of the initial level x . This is what makes it such an interesting object from a risk management perspective. Indeed, the EDPF gives a quantifiable relation between initial reserve requirements and risk measures constructed from ruin-related random variables. From their very definition we can see that the EDPFs in (2.2) and (2.3) contain relevant information on the ruin event. Indeed, if $w(x_1, x_2, x_3) \equiv 1$ and $q = 0$ and the EDPFs in (2.2) and (2.3) reduce respectively to the ultimate and finite-time ruin probabilities. A second relevant object is the joint distribution of the deficit at ruin, the surplus before ruin and the last minimum before ruin. Notice that the value at (y, z, u) of infinite- and finite-time versions of the joint distribution of the deficit at ruin, the surplus before ruin and the last minimum before ruin, $F_x(y, z, u)$ and $F_{x,t}(y, z, u)$, can be recovered from the EDPF's in (2.2) and (2.3). Indeed, when $q = 0$ and when w is an indicator function

$$w(x_1, x_2, x_3) = \mathbb{I}_{[0,y]}(x_1) \mathbb{I}_{[0,z]}(x_2) \mathbb{I}_{[0,u]}(x_3) , \quad (2.4)$$

we have $\phi(x) = F_x(y, z, u)$ and $\phi_t(x) = F_{x,t}(y, z, u)$.

This last joint distribution is a very interesting object since it contains information on the path leading to ruin. Of particular interest is the marginal giving the distribution of the deficit at ruin. For it is not only of importance to know how likely the ruin event is but also how large the deficit can be at the moment of ruin. For instance, this joint distribution could be used to set capital requirements in terms of VaR values (or other suitable risk measure) for the deficit at ruin. In order to make this idea somehow clearer, we have to take into account that the new EDPF contains the conditional marginal distribution of the deficit at ruin as a particular case. If we denote this distribution function by $F_x(y) := \mathbb{P}_x(-X_\tau < y | \tau < \infty)$, we can see that any given level VaR_α such that $F_x(\text{VaR}_\alpha) = \alpha$ can be interpreted as follows: *1 - $\alpha \times 100\%$ of the times when ruin occurs, the deficit at ruin would be larger than VaR_α .* The threshold VaR_α is, for a fixed confidence level α , a function of the initial reserve. This gives us a rule to set adequate initial levels as measured by the risk measure VaR_α .

But for these kind of applications to be feasible we need to be able to compute the EDPF for a suitable family of models. This is the ultimate goal of this paper. We aim at introducing a family of Lévy risk models with appealing features and for which expressions of the EDPF can be readily computed. As we have discussed, this will immediately endow us with a way to numerically evaluate ruin-related risk measures based on cases of interest of the EDPF.

At this point, it should be clear that in order to evaluate the EDPF's in Definition 1 and Definition 2 we need the joint distribution of all random variables involved, i.e. τ , X_τ , $X_{\tau-}$ and $\underline{X}_{\tau-}$. This law has been studied in [7]. In fact, it turns out that the EDPF is actually another way in which the so-called quintuple law expresses itself. We note that results in ruin theory have commonly appeared in terms of renewal equations, for instance [5] give a characterization of the EDPF in (2.2) in terms of convolutions and renewal equations.

In this paper we use results from [4], [7] and [20] in order to compute the EDPF. In fact, expressions for (2.2) can be written out using the theory of fluctuations for one-sided Lévy processes as we discuss in Section 4. Then expressions for (2.3) can be computed numerically as we discuss in Section 5.

One feature that has to be highlighted at this point about evaluating (2.2) is that, unlike the classical compound Poisson case, now there is a possibility that X enters $(-\infty, 0)$ continuously, a situation referred to as 'creeping' of X below zero and characterized by the event $\{X_\tau = 0\}$ having strictly positive probability (see [21] for a discussion). In this paper, the reserve of an insurance company X , as defined in (2.1), is a spectrally negative Lévy process and therefore down-creeping can only occur if there is a Brownian component in the process X (see Example 7.6 in [21]). When X creeps below zero,

the components of the triplet $(-X_\tau, X_{\tau-}, \underline{X}_{\tau-})$ are all zero, since $\underline{X}_{\tau-} = X_{\tau-} = 0$ on $\{X_\tau = 0\}$ and the EDPF in (2.2) becomes

$$\phi(x) = w_0 \mathbb{E}_x [e^{-q\tau} \mathbb{I}(X_\tau = 0)] + \mathbb{E}_x [e^{-q\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbb{I}_{\{\tau < \infty, X_\tau < 0\}}] , \quad (2.5)$$

where $w(0, 0, 0) = w_0$.

3 Beta and theta Families of Lévy Processes

The model in (2.1) is quite general. There are an infinite number of choices for the process Y in equation (2.1). The first reason why a Lévy risk process as a model for Y is appealing comes from the fact that the distribution of aggregate net cash inflow in $[0, t]$ can be chosen to be any infinitely divisible distribution. Natural choices seem to be the inverse Gaussian, generalized inverse Gaussian or gamma distributions (see [12]). One practical problem with these choices is that even the computation of the infinite-time EDPF requires Laplace inversion techniques, and the computation of the finite-time EDPF would be a formidable challenge. In this section we present several families of Lévy processes whose associated infinitely divisible distributions are also suitable as insurance models, but for which the infinite-time EDPF can be computed almost explicitly and a finite-time EDPF can be computed with a single numerical inversion of Laplace transform. These new processes share key features with the generalized inverse Gaussian (GIG) family of processes; in particular they have similar tail behavior of their Levy measures. This alone might not make for such a strong case for these new family as risk models but, when combined with their numerical tractability, the potential of these processes becomes more transparent. Indeed, we are in the presence of a family of process that can model a wide range of aggregate claims behavior (like the GIG family of processes) but for which we can readily compute the EDPF and its particular cases of interest (unlike the GIG family of processes).

Consider the model in (2.1), where Y is a spectrally negative Lévy process, and let ψ denote its Laplace exponent defined as $\psi(z) = \ln \mathbb{E}[\exp(zY_1)]$. As was shown in [20], many results of fluctuation theory can be given rather explicitly for the so-called meromorphic processes. These processes are defined by requiring that their Lévy measure $\Pi(dx) = \pi(x)dx$ is essentially a “mixture” of exponential distributions, and in the spectrally negative case this translates into the following definition

$$\pi(x) = \mathbb{I}_{\{x < 0\}} \sum_{m=1}^M b_m e^{\rho_m x}, \quad (3.1)$$

where all the coefficients b_m and ρ_m are positive and we assume that $\rho_1 < \rho_2 < \dots$. When M is finite we have a process with hyper-exponential jumps, and it is clear that since $\Pi(\mathbb{R}) < \infty$ the process has finite activity of jumps. Things become considerably more interesting if we allow $M = +\infty$, in this case we will additionally assume that $\rho_m \rightarrow +\infty$ as $m \rightarrow +\infty$. One can see that if

$$\int_{-\infty}^0 \pi(x) dx = \sum_{m \geq 1} \frac{b_m}{\rho_m} < \infty,$$

then we have compound Poisson jumps; if the above expression is infinite we have jumps of infinite activity; and if

$$\int_{-\infty}^0 |x| \pi(x) dx = \sum_{m \geq 1} \frac{b_m}{\rho_m^2} = \infty,$$

we have jumps of infinite variation. Note that if we want $\pi(x)$ given by (3.1) to define a density of a Lévy measure, it must satisfy the integrability condition

$$\int_{-\infty}^0 |x|^2 \pi(x) dx = \sum_{m \geq 1} \frac{b_m}{\rho_m^3} < \infty.$$

Next, using the Lévy-Khintchine formula we find that the Laplace exponent $\psi(z)$ for a process with the Lévy measure (3.1) is a rational ($M < \infty$) or meromorphic ($M = +\infty$) function given by the partial fraction decomposition

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \tilde{\mu} z + \int_{-\infty}^0 (e^{zx} - 1 - zx) \pi(x) dx = \frac{\sigma^2}{2} z^2 + \tilde{\mu} z + z^2 \sum_{m=1}^M \frac{b_m}{\rho_m^2 (\rho_m + z)}. \quad (3.2)$$

Moreover, as was established in [20], the function $\psi(z)$ given by (3.2) satisfies the following properties,

- (i) $\psi(z)$ has simple poles at points $\{-\rho_m\}_{m=1, \dots, M}$;
- (ii) for $q > 0$ the function $\psi(z) - q$ has a unique positive root $z = \Phi(q)$;
- (iii) if $M < \infty$, $q \geq 0$ and $\sigma = 0$ (resp. $\sigma > 0$) then $\psi(z) - q$ has M (resp. $M + 1$) negative simple roots $\{-\zeta_n(q)\}$;
- (iv) if $M = \infty$, $q \geq 0$ then $\psi(z) - q$ has an infinite number of negative simple roots $\{-\zeta_n(q)\}$;
- (v) for $q \geq 0$ the function $\psi(z) - q$ has no other roots in the entire complex plane, except for $\Phi(q)$ and $\{-\zeta_n(q)\}$;
- (vi) assume that $\zeta_n(q)$ are labeled so that they increase in n , then we have the following interlacing property

$$0 < \zeta_1(q) < \rho_1 < \zeta_2(q) < \rho_2 < \dots \quad (3.3)$$

While all the results presented in this paper are true for processes with compound Poisson jumps, which corresponds to the case $M < \infty$, we will be more interested in Lévy processes whose jumps have infinite activity or even infinite variation. This case is more delicate than the compound Poisson case, as dealing with function $\psi(z)$ is nontrivial when $M = \infty$. As we will see later, function $\psi(z)$ is the key to efficient numerical computations, and it would be extremely inefficient if we had to compute $\psi(z)$ using the series expansion (3.2), since this series would converge rather slowly. Therefore, if we want to develop efficient numerical schemes, it is crucial to choose coefficients b_m and ρ_m so that one can compute the Laplace exponent $\psi(z)$ explicitly. It turns out that this is possible for meromorphic processes which belong to beta and theta families, introduced in [18, 19]. These families of processes have a Lévy measure of the form (3.1) and they can be defined through their Laplace exponents ψ as follows:

- (i) θ -process with parameter $\lambda = 3/2$

$$\psi(z) = \frac{1}{2} \sigma^2 z^2 + \mu z - c \sqrt{\alpha + z/\beta} \coth \left(\pi \sqrt{\alpha + z/\beta} \right) + c \sqrt{\alpha} \coth \left(\pi \sqrt{\alpha} \right), \quad (3.4)$$

(ii) θ -process with parameter $\lambda = 5/2$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + c(\alpha + z/\beta)^{\frac{3}{2}} \coth\left(\pi\sqrt{\alpha + z/\beta}\right) - c\alpha^{\frac{3}{2}} \coth\left(\pi\sqrt{\alpha}\right), \quad (3.5)$$

(iii) β -process with parameter $\lambda \in (1, 2) \cup (2, 3)$

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + cB(1 + \alpha + z/\beta, 1 - \lambda) - cB(1 + \alpha, 1 - \lambda). \quad (3.6)$$

where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the Beta function.

For these families of processes, the range of parameters in the above equations are $\sigma \geq 0$, $\mu \in \mathbb{R}$, $c > 0$, $\alpha > 0$ and $\beta > 0$. However, when seen as the net aggregate claims process Y in (2.1), all parameters must be set such that there is a loaded premium $\mu > 0$ and the net-profit condition $\mathbb{E}[Y_1] > 0$ is satisfied. Note that here we use slightly different parametrization compared to the original definitions in [18, 19], however, as we will see later, this parametrization makes it easier to compare different beta and theta processes.

All three families have a Gaussian component that can be switched on and off using the σ parameter making our risk model a member of the so-called perturbed family of models. In general, the drift parameter μ of this family can be chosen arbitrarily but in this application it is seen as a premium containing the safety loading factor $\mu > 0$ and it must be chosen such that the net-profit condition $\mathbb{E}[Y_1] > 0$ is satisfied. Parameters α and β are responsible for the rate of decay of the tail of the Lévy measure and for the shape of this measure, parameter c controls the overall “intensity” of jumps, while the parameter λ describes the singularity of the Lévy measure at zero and therefore controls the intensity of the small jumps. It can be shown (see [18, 19]) that in all three cases the Lévy measure has a density with respect to the Lebesgue measure that satisfies (up to a multiplicative constant)

$$\begin{aligned} \pi(x) &\sim |x|^{-\lambda}, \quad \text{as } x \rightarrow 0^-, \\ \pi(x) &\sim e^{\beta(1+\alpha)x}, \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

In particular, we see that the Lévy measure always has exponential tails, and if $\lambda \in (1, 2)$ then the process has jump part of infinite activity and finite variation, and in this case the parameter μ is just the linear drift of the process. On the other hand, when $\lambda \in (2, 3)$, the jump part of the process will be of infinite variation. Notice that if $\sigma > 0$, then the process Y always has paths of infinite variation.

One can see that the beta family of processes is somewhat more general than the theta family, as it allows a greater range of parameter λ . This allows us to “fine-tune” the behaviour of small jumps. However, processes in the theta family have the great advantage of having a Laplace exponent given in terms of elementary functions. This allows for the implementation of very efficient numerical algorithms.

The parameters b_m and ρ_m , which define the Lévy measure via (3.1), are given as follows (see [18, 19]): in the case of θ -process we have

$$b_m = \frac{2}{\pi}c\beta m^{2\lambda-1}, \quad \rho_m = \beta(\alpha + m^2), \quad (3.7)$$

and in the case of β -process

$$b_m = c\beta \binom{m + \lambda - 2}{m - 1}, \quad \rho_m = \beta(\alpha + m). \quad (3.8)$$

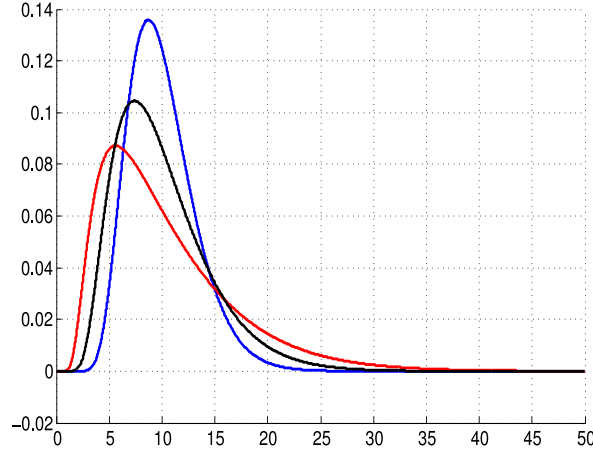


Figure 1: The probability density function of the aggregate claims process $\mu t - Y_t$ at $t = 1$ for the theta-process with $\lambda = 3/2$ and $\sigma = 0$. The parameters are chosen so that $\mathbb{E}[\mu - Y_1] = 10$ and $\text{Var}[Y_1] \in \{10, 20, 40\}$, which corresponds to blue, black and red curve.

In the case of β -process we also have a nice explicit formula for the density of the Lévy measure (see [18])

$$\pi(x) = c\beta \frac{e^{(1+\alpha)\beta x}}{(1 - e^{\beta x})^\lambda}, \quad x < 0.$$

Surprisingly, this last simple expression will not be used in the current paper. In the case of θ -processes the density of the Lévy measure can be expressed in terms of derivatives of the theta function $\theta_3(0; e^{-x})$, see [19] for all the details.

The choice of notation in model (2.1) is quite general. It is only when the Laplace exponent of such a risk model (3.2) is exhibited that all components are clearly visible. Indeed, in (3.4), (3.5) and (3.6), we can always identify three components for each one of the processes that are studied in this paper. The process Y contains a loaded premium expressed as the drift term μ and characterized by the net profit condition $\mathbb{E}[Y_1] > 0$. A perturbation component is also included through the Brownian term σ . Finally, the third term characterizes the jumps of the pure aggregate claims component of Y . This component is often studied separately in the literature but here we chose to integrate it along with the other two in one single object Y . Notice that if one wants to recuperate the pure aggregate claims process inside Y , it is enough to set $\sigma = 0$ and then look at the process $\mu t - Y_t$.

Recall that the feature of having an exponentially decaying Lévy density is one of the appealing traits of the generalized inverse Gaussian family of Lévy processes. Along the same lines, these three examples of processes are suitable choices for Y in model (2.1) since they behave very much like these popular choices of processes in an insurance context. The infinitely divisible distributions associated with these processes have semi-heavy exponential tails just like generalized inverse Gaussian family of distributions and are therefore suitable for modeling aggregate claims. Moreover, moments of these distributions can be readily computed by taking derivatives of the Laplace exponents, and estimation of parameters can be carried out using method of moments. In Figure 1 we show graphs of densities of the pure aggregate claim process $\mu t - Y_t$ at $t = 1$ for the theta-process with $\lambda = 3/2$ and $\sigma = 0$; for which $\mathbb{E}[\mu - Y_1] = 10$ and $\text{Var}[Y_1] \in \{10, 20, 40\}$. This choice of parameters produces a density that could be a model for the aggregate claims with tails that decay like the one in the inverse Gaussian family.

4 Computing the EDPF

In this section we discuss how the EDPF in (2.2) can be computed for the risk model in (2.1) for the general meromorphic processes, and in particular, for the three choices of processes discussed in the previous section and defined through (3.4), (3.5) and (3.6). We recall that we assume the net-profit condition $\mathbb{E}[Y_1] > 0$ so that the risk process X drifts to $+\infty$. Moreover, for $q > 0$ we have defined $\Phi(q)$ as the unique positive solution to $\psi(z) = q$ (and by continuity $\Phi(0) = 0$), and the increasing sequence of positive numbers $\{\zeta_n(q)\}_{n \geq 1}$ as solutions to $\psi(-z) = q$. All the formulas presented below will be expressed in terms of these numbers, and also in terms of the Laplace exponent $\psi(z)$ and parameters b_m and ρ_m which define the Lévy measure of these processes, see (3.1). For all the processes considered in this paper, the Laplace exponent and the parameters b_m and ρ_m are known explicitly (see formulas (3.4)-(3.6) and (3.7), (3.8)), yet the numbers $\Phi(q)$ and $\zeta_n(q)$ have to be found numerically. We will give more details on this in Section 5.3, here we will just mention that it is not hard to do.

Remark 1. We would like to emphasize again that all the results presented below are also valid for process with compound Poisson jumps, whose Lévy measure (3.1) is a *finite* mixture of exponentials. One should just replace all the infinite summations in the index n by the finite ones, where the upper limit is either M or $M + 1$ depending on whether $\sigma = 0$ or $\sigma > 0$, and all the infinite summations in the index m by the finite sums with the upper limit M . Since all the summations are finite and the Laplace exponent $\psi(z)$ is just a rational function given by (3.2), this special case will be the most convenient and efficient for numerical computations. In this sense, our model generalizes the well-known results in the actuarial literature, indeed the case of a Cramer-Lundberg process with mixture of exponentials has been extensively studied in ruin theory (see [8]).

We recall that for $q \geq 0$ the scale function $W^{(q)}$ is defined as follows (see [21], p. 214): $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ it is characterized by a Laplace transform

$$\int_0^{\infty} e^{-zx} W^{(q)}(x) dx = \frac{1}{\psi(z) - q}, \quad z > \Phi(q). \quad (4.1)$$

The scale function $W^{(q)}(x)$ plays a key role in understanding of exit-problems for spectrally negative Lévy processes. In fact, it contains all relevant information needed to describe the first-passage of a process. This has made it a favorite tool in recent applications as a glance to the current literature shows. For a thorough account on scale functions for spectrally negative Lévy processes we refer to [15]. Recently, we find several articles that use the scale function as the main tool to give simple and elegant solutions to old and new problems in ruin theory. For instance, [4] gives an analytical characterization of the generalized EDPF (2.2) in terms of scale functions. An application of scale functions in the context of expected dividends is given in [26] and [27]. In [22] we find an application of scale functions in a taxed Lévy risk process. And in [24] we find a discussion of DeFinetti's dividend problem for a Lévy insurance risk process in terms of scale functions.

At this point we should remark that all the results in this paper can also be obtained via scale functions. Indeed, the following proposition gives an expression of the scale function for arbitrary meromorphic processes defined through (3.1).

Proposition 1. *Assume that X is a meromorphic spectrally negative process. Then for $q > 0$*

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \sum_{n \geq 1} \frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)}, \quad x > 0. \quad (4.2)$$

Proof. From Corollary 8.9 in [21] we find that for $x > 0$

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} - u^{(q)}(-x), \quad (4.3)$$

where $u^{(q)}(x)$ is the q -potential density of the process x defined as

$$u^{(q)}(x)dx = \int_0^\infty e^{-qt} \mathbb{P}(X_t \in dx) dt.$$

The expression for $u^{(q)}(x)$ for meromorphic processes is given in part (v) of Theorem 2 in [20], which, combined with (4.3), proves this proposition. \square

Remark 2. Note that in the case when $M < \infty$ formula (4.2) follows easily from (4.1) and from the partial fraction decomposition of the rational function $1/(\psi(z) - q)$. Also, formula (4.2) holds true in the case $q = 0$ unless $\mathbb{E}[X_1] = \psi'(0) = 0$. In the latter case one can obtain an expression for the scale function by taking the limit of (4.2) as $q \rightarrow 0^+$ and using part (ii) of Corollary 2 in [20].

Once we have a formula for the scale function, we can obtain the EDPF using Theorem 1 from [4]. However, as we will see, it is considerably easier to use results in [20]. We begin by stating a result that brings a first insight to the discussion. We recall at this point that the ruin probability is actually equivalent to the probability that the overall infimum of the process Y is larger than $-x$, i.e.

$$\mathbb{P}_x(\tau < \infty) = \mathbb{P}(-\underline{Y}_\infty > x).$$

The following proposition gives an expression for the distribution of the running infimum of the process Y sampled at an exponential random time $e(q)$ with parameter q . This holds for any meromorphic process and it brings us one step closer to an expression for the ruin probability for the risk models studied in this paper. Recall that we have defined $\Phi(q)$ as the unique positive solution to $\psi(z) = q$ for $q \geq 0$.

Proposition 2. *Assume that Y is a meromorphic spectrally negative process. Then for $q > 0$ we have $\bar{Y}_{e(q)} \sim \text{Exp}(\Phi(q))$ and the density of infimum is a mixture of exponential distributions*

$$\mathbb{P}(-\underline{Y}_{e(q)} \in dx) = \left[\sum_{n \geq 1} c_n(q) \zeta_n(q) e^{-\zeta_n(q)x} \right] dx, \quad x > 0, \quad (4.4)$$

where

$$c_n(q) := - \left(\frac{1}{\zeta_n(q)} + \frac{1}{\Phi(q)} \right) \frac{q}{\psi'(-\zeta_n(q))}. \quad (4.5)$$

If the process has bounded variation then the atom at zero is

$$c_0(q) := \mathbb{P}(\underline{Y}_{e(q)} = 0) = \frac{q}{\Phi(q)\mu}.$$

Proof. Formula (8.20) in [21] tells us that

$$\mathbb{P}(-\underline{Y}_{e(q)} \in dx) = \left[\frac{q}{\Phi(q)} \frac{d}{dx} W^{(q)}(x) - qW^{(q)}(x) \right] dx.$$

The results of Proposition 2 follow at once from the above formula and Proposition 1. \square

If we notice that when $q \rightarrow 0^+$ we have $e(q) \rightarrow +\infty$ (in the sense of the weak convergence), then the following result is all we need to give an expression for the ruin probability for the models studied in this paper.

Proposition 3. *As $q \rightarrow 0^+$ we have $\zeta_n(q) \rightarrow \zeta_n(0) \neq 0$, $\Phi(q) \rightarrow 0^+$ and $q/\Phi(q) \rightarrow \mathbb{E}[Y_1]$.*

Proof. We have $\psi(z) = \mathbb{E}[Y_1]z + O(z^2)$ as $z \rightarrow 0$, and we also know that $\mathbb{E}[Y_1] > 0$, therefore $\Phi(q)$, which is a unique positive solution to $\psi(z) = q$, will satisfy $\Phi(q) = q/\mathbb{E}[Y_1] + O(q^2)$ as $q \rightarrow 0^+$, which implies $\Phi(q) \rightarrow 0^+$ and $q/\Phi(q) \rightarrow \mathbb{E}[Y_1]$ as $q \rightarrow 0^+$. In fact, $q/\Phi(q) \rightarrow \mathbb{E}[Y_1]$ as $q \rightarrow 0^+$ holds for any spectrally negative Lévy process Y with $\mu > 0$ and $\mathbb{E}[Y_1] > 0$. The fact that $\zeta_n(q)$ tend to strictly positive numbers follows from the fact that $-\zeta_n$ are negative solutions to $\psi(z) = q$ and that $\psi(0) = 0$, $\psi'(0) > 0$. \square

This yields the following corollary that gives a simple expression for the ruin probability.

Corollary 1. *Assume that Y is a meromorphic spectrally negative process. The probability of ruin is*

$$\mathbb{P}_x(\tau < \infty) = \mathbb{P}(-\underline{Y}_\infty > x) = \sum_{n \geq 1} c_n(0) e^{-\zeta_n(0)x}, \quad x > 0,$$

where

$$c_n(0) = -\frac{\mathbb{E}[Y_1]}{\psi'(-\zeta_n(0))}.$$

If the process has bounded variation then the atom at zero is

$$c_0(0) = \mathbb{P}_0(\tau < \infty) = \mathbb{P}(\underline{Y}_\infty = 0) = \frac{\mathbb{E}[Y_1]}{\mu},$$

which also gives the probability of ruin when the initial surplus is equal to zero.

Notice that this is a closed-form expression for the ultimate ruin probability under meromorphic Lévy risk processes and in fact the last expression for $\mathbb{P}_0(\tau < \infty)$ holds for a general spectrally negative Lévy process Y with $\mu > 0$ and $\mathbb{E}[Y_1] > 0$.

It is the form of these expressions in terms of exponentially converging infinite series that allows us to numerically compute the EDPF and related quantities. In the following, we show how the EDPF in (2.2) preserves the same structure as the ruin probability for the models studied in this paper. In order to simplify notations, from now on we will write $\zeta_n = \zeta_n(q)$ and $c_n = c_n(q)$ since in all computations it is always assumed that $q \geq 0$ is a fixed parameter.

We start by stating a result that gives an expression for the probability measure of the EDPF under the risk models (3.4), (3.5) and (3.6). Define the Gerber-Shiu measure

$$P^{x,q}(dy, dz, du) = \mathbb{E}_x \left[e^{-q\tau} \mathbb{I}(-X_\tau \in dy; X_{\tau-} \in dz; \underline{X}_{\tau-} \in du) \right]. \quad (4.6)$$

Here and everywhere else in this paper we assume that for $q \geq 0$ we have $e^{-q\tau} = 0$ on the event $\{\tau = \infty\}$. One can see that $P^{x,q}(dy, dz, du)$ is a measure on the set $\{y \geq 0, z \geq 0, u \in [0, z \wedge x]\} \subset \mathbb{R}^3$ and $P^{x,q}$ which is well defined for $q \geq 0$ and $x \geq 0$.

Proposition 4. (Gerber-Shiu measure) Assume that X is a meromorphic spectrally negative process. For $q \geq 0$, $x > 0$, $y > 0$, $z > 0$ and $u \in (0, z \wedge x)$

$$P^{x,q}(dy, dz, du) = \left[\frac{\Phi(q)}{q} \sum_{m,n \geq 1} c_n \zeta_n b_m \exp(-\zeta_n x - \rho_m y - (\Phi(q) + \rho_m)z + (\Phi(q) + \zeta_n)u) \right] dy dz du. \quad (4.7)$$

If $\sigma > 0$ then $P^{x,q}$ has an atom at $y = z = u = 0$:

$$\mathbb{E}_x [e^{-q\tau} \mathbb{I}(X_\tau = 0)] = \frac{\sigma^2 \Phi(q)}{2q} \sum_{n \geq 1} c_n \zeta_n e^{-\zeta_n x}.$$

If we have a process of bounded variation, then for $y > 0$ and $z > x$ the measure $P^{x,q}$ has positive mass on the plane $u = x$ and

$$\mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_\tau \in dy; X_{\tau-} \in dz; \underline{X}_{\tau-} = x)] = \frac{1}{\mu} e^{\Phi(q)x} \sum_{m \geq 1} b_m e^{-\rho_m y - (\Phi(q) + \rho_m)z}, \quad (4.8)$$

where μ is the linear drift of the process.

Proof. This result can be considered as a special case of Lemma 2 and Corollary 5 in [20]. We present here the main steps of the proof for the sake of completeness. Let us define the first passage above a level h as $\tau_h^+ = \inf\{t > 0 : X_t > h\}$. Then using the quintuple law at first passage (see [7]) we find

$$\begin{aligned} & \mathbb{E} \left[e^{-q\tau_h^+} \mathbb{I}(X_{\tau_h^+} - h \in dy; h - X_{\tau_h^+} \in dz; x - \bar{X}_{\tau_h^+} \in du) \right] \\ &= \int_0^\infty e^{-qs} \mathcal{U}(ds, h - du) \int_0^\infty e^{-qt} \hat{\mathcal{U}}(dt, dz - u) \Pi(dy + z). \end{aligned}$$

Using the definition of the bivariate renewal measure we find

$$\int_0^\infty e^{-qs} \mathcal{U}(ds, h - du) = \frac{1}{\kappa(q, 0)} \mathbb{P}(\bar{X}_{e(q)} \in h - du), \quad \int_0^\infty e^{-qt} \hat{\mathcal{U}}(dt, dz - u) = \frac{1}{\hat{\kappa}(q, 0)} \mathbb{P}(-\underline{X}_{e(q)} \in dz - u).$$

Since $\kappa(q, 0)\hat{\kappa}(q, 0) = q$ we obtain

$$\begin{aligned} & \mathbb{E} \left[e^{-q\tau_h^+} \mathbb{I}(X_{\tau_h^+} - h \in dy; h - X_{\tau_h^+} \in dz; x - \bar{X}_{\tau_h^+} \in du) \right] \\ &= \frac{1}{q} \mathbb{P}(\bar{X}_{e(q)} \in h - du) \mathbb{P}(-\underline{X}_{e(q)} \in dz - u) \Pi(dy + z). \end{aligned}$$

now, by considering the dual process we conclude

$$\begin{aligned} & \mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_\tau \in dy; X_{\tau-} \in dz; \underline{X}_{\tau-} \in du)] \\ &= \frac{1}{q} \mathbb{P}(-\underline{X}_{e(q)} \in x - du) \mathbb{P}(\bar{X}_{e(q)} \in dz - u) \Pi(-z - dy). \end{aligned}$$

In order to finish the proof one should apply results of Proposition 2. □

Equation (4.7) is the first main result of this paper. It gives a closed-form expression for the discounted joint density of the three ruin-related quantities. Notice that the expression involves different series of exponential terms which can be easily implemented. The creeping probability gives the discounted probability that ruin is attained by continuously crossing the level zero.

Using Proposition 4 we can derive expressions for the EDPF in (2.2) for a general penalty w . Let us denote the joint density (4.7) by $P^{x,q}$. When X creeps below zero, the components of the triplet $(-X_\tau, X_{\tau-}, \underline{X}_{\tau-})$ are all zero, since $\underline{X}_{\tau-} = X_{\tau-} = 0$ on $\{X_\tau = 0\}$. In that case, $\{0\} \times \{0\} \times \{0\}$ is an atom for $P^{x,q}$. Setting $w(0, 0, 0) = w_0$, from (2.2) we have for $q > 0$

$$\phi(x) = w_0 \mathbb{E}_x [e^{-q\tau} \mathbb{I}(X_\tau = 0)] + \int_{0+}^{+\infty} \int_{0+}^{+\infty} \int_{0+}^{z \wedge x} w(\tilde{y}, \tilde{z}, \tilde{u}) P^{x,q}(d\tilde{y}, d\tilde{z}, d\tilde{u}), \quad (4.9)$$

where the first term is given by the probability of creeping and the second term is given in terms of an integral of the Gerber-Shiu measure (4.7).

An important result that can be obtained from Proposition 4 is the discounted joint distribution of these three ruin-related quantities. This is a particular case of the EDPF defined in (2.2) when the penalty w is the indicator function defined in (2.4) and is given in the following two Theorems.

Theorem 1. *Assume that X is a meromorphic spectrally negative process. For $q \geq 0$, $x > 0$, $y > 0$, $z > 0$ and $u \in (0, z \wedge x)$*

$$\begin{aligned} \mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_\tau < y; X_{\tau-} < z; \underline{X}_{\tau-} < u)] &= \sum_{n \geq 1} c_n e^{-\zeta_n x} \\ &+ \frac{\Phi(q)}{q} \sum_{m, n \geq 1} \frac{c_n \zeta_n b_m e^{-\zeta_n x} (1 - e^{-\rho_m y})}{\rho_m (\Phi(q) + \rho_m)} \left[\frac{e^{(\zeta_n - \rho_m)u} + (e^{\rho_m y} - 1)^{-1}}{\zeta_n - \rho_m} - \frac{e^{-(\Phi(q) + \rho_m)z} (e^{(\Phi(q) + \zeta_n)u} - 1)}{\Phi(q) + \zeta_n} \right]. \end{aligned} \quad (4.10)$$

If X is a beta or theta process, then all infinite series converge exponentially fast, uniformly on compact subsets of the admissible set of variables (x, y, z, u) .

Proof. Consider a function $w(\tilde{y}, \tilde{z}, \tilde{u}) = \mathbb{I}(\tilde{y} < y, \tilde{z} < z, \tilde{u} < u)$. Using the expression for the Gerber-Shiu measure (4.7) and integrating (4.9) we find that the quantity in the left hand-side of (4.10) is equal to

$$\begin{aligned} \phi(x) &= \frac{\sigma^2 \Phi(q)}{2q} \sum_{n \geq 1} c_n \zeta_n e^{-\zeta_n x} + \frac{\Phi(q)}{q} \sum_{m, n \geq 1} c_n \zeta_n \frac{b_m e^{-\zeta_n x}}{\rho_m (\Phi(q) + \zeta_n)} (1 - e^{-\rho_m y}) \\ &\times \left[\frac{e^{(\zeta_n - \rho_m)u} - 1}{\zeta_n - \rho_m} - \frac{1 - e^{-(\Phi(q) + \rho_m)u}}{\Phi(q) + \rho_m} + \frac{(e^{(\Phi(q) + \zeta_n)u} - 1) (e^{-(\Phi(q) + \rho_m)u} - e^{-(\Phi(q) + \rho_m)z})}{\Phi(q) + \rho_m} \right]. \end{aligned} \quad (4.11)$$

We could have stopped here, however the above expression is not convenient for numerical computation. This is due to the fact that in the above double series the coefficients do not decrease sufficiently fast as $m \rightarrow +\infty$. Thus we have to do some extra work and rearrange this series so that everything converges exponentially fast.

First of all, one can easily check that

$$\sum_{m \geq 1} \frac{b_m}{\rho_m (\rho_m + \Phi(q)) (\rho_m - \zeta_n)} = \frac{\zeta_n}{\zeta_n + \Phi(q)} \sum_{m \geq 1} \frac{b_m}{\rho_m^2 (\rho_m - \zeta_n)} + \frac{\Phi(q)}{\zeta_n + \Phi(q)} \sum_{m \geq 1} \frac{b_m}{\rho_m^2 (\rho_m + \Phi(q))}.$$

Using the above identity, formula (3.2) and the fact that $-\zeta_n$ and $\Phi(q)$ are solutions to the equation $\psi(z) = q$, we obtain

$$\sum_{m \geq 1} \frac{b_m}{\rho_m(\rho_m + \Phi(q))(\rho_m - \zeta_n)} = \frac{q}{\zeta_n \Phi(q)} - \frac{\sigma^2}{2}.$$

Now, the final expression (4.10) follows from the above identity and (4.11) after some straightforward (but tedious) algebraic manipulations. \square

An important feature of theorem 1 is that the series converge exponentially fast in the case of beta processes, and even faster in the case of theta process. This is due to the linear/quadratic growth of ρ_n and ζ_n , see (3.7), (3.8) and the interlacing property (3.3). As we will see later in Section 6 this leads to very efficient numerical schemes. Another important feature is that one can set $y = +\infty$ and/or $z = +\infty$ in (4.10) and obtain exponentially convergent series for the marginal distribution of $(X_{\tau-}, \underline{X}_{\tau-})$ or $(-X_{\tau}, \underline{X}_{\tau-})$ or just $\underline{X}_{\tau-}$. However it is not so easy to obtain the marginal distribution of deficit at ruin and the surplus before ruin $(-X_{\tau}, X_{\tau-})$: if we set $u = z \wedge x$ in (4.10) then we observe that the resulting series will not be exponentially convergent when $z \geq x$. This issue is resolved by the following theorem.

Theorem 2. *Assume that X is a meromorphic spectrally negative process and that $q \geq 0$, $x > 0$ and $y > 0$.*

(i) *If $z < x$ then*

$$\mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_{\tau} < y ; X_{\tau-} < z)] = \mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_{\tau} < y ; X_{\tau-} < z ; \underline{X}_{\tau-} < z)] ,$$

where the right-hand side can be computed via (4.10).

(ii) *If $z \geq x$ then*

$$\begin{aligned} \mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_{\tau} < y ; X_{\tau-} < z)] &= \sum_{n \geq 1} c_n e^{-\zeta_n x} - \frac{e^{-\Phi(q)(z-x)}}{\psi'(\Phi(q))} \sum_{m \geq 1} \frac{b_m e^{-\rho_m z} (1 - e^{-\rho_m y})}{\rho_m (\Phi(q) + \rho_m)} \\ &+ \frac{\Phi(q)}{q} \sum_{m, n \geq 1} \frac{c_n \zeta_n b_m e^{-\zeta_n x}}{\rho_m (\Phi(q) + \rho_m)} \left[\frac{e^{-\rho_m y}}{\zeta_n - \rho_m} + \frac{e^{-(\Phi(q) + \rho_m)z} (1 - e^{-\rho_m y})}{\Phi(q) + \zeta_n} \right]. \end{aligned} \quad (4.12)$$

If X is a beta or theta process, then in both cases all infinite series converge exponentially fast, uniformly on compact subsets of the admissible set of variables (x, y, z) .

Proof. The proof is very similar to the proof of Theorem 1. First of all, let us establish the following two identities:

$$\sum_{n \geq 1} c_n \frac{\zeta_n}{\zeta_n - \rho_m} = -c_0, \quad \sum_{n \geq 1} c_n \frac{\zeta_n}{\zeta_n + \Phi(q)} = -c_0 + \frac{q}{\Phi(q)} \frac{1}{\psi'(\Phi(q))}. \quad (4.13)$$

We use proposition 2 and formula (8.2) in [21] to find that for $\text{Re}(z) \geq 0$

$$\mathbb{E} [e^{zY_{e(q)}}] = c_0 + \sum_{n \geq 1} c_n \frac{\zeta_n}{\zeta_n + z} = \frac{q}{\Phi(q)} \frac{\Phi(q) - z}{q - \psi(z)}.$$

For meromorphic processes the last two expressions are well defined in the entire complex plane, thus we can extend the second equality to all complex numbers z . In particular, as $z \rightarrow -\rho_m$ we have $\psi(z) \rightarrow \infty$ (see (3.2)), and one obtains the first identity in (4.13). Similarly, taking limit as $z \rightarrow \Phi(q)$ we obtain the second identity in (4.13).

Now, in order to prove part (ii) of Theorem 2, we start with (4.10), set $u = x$, taking into account the atom on the plane $u = x$ if we have a process of bounded variation, and with the help of the two identities (4.13) and some straightforward algebraic manipulations we obtain (4.12). Note that due to proposition 2 the coefficient c_0 is non-zero if and only the process is of bounded variation, and it turns out that the contribution coming from (4.8) is cancelled in the resulting expression. \square

The above two theorems give closed-form expressions for the joint distribution of the three ruin-related quantities that, as we discussed, are of potential interest in risk management applications. The expressions involve infinite series, but in all cases these series converge exponentially fast, and as we will see in Section 6, we can usually truncate these series and keep only the first few terms. Also, it is an important feature of these results that they easily give us all possible combinations of marginal distributions. As we will see in Section 6, these Theorems allow us to compute popular distribution-based risk measures like VaR or CVaR for these random variables. Recall however, that these formulas are for the infinite-time horizon. In the following section we look at the more interesting, and more challenging, case of the finite-time horizon.

5 Computing the finite-time EDPF

As we have shown in the previous section, the infinite-time EDPF $\phi(x; q)$ can be computed in closed form, at least for functions w which can be integrated explicitly against exponential functions. Computing the finite-time EDPF $\phi_t(x; q)$ is a much harder problem. There is no known example of a Lévy process X , for which this object can be computed explicitly. At the same time, the Laplace transform in the t -variable of $\phi_t(x; q)$ is given by a closed form expression

$$\begin{aligned} \int_0^\infty e^{-\xi t} \phi_t(x; q) dt &= \int_0^\infty e^{-\xi t} \mathbb{E} [e^{-q\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbb{I}_{\{\tau < t\}} | X_0 = x] dt \\ &= \frac{1}{\xi} \mathbb{E} [e^{-q\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) e^{-\xi\tau} | X_0 = x] = \frac{\phi(x; q + \xi)}{\xi}. \end{aligned} \tag{5.1}$$

Thus we see that computing the finite time EDPF is equivalent to inverting the Laplace transform in ξ -variable in (5.1), and this has to be done numerically. The good news is that there exist a great variety of efficient methods for numerical Laplace transform inversion, see an excellent book by Cohen [6].

In this section we will discuss two methods for computing the finite-time EDPF, which we found to be particularly well suited for our problem. In the first method we express $\phi_t(x; q)$ as a cosine transform and then use Filon's approach to compute it efficiently. An important feature of this method is that it requires computing $\phi(x; q)$ for complex values of q . In the second method we use Gaver-Stehfest approach, which does not involve the use of complex numbers, however usually it would require high-precision arithmetic.

5.1 Method based on the cosine transform

We start with equation (5.1) and express $\phi_t(x; q)$ as a cosine transform: for any $q > 0$, $q_0 > 0$ we have

$$\phi_t(x; q) = \frac{2}{\pi} e^{q_0 t} \int_0^{\infty} \operatorname{Re} \left[\frac{\phi(x; q_0 + q + iu)}{q_0 + iu} \right] \cos(ut) du. \quad (5.2)$$

This reduces the problem of evaluating the finite-time EDPF $\phi_t(x; q)$ to computing the cosine transform of a function

$$G(u) := \operatorname{Re} \left[\frac{\phi(x; q_0 + q + iu)}{q_0 + iu} \right]. \quad (5.3)$$

Note, that the function $\phi(x; q)$ was introduced in Definition 1 only for $q \geq 0$, however it is clear that it can be analytically continued into the half-plane $\operatorname{Re}(q) \geq 0$, thus the expression in the right-hand side of (5.3) is well-defined. This expression is relatively easy to evaluate: one has to integrate the function w (see Definition 1) against the Gerber-Shiu measure (4.7), and in certain cases this can be done explicitly, see Theorems 1 and 2. The only unknown numbers in (4.7) are $\Phi(q)$ and $-\zeta_n$, which are the solutions to the equation $\psi(z) = q$. We will describe in detail how to compute these numbers efficiently for the case when $q \geq 0$ or $q \in \mathbb{C}$ below in Section 5.3.

While $G(u)$ is relatively easy to compute numerically, in order to compute the cosine transform in (5.2) we will need to evaluate the function $G(u)$ many times, and it is obvious that evaluating $G(u)$ is much harder than evaluating trigonometric functions. Thus we would want to use an algorithm which gives a good precision while minimizing the number of evaluations of $G(u)$. Filon's method is a great way to compute Fourier transforms, see [9, 10, 17], which help to achieve this objection. Let us briefly explain the main ideas behind this method and how one could implement it efficiently.

First, let us consider a simpler problem of how to approximate the cosine transform over a finite interval $[a, c]$:

$$I(a, c; t) = \int_a^c G(u) \cos(ut) du.$$

The main idea of Filon's method is to approximate $G(u)$ by a second-order Lagrange polynomial based on the three values at points a , $b = (a + c)/2$ and c :

$$G(u) = g_0 + g_1(u - b) + g_2(u - b)^2 + R(u), \quad (5.4)$$

where the constants g_i are defined as

$$g_0 = G(b), \quad g_1 = \frac{G(c) - G(a)}{2h}, \quad g_2 = \frac{G(c) - 2G(b) + G(a)}{2h^2}.$$

and $h = (b - c)/2$. Note that by construction we have $R(u) = 0$ at the three points a , b and c . It is well-known that the error of this approximation can be uniformly bounded by

$$\max\{|R(u)| : a \leq u \leq c\} < \frac{h^3}{12} \max\{|G^{(3)}(u)| : a \leq u \leq c\}.$$

Next, we integrate explicitly the quadratic polynomial in the approximation (5.4) against $\cos(ut)$ and after some simplification we obtain

$$\begin{aligned} I(a, c; t) &= g_0 f_s^-(t) t^{-1} + g_1 (f_c^-(t) t^{-2} + h f_s^+(t) t^{-1}) \\ &+ g_2 (2h f_c^+(t) t^{-2} + f_s^-(t) (h^2 t^{-1} - 2t^{-3})) + \mathcal{E}(t), \end{aligned} \quad (5.5)$$

where we have defined

$$f_s^\pm(t) = \sin(ct) \pm \sin(at), \quad f_c^\pm(t) = \cos(ct) \pm \cos(at).$$

Formula (5.5) is the main building block of approximating the cosine transform. An important feature of this algorithm is that the error of this approximation can be bounded by

$$|\mathcal{E}(t)| \leq \int_a^c |R(u)| |\cos(ut)| du < (c - a) \max\{|R(u)| : a \leq u \leq c\} < \frac{h^4}{6} \max\{|G^{(3)}(u)| : a \leq u \leq c\}, \quad (5.6)$$

and this bound does not depend on t .

Now the full algorithm for computing the finite-time EDPF can be described as follows. First we choose a small positive number q_0 , so that $q_0 t$ is not too large for the values of t that interest us. Second, we truncate the domain of integration in (5.2) at some large positive number U_{\max} . Third, we divide the interval $[0, U_{\max}]$ into N small subintervals $0 = u_0 < u_1 < \dots < u_N = U_{\max}$ and on each of them we approximate the cosine transform via Filon's formula (5.5). Note that this algorithm requires $2N + 1$ evaluation of $G(u)$.

There is another trick that can considerably speed up the computations. Usually the function $G(u)$ is changing rapidly for small values of u but it changes rather slowly when u is large, which implies that $G^{(3)}(u)$ is very small for u large. Therefore, from the error bound (5.6) we see that we can gradually increase the spacing between points u_i , which allows us to reduce the total number of evaluations of the integrand function $G(u)$ while keeping the error under control. The details and some examples of this technique will be presented in Section 6.

5.2 Method based on the Gaver-Stehfest algorithm

Gaver-Stehfest algorithm is a general technique for inverting Laplace transforms, which produces excellent results in many different examples. The detailed discussion of this algorithm will take us too far away from our immediate goal of computing the finite-time EDPF, thus we refer the interested reader to Abate and Whitt [2] and Section 7.2 in [6] for all the details and for the derivation of this algorithm, and here we will only present the final formula. The function $\phi_t(x; q)$ is approximated by $\phi_t^{GS}(x; q; M)$ (here M is a large integer number), and the approximation is defined as

$$\phi_t^{GS}(x; q; M) = \sum_{n=1}^{2M} \frac{a_n}{n} \phi(x; q + n \ln(2)t^{-1}), \quad (5.7)$$

where the coefficients a_n are given explicitly

$$a_n = (-1)^{M+n} \sum_{j=\lceil (n+1)/2 \rceil}^{n \wedge M} \frac{j^{M+1}}{M!} \binom{M}{j} \binom{2j}{j} \binom{j}{n-j}. \quad (5.8)$$

In order to ensure that the Gaver-Stehfest approximation $\phi_t^{GS}(x; q; M)$ is close to the true value of $\phi_t(x; q)$ we will have to use reasonably large values of M . One can see then that the coefficients a_n will be very large numbers, alternating in sign, which implies that we will lose accuracy in (5.7) due to subtracting very large numbers. The only way to deal with this problem is to use high-precision arithmetic and to compute a_n and $\phi(x, q)$ very accurately. Abate and Valko [1] (see also [2]) recommend the following “rule of thumb”: if we want j significant digits in our approximation, we should set $M = \lceil 1.1j \rceil$ (the least integer greater than or equal to $1.1j$) and set the system precision at $\lceil 2.2M \rceil$. It is also useful to check the accuracy of computation of a_n using the fact that the sum of these coefficients must be zero.

Several examples which illustrate the performance and efficiency of the Gaver-Stehfest algorithm will be discussed later in Section 6.

5.3 Computing $\zeta_n(q)$ and $\Phi(q)$ efficiently

As we have seen in Section 4, the solutions to the equation $\psi(z) = q$, which are given by $\{\Phi(q), -\zeta_n(q)\}$, are the most important objects for computing the expected discounted penalty function. Therefore it is crucial to be able to compute these numbers quickly and accurately. The good news is that it is very easy to do for the particular examples of beta and theta processes.

First, let us consider the simple case when q is real and positive. We will present an algorithm for computing ζ_n for $n \geq 2$, and the computation of $\Phi(q)$ and ζ_1 can be done along similar lines. The numbers ζ_n are the positive solutions to the equation $\psi(-z) = q$, where the Laplace exponent $\psi(z)$ is known in closed form (see formulas (3.4), (3.5) and (3.6)). Interlacing property (3.3) guarantees that ζ_n lies in the interval (ρ_{n-1}, ρ_n) , and formula (3.2) shows that $\psi(-z) \rightarrow -\infty$ as $z \rightarrow \rho_{n-1}^+$ and $\psi(-z) \rightarrow +\infty$ as $z \rightarrow \rho_n^-$. Moreover, it is easy to see from (3.2) that $\psi(-z)$ is strictly increasing on the interval (ρ_{n-1}, ρ_n) . This means that we can take a small positive ϵ (say, $\epsilon = 10^{-10}$) and use a standard bisection algorithm on the interval $(\rho_{n-1} + \epsilon, \rho_n - \epsilon)$ in order to approximate a solution ζ_n to the equation $\psi(-z) = q$. After several steps of bisection algorithm, once we have a reasonably good approximation of ζ_n , we apply a few steps of Newton’s method to obtain ζ_n to any desired precision.

There is another very useful trick that can speed up the computations of the roots ζ_n considerably. Let us consider the case of a beta process, for which the Laplace exponent is given by (3.6). We know that in this case the poles of $\psi(-z)$ are located at equally spaced points $\rho_m = \beta(\alpha + m)$. One can show (see [18]) that as $n \rightarrow +\infty$ the spacing $\zeta_{n+1} - \zeta_n$ converges to β . In particular, for n large enough, the number $\zeta_n + \beta$ will be a good approximation to ζ_{n+1} , and one can start Newton’s method from this point and obtain ζ_{n+1} to any desired precision in just a few iterations. This means that we don’t need to use bisection method for all roots ζ_n , but only for the first few of them.

Second, let us consider the more complicated problem of computing $\zeta_n(q)$ for $q \in \mathbb{C}$. This problem arises when we compute the finite time EDPF using cosine transform method, as outlined in section 5.1. One can see that this is a much harder problem compared to the case when q is real and positive. The reason is that now the solutions are complex numbers and we don’t have any localization results for them, such as the interlacing property (3.3). However there does exist a simple and efficient method, which was first presented in [18]. Assume that we want to compute the solutions of equation $\psi(-z) = q$ for all q in the interval with $\text{Re}(q) = q_0$ and $\text{Im}(q) \in [0, u_0]$ for some $q_0 > 0$ and $u_0 > 0$. Our first step is to compute “the initial values” $\zeta_n(q_0)$ using the method discussed in the previous two paragraphs. Next we consider each root $z(u) = \zeta_n(q_0 + iu)$ as a function of u , and applying implicit differentiation to the equation $\psi(-z) = q$ we obtain a first order differential equation

$$\frac{dz(u)}{du} = -\frac{i}{\psi'(-z(u))},$$

with the initial condition $z|_{u=0} = \zeta_n(q_0)$. Now we can compute the solution to this ODE using a numerical scheme, for example an adaptive Runge-Kutta or even a simple Euler's method, and at each step of this scheme we correct the solution to make sure that $\psi(-z(u)) = q_0 + iu$ by applying several iterations of Newton's method. Even though this method seems complicated, as we will see in Section 6, it runs very fast and is capable of producing very accurate results.

6 Numerical results

In this section we illustrate the models and methods discussed in this paper by presenting the results of several numerical experiments on computing both the finite and infinite time EDPF. We work with the theta process with parameter $\lambda = 3/2$, whose Laplace exponent is given by (3.4), although similar implementation can be carried out for the other two examples of processes discussed in this paper. We recall that (3.4) defines a process of infinite activity but finite variation of jumps, and its Lévy measure has a singularity at zero of the order $3/2$. We fix the two sets of parameters as follows:

$$\begin{aligned} \text{Parameter Set 1:} & \quad \sigma = 0, \mu = 15, c = 5.4, \alpha = 0.5, \beta = 0.35, \\ \text{Parameter Set 2:} & \quad \sigma = 0, \mu = 20, c = 5.4, \alpha = 0.5, \beta = 0.35. \end{aligned}$$

These parameters correspond to the mean/variance rate of the negative pure jump process being approximately 10 and 20, and $\mathbb{E}[Y_1] = 5$ $\{\mathbb{E}[Y_1] = 10\}$ in the case of parameter Set 1 $\{2\}$. This choice corresponds to a mean aggregate claim per time unit of 3.33 $\{5\}$ i.e. safety loading of 1.5 $\{2.0\}$. These parameter sets define a process of bounded variation; in order to illustrate the qualitative difference between bounded/unbounded variation (unperturbed/perturbed) case we will set $\sigma = 1$ in some examples below.

The code for all numerical experiments was written in Fortran90 and the computations were performed on a standard laptop (Intel Core i5 2.6 GHz processor and 4 GB of RAM).

Numerical Example 1.

As our first numerical example we compute the marginal distribution of the deficit at ruin

$$\mathbb{P}_x(-X_\tau < y, \tau < \infty),$$

and the distribution of the last minimum before ruin

$$\mathbb{P}_x(\underline{X}_{\tau-} < u, \tau < \infty),$$

for the parameter Set 1 and the two cases $\sigma = 0$ and $\sigma = 1$. The formula for $\mathbb{P}_x(X_\tau < y, \tau < \infty)$ can be obtained by setting $q = 0$ and $z = +\infty$ in (4.12), while the formula for $\mathbb{P}_x(\underline{X}_{\tau-} < u, \tau < \infty)$ is obtained by setting $q = 0$ and $y = z = +\infty$ in (4.10). We truncate all infinite series in these expressions at $n = 150$ and $m = 150$. Note that in the case of theta process both ρ_m and ζ_n increase quadratically in n or m (see (3.7) and (3.3)), which means that we do not need to take too many terms in the infinite series. As we will see later, sometimes even a much smaller number of terms can produce surprisingly accurate results.

We compute these quantities on a grid of 70×70 uniformly spaced points on the rectangle (x, y) or $(x, u) \in [0, 10] \times [0, 16]$. The results are presented on figure (2). Note that the two cases corresponding to $\sigma = 0$ and $\sigma = 1$ have different qualitative behavior, which agrees with what would be predicted by the

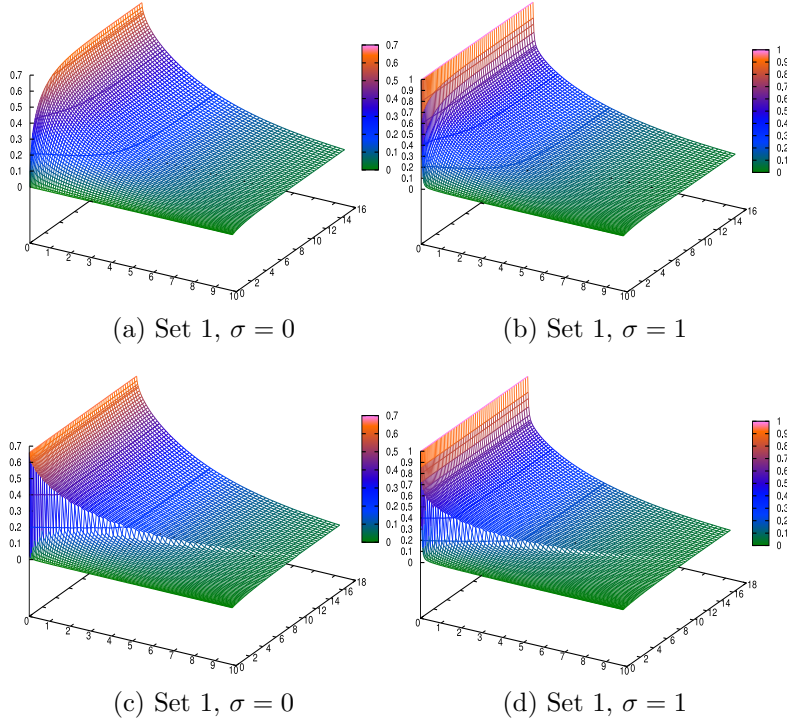


Figure 2: Graphs (a) and (b): computing $\mathbb{P}(X_\tau < y, \tau < \infty \mid X_0 = x)$ for $0 \leq x \leq 10$ and $0 \leq y \leq 16$. Graphs (c) and (d): computing $\mathbb{P}(\underline{X}_{\tau-} < u, \tau < \infty \mid X_0 = x)$ for $0 \leq x \leq 10$ and $0 \leq u \leq 16$.

general results of the fluctuation theory of Lévy processes. When $\sigma = 0$ we have a process of bounded variation, therefore the distribution of the last minimum has an atom along the line $u = x$, which is seen as a discontinuity of the function on the figure 2(c). This atom disappears in the regular case when $\sigma = 1$. Also, when $\sigma = 0$ $\{\sigma = 1\}$ we have $\lim_{x \rightarrow 0^+} \mathbb{P}_x(\tau < \infty) < 1$ $\{= 1\}$, and this results in a different behavior which is seen on all four graphs as $x \rightarrow 0^+$.

Next we would like to discuss performance issues related to these computations. In the first part of the algorithm we compute $\Phi(0)$ and $\zeta_n(0)$ for $1 \leq n \leq 150$ using the technique outlined in section 5.3. This part of the algorithm is extremely fast, it takes just 1.2×10^{-4} second to find the first 150 solutions.

Performing the second step, i.e. computing the single and double sums in (4.12) and (4.10), is also quite straightforward; the total computation time needed to produce each of the graphs (c) and (d) in figure 2 is around 3 seconds (note that we have to do computations for each point in the 70×70 grid). However it is well known that many modern programming languages, including Fortran90 and Matlab, can have much higher performance when computations are vectorized, which means that we work with vectors and matrices instead of numbers. We did this optimization exercise for computing the distribution of the deficit at ruin, which is shown on figure 2 (a) and (b), and in this case the time needed to produce each surface plot was reduced to just 0.1 second.

Numerical Example 2.

Now we would like to test the two methods for computing the finite time EDPF, which were discussed in section 5. In order to compare the accuracy and performance of these two methods we will consider first a rather simple problem of computing the finite time ruin probability $\mathbb{P}_x(\tau < t)$. Our starting point

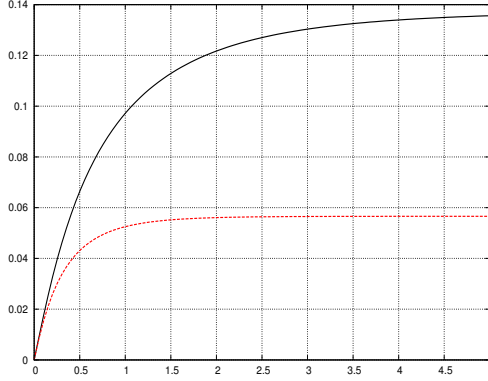


Figure 3: Ruin probability $\mathbb{P}(\tau < t | X_0 = 5)$ for parameter set 1 (black) and set 2 (red)

will be the following Laplace transform identity, which follows from Proposition 2

$$\int_0^{\infty} e^{-qt} \mathbb{P}_x(\tau < t) dt = q^{-1} \mathbb{P}_x(\tau < e(q)) = q^{-1} \mathbb{P}(-\underline{Y}_{e(q)} > x) = q^{-1} \sum_{n \geq 1} c_n e^{-\zeta_n x}, \quad (6.1)$$

where the coefficients c_n are defined by (4.5).

In order to compare the performance of the two methods presented in section 5 we assume that the initial surplus is $X_0 = 5$ and we will compute the maximum absolute error

$$\text{Error} = \max\{|\mathbb{P}_{\text{exact}}(\tau < t | X_0 = 5) - \mathbb{P}_{\text{approx}}(\tau < t | X_0 = 5)| : t = j/10, \quad 1 \leq j \leq 50\}.$$

Since we do not have an explicit expression for the exact value of the finite time ruin probability, we'll have to compute the benchmark $\mathbb{P}_{\text{exact}}(\tau < t | X_0 = 5)$ numerically, and we will use the method based on the cosine transform (see section 5.1). From our later results we will have sufficient evidence that our benchmark is in fact good enough.

Let us describe the parameters we have used to compute this benchmark. We truncate the series in (6.1) at $N_{\zeta} = 500$. We take a large cutoff $U_{\max} = 10^5$. Now we need to partition the interval of integration $[0, U_{\max}]$ into a large number of smaller subintervals. As we have discussed at the end of Section 5.1, we do not need to keep the width of all subintervals constant, it is in fact more efficient if we slowly increase their width. We use the following simple rule: we set the width of the first subinterval $\Delta u_1 := u_1 - u_0 = 10^{-4}$, and then increase the width of each next subinterval by $100 \times \epsilon \%$ up to a maximum of Δu_{\max} , thus

$$\Delta u_{n+1} := u_{n+1} - u_n = \min\{\Delta u_n(1 + \epsilon), \Delta u_{\max}\}.$$

where for our computations we have set $\epsilon = 0.01$ and $\Delta u_{\max} = 0.1$. This gives us a partition of the interval $[0, U_{\max}]$ into subintervals $[u_n, u_{n+1}]$, and on each of these subintervals we apply the three point Filon's integration rule (5.5) (using the middle point of $[u_n, u_{n+1}]$ as the third point).

The overall time needed to compute this benchmark was around 14 minutes. As we will see later, there seems to be enough evidence that this benchmark is correct to at least 9 decimal digits. The graphs of the benchmark for the two parameter sets are presented in the figure 3. Note that, as we would

Method based on the cosine transform						
Δu_1	ϵ	Δu_{\max}	N_ζ	U_{\max}	time (sec.)	error
0.001	0.01	1	50	10 0000	1.28	1.0e-8
0.01	0.01	5	50	5 000	0.17	5.7e-8
0.01	0.03	10	20	1 000	0.012	3.8e-7
0.01	0.03	10	3	1 000	0.002	3.8e-7
0.01	0.03	10	2	1 000	0.0015	3.8e-7
0.01	0.03	10	1	1 000	0.001	1.2e-4

Table 1: Computing the finite time ruin probability $\mathbb{P}(\tau < t|X_0 = 5)$ (Parameter set 1)

expect, the ruin probability for parameter set 1 (safety loading equals 1.5) is higher than in the case of parameter set 2 (safety loading equal to 2).

First we investigate the accuracy and the performance of the method based on the cosine transform. The numerical results and the values of all relevant parameters are presented in table 1. From these results we see that one can compute the finite time ruin probability to a reasonably high accuracy of 7-8 decimal digits in just a fraction of a second. We would like to emphasize that we present the time it takes to compute the values of the approximation to $\mathbb{P}(\tau < t_j|X_0 = 5)$ at fifty points $t_j = j/10$, $1 \leq j \leq 50$, so the algorithm is indeed extremely fast and efficient. A rather surprising fact is that we still have a very good approximation, with the error of just $3.8e - 7$, even if we truncate the series in (6.1) at $N_\zeta = 2$ terms!

Method based on the Gaver-Stehfest algorithm					
	$N_\zeta = 50$	$N_\zeta = 20$	$N_\zeta = 3$	$N_\zeta = 2$	$N_\zeta = 1$
$M = 20$	error=4.4e-10 time=0.90	error=4.4e-10 time=0.37	error=4.4e-10 time=0.07	error=8.8e-9 time=0.05	error=1.2e-4 time=0.034
$M = 10$	error=4.8e-9 time=0.44	error=4.8e-9 time=0.18	error=4.8e-9 time=0.04	error=9.7e-9 time=0.026	error=1.2e-4 time=0.016

Table 2: Computing the finite time ruin probability $\mathbb{P}(\tau < t|X_0 = 5)$ (Parameter set 1). The time is given in seconds.

Next we turn our attention to the Gaver-Stehfest algorithm described in Section 5.2. As we have discussed in Section 5.2, Gaver-Stehfest algorithm normally requires higher-precision arithmetic, therefore for this experiment we wrote the code in Fortran90 using the “quad” data type, which represents numbers with the precision of 32 decimal digits (compared to 16 decimal digits of the standard, “double” data type).

The numerical results are presented in table 2. Recall, that Gaver-Stehfest approximation depends on a single integer parameter M , see (5.7). First of all, note that when we take $M = 20$, the results produced by the Gaver-Stehfest algorithm are within $4.4e-10$ of our benchmark. Of course this tells us that the Gaver-Stehfest method is very accurate, but more importantly, it also confirms that our benchmark is correct up to at least 9 decimal digits. If this was not correct, it would mean that two completely different

numerical algorithms produce almost identical incorrect results, which is a very unlikely scenario. Also note that, as in the previous case, the results are quite accurate even if we truncate the infinite series (6.1) at $N_\zeta = 2$ terms, and of course the run time of the algorithm is extremely small.

Overall, as we see from tables 1 and 2, the accuracy and performance of the Gaver-Stehfest algorithm is comparable to the method based on cosine transform and Filon's rule. The first method has an important advantage that it is quite robust: we can be certain that we will achieve good accuracy provided that U_{\max} is sufficiently large and that we create a sufficiently fine partition of the interval $[0, U_{\max}]$ (though it is hard to say apriori how large U_{\max} should be and how small the partition should be). The second method has an advantage that it depends only on a single parameter M , it is easier to program, it does not require complex numbers and we don't have to solve equation $\psi(z) = q$ for complex values of q . The price we have to pay for this luxury is that we have to use higher-precision arithmetic. However as we see from our results, the "quad" data type (which represents numbers with the precision of 32 decimal digits) seems to be enough for our purposes.

Numerical Example 3.

For our final numerical experiment we compute the value at risk for the deficit at ruin, conditional on the event that the ruin happens before deterministic time t . That is, first we compute the conditional distribution function

$$F_{x,t}(y) = \mathbb{P}_x(-X_\tau < y | \tau < t)$$

and then for a given confidence level $\alpha \in (0, 1)$ we find VaR_α by solving equation

$$F_{x,t}(\text{VaR}_\alpha) = \alpha. \tag{6.2}$$

We see that in order to find VaR_α first we need to be able to compute function $F_{x,t}(y)$. Clearly, the conditional distribution function is given by $F_{x,t}(y) = f(x; t, y)/f(x; t, +\infty)$ where we have defined

$$f(x; t, y) = \mathbb{P}_x(-X_\tau < y, \tau < t).$$

As in the previous section, we find that function $f(x; t, y)$ has an explicit Laplace transform in t -variable:

$$\begin{aligned} \int_0^\infty e^{-qt} f(x; t, y) dt &= q^{-1} \mathbb{P}_x(-X_\tau < y, \tau < e(q)) = q^{-1} \mathbb{E}_x [e^{-q\tau} \mathbb{I}(-X_\tau < y)] \\ &= \frac{1}{q} \sum_{n \geq 1} c_n e^{-\zeta_n x} + \frac{\Phi(q)}{q^2} \sum_{m, n \geq 1} \frac{c_n \zeta_n b_m e^{-\zeta_n x - \rho_m y}}{\rho_m (\Phi(q) + \rho_m) (\zeta_n - \rho_m)}. \end{aligned} \tag{6.3}$$

The last equality in the above expression follows by setting $z = +\infty$ in formula (4.12). Therefore, we can compute $f(x; t, y)$ and $f(x; t, +\infty)$ by evaluating the inverse Laplace transform using one of the methods presented in Section 5 and then obtain the conditional cdf $F_{x,t}(y)$ via $F_{x,t}(y) = f(x; t, y)/f(x; t, +\infty)$.

In order to compute the value at risk, we note that (6.2) tells us that VaR_α as a function of α is just an inverse function of $F_{x,t}(y)$, and we use the following standard technique for computing the values of the inverse function to high precision. We first evaluate the conditional cdf $F_{x,t}(y)$ at 200 equally spaced points $y_j = j/20$ in the interval $[0, 10]$. The definition of the value at risk (6.2) then tells us that

$$\text{VaR}_{\alpha_j} = y_j,$$

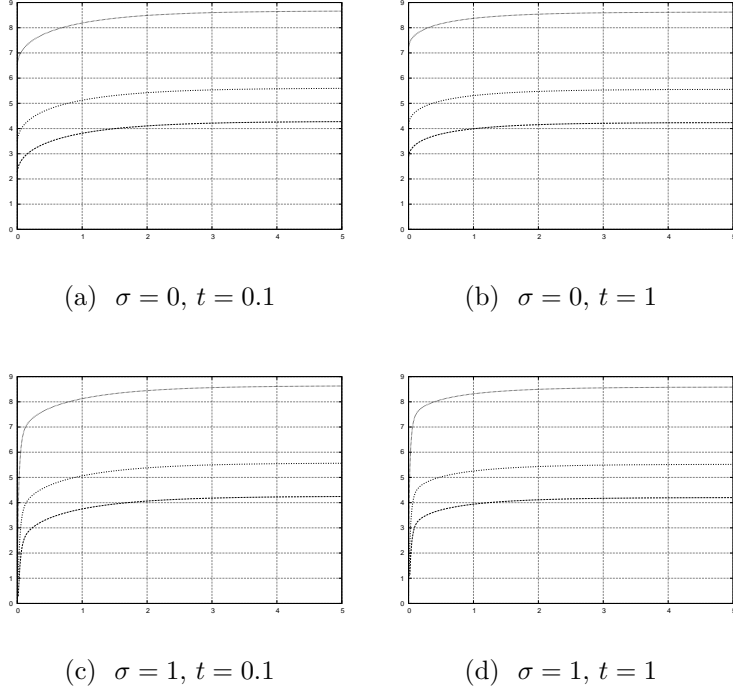


Figure 4: Computing finite time VaR_α for the overshoot $-X_\tau$ for $\alpha \in \{0.9, 0.95, 0.99\}$. The initial surplus is $X_0 \in [0, 5]$. We use parameter set 2.

where $\alpha_j := F_{x,t}(y_j)$. Finally, we find five points α_j closest to α and use the fourth order Lagrange polynomial interpolation (with nodes as these five points α_j) to obtain VaR_α with better precision.

In order to be able to compare our results, first of all we compute the benchmark values of VaR_α for the three values of $\alpha \in \{0.9, 0.95, 0.99\}$. As in the previous section, we use the method based on the cosine transform and Filon's rule with a very large cutoff U_{\max} and very fine partition of the interval of integration $[0, U_{\max}]$. We truncate the series in (6.3) at $n = m = N_\zeta$. On figure 4 we present the results produced by this benchmark for different values of the initial surplus $X_0 \in [0, 5]$, and for further investigation we will use the following benchmark values for the initial surplus $X_0 = 2$:

$$\text{VaR}_{0.9} = 4.154505818\dots, \quad \text{VaR}_{0.95} = 5.472856602\dots, \quad \text{VaR}_{0.99} = 8.537833156\dots$$

Method based on the cosine transform						
Δu_1	ϵ	Δu_{\max}	N_ζ	U_{\max}	time (sec.)	error
0.01	0.01	5	50	5 000	93	7.0e-9
0.01	0.03	10	20	1 000	0.6	4.7e-8
0.01	0.03	10	3	1 000	0.02	3.9e-6
0.01	0.03	10	2	1 000	0.01	9.6e-4

Table 3: Computing the finite time VaR_α for $X_0 = 2$ (Parameter set 2).

Method based on the Gaver-Stehfest algorithm				
	$N_\zeta = 50$	$N_\zeta = 20$	$N_\zeta = 3$	$N_\zeta = 2$
$M = 20$	error=6.5e-4 time=1.4	error=1.6e-3 time=0.24	error=7.1e-3 time=0.02	error=3.4e-2 time=0.012
$M = 10$	error=6.5e-4 time=0.68	error=1.6e-3 time=0.12	error=7.1e-3 time=0.008	error=3.4e-2 time=0.005

Table 4: Computing the finite time VaR_α for $X_0 = 2$ (Parameter set 2). The time is given in seconds.

Once we have our benchmark, we can compare it with the results produced by the two methods. In tables 3 and 4 we present the error of the approximation (the maximum absolute error over the three numbers) and the running time of the algorithm. Qualitatively the results are very similar to the ones presented in the previous section. One difference is that the Gaver-Stehfest algorithm is not as accurate as before, and it produces only 3-4 decimal digits. It seems that running the Gaver-Stehfest algorithm with larger values of N_ζ improves accuracy, but at the expense of longer running time. Overall, the Gaver-Stehfest algorithm confirms that our benchmark is correct to at least 3-4 decimal digits (however it is more likely to be correct to at least 7 digits). Also note that, similarly to the results in the previous section, both algorithms produce reasonably accurate results even when we truncate the series in (6.3) at just $m = n = N_\zeta = 3$ terms, and the run time of the program in this case is extremely small.

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