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ON THE FIRST PASSAGE TIME FOR BROWNIAN MOTION SUBORDINATED BY A LÉVY PROCESS

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Abstract

This paper considers the class of Lévy processes that can be written as a Brownian motion time changed by an independent Lévy subordinator. Examples in this class include the variance gamma model, the normal inverse Gaussian model, and other processes popular in financial modeling. The question addressed is the precise relation between the standard first passage time and an alternative notion, which we call first passage of the second kind, as suggested by [12] and others. We are able to prove that standard first passage time is the almost sure limit of iterations of first passage of the second kind. Many different problems arising in financial mathematics are posed as first passage problems, and motivated by this fact, we are lead to consider the implications of the approximation scheme for fast numerical methods for computing first passage. We find that the generic form of the iteration can be competitive with other numerical techniques. In the particular case of the VG model, the scheme can be further refined to give very fast algorithms.

Keywords: Brownian motion; first passage; time change; Lévy subordinators; stopping times; financial models.

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1. Introduction

First passage problems are a classic aspect of stochastic processes that arise in many areas of application. In mathematical finance, for example, first passage problems lie at the heart of such issues as credit risk modelling, pricing barrier options, and the optimal exercise of american options. If X_t is any process with initial value $X_0 = x_0$, the first passage time to a lower level b is defined to be the stopping time

$$t_b^*(x_0) = \inf\{t \geq 0 | X_t \leq b\} \tag{1}$$

The distributional properties of t^* can be easily obtained when the underlying process X is a diffusion (see [7]), but when X has jumps the situation is much more challenging.

Results on Wiener-Hopf type factorizations (see [3], [6], [15], [16]) have proved to be very useful for studying first-passage time problems for a Lévy processes. Probably the best known result of this type is the following identity (see [6],[15]):

$$\frac{q}{q + \psi(\lambda)} = \Phi_q^+(\lambda)\Phi_q^-(\lambda). \tag{2}$$

Here $\psi(\lambda)$ is the characteristic exponent of X_t; $\Phi_q^+(\lambda)$ and $\Phi_q^-(\lambda)$ are characteristic functions of infinitely divisible random variables $S_{\tau(q)}$ and $X_{\tau(q)} - S_{\tau(q)}$, where $S_t = \sup\{X_s : s \leq t\}$ is the supremum process and $\tau(q)$ is an exponential random variable with parameter q, independent of X_t.

We can efficiently recover functions $\Phi^{\pm}$ using equation (2) when $\psi(\lambda)$ is a rational function. A well studied class of processes for which this approach works well consists of Lévy processes with *phase-type distributed jumps* (see [3], [4], [14], [1]). Phase-type distributions are defined as the first passage time for a continuous time, finite state Markov chain, they form a dense class in the set of all distributions on $\mathbb{R}^+$ and, most importantly, if a Lévy process has phase-type jumps, it's characteristic exponent is a rational function (though the converse is not true, see [6], [19]).

However, if the jumps of process X_t are not of phase-type, we would need to approximate the jump measure of X_t with a sequence of phase-type measures. The first problem with this approach is that there do not exist any efficient algorithms on how to achieve this. The second problem is that the degree of polynomial equation $q + \psi(\lambda) = 0$ would necessarily grow to infinity, which will make solution of this equation very complicated. Also, see [3] and [19] for an interesting example of a distribution with rational transform which would require an infinite degree phase-type representation.

A second general approach to first passage is to solve the Fokker-Planck equation for the probability density of X_t conditioned on the set $\{t^* > t\}$. For Lévy processes this amounts to solving a certain linear partial integral differential equation (PIDE) with nonlocal Dirichlet conditions (see [9], [10]). In the case of Lévy processes the PIDE approach has an advantage that we can utilize the FFT to perform efficient computation of convolutions involved, however the method also involves the truncation of the state space (real line in our case), discretization in x and t variables, and the resulting errors are not easy to control.

Our purpose here is to present a new approach to first passage problems applicable whenever the underlying Lévy process can be realized as a Lévy subordinated Brownian motion (LSBM), that is, whenever X can be constructed as $\tilde{W} \circ T$ where $\tilde{W}$ is a standard drifting Brownian motion and T is a non-decreasing Lévy process independent of $\tilde{W}$. The class of Lévy processes that are realizable as LSBMs is identified in [9][Theorem 4.3] , and is broad enough to include many of the Lévy processes that have so far been used in finance, such as a four-parameter subclass of the Kou-Wang model, the variance gamma (VG) model, the normal inverse Gaussian (NIG) model and a four-parameter subclass of the generalized tempered stable process.

The basis for our approach is that for processes that are realizable as time changes of Brownian motions, there is an alternative notion that is also relevant, namely, the first time the time change exceeds the first passage time of the Brownian motion. This notion, called *first passage of the second kind* in [12], shares some characteristics with the usual first passage time and can be applied in a similar way. The usefulness of this new concept is that it can be computed efficiently in many cases where the usual first passage time cannot.

In the present paper, we study first passage for LSBMs and show how the first passage of the second kind is the first of a sequence of stopping times that converges almost surely to the first passage time. Expressed differently, first passage can be viewed as a stochastic sum of first passage times of the second kind. This sequence leads to a convergent and computable expansion for the first passage probability distribution function p^* in terms of a similar function p^*_1 that describes the first passage distribution of the second kind. The outline of the paper is as follows. In Section 2, we define the objects needed to understand first passage time, and prove the expansion formula for first passage. In Section 3, we demonstrate the usefulness of this expansion by proving several explicit two dimensional integral formulas for p_1^*, the first passage distribution of the second kind. Section 4 provides two proofs of the convergence of the expansion. The first proof is a proof of convergence in distribution, the second is in the pathwise (almost sure) sense. Section 5 focusses on the special case of the variance gamma (VG) model. In this important example, the formula for p_1^* is reduced to a one-dimensional integral (involving the exponential integral function). In Section 6, the expansion of the function p^* is studied numerically, and found to be numerically stable and efficient.

2. First Passage for LSBMs

Let X_t be a general Lévy process with initial value $X_0 = x_0$ and characteristics $(b, c, \nu)_h$ with respect to a truncation function $h(x)$ (see [2] or [13]). This means X is an infinitely divisible process with identical independent increments and cádlág paths almost surely. $b, c \geq 0$ are real numbers and ν is a sigma finite measure on $\mathbb{R} \setminus 0$ that integrates the function $1 \wedge x^2$. By the Lévy-Khintchine formula, the log-characteristic function of X_1 is

$$\log E[e^{iuX_1}] = iub - cu^2/2 + \int_{\mathbb{R}\setminus 0} \left(e^{iux} - 1 - xh(x)\right) \nu(dx). \tag{3}$$

In what follows we will find it convenient to focus on the Laplace exponent of X:

$$\psi_X(u) := -\log E[e^{-uX_1}]$$

For simplicity of exposition, we specialize slightly by assuming that ν is continuous with respect to Lebesgue measure $\nu(dx) = \nu(x)dx$, and integrates $1 \wedge |x|$, allowing us to take $h(x) = 0$. In this setting, the Markov generator of the process X_t applied to any sufficiently smooth function $f(x)$ is

$$[\mathcal{L}f](x) = b\partial_x f + \frac{c}{2}\partial_{xx}^2 f + \int_{\mathbb{R}\setminus 0} (f(x+y) - f(x))\, \nu(y)dy \tag{4}$$

Definition 1. For any $b \in \mathbb{R}$, the random variable $t_b^* = t_b^*(x_0) := \inf\{t \mid X_t \leq b\}$ is called *the first passage time for level* b. When $b = 0$, we drop the subscript and $t^* := \inf\{t \mid X_t \leq 0\}$ is called simply *the first passage time of* X.

Remarks 1. 1. Since distributions of the increments of X are invariant under time and state space shifts, we can reduce computations of $t_b^*(x_0)$ to computations of $t^*(x_0 - b)$.

2. A general Lévy process is a mixture of a continuous Brownian motion with drift and a pure jump process. One says that “downward creeping” occurs if $X_{t^*} = 0$ and does not occur if $X_{t^*} < 0$. Under the assumption that ν integrates $1 \wedge |x|$, Corollaries 3 and 4 of Rogers [21] prove that there is almost surely no downward creeping if and only if the diffusive part is zero (i.e. $c = 0$) and the drift $b \geq 0$. In what follows we shall exclude the possibility of downward creeping. In this case, $X_{t^*} - X_{t^*-} \neq 0$, so X jumps across 0, and we can define the *overshoot* to be X_{t^*}.

The central object of study in this paper is the joint distribution of t^* and the overshoot X_{t^*}, in particular the joint probability density function

$$p^*(x_0; s, x_1) = E_{x_0}[\delta(t^* - s)\delta(X_{t^*} - x_1)]. \tag{5}$$

The marginal density of t^* is $p^*(x_0; s) = \int_{-\infty}^{0} p^*(x_0; s, x_1)\, dx_1$.

In the introduction, we noted that results on first passage for general Lévy processes, in particular results on the functions p^*, are difficult. For this reason, we now focus on a special class of Lévy processes that can be expressed as a drifting Brownian motion subjected to a time change by an independent Lévy subordinator. Such *Lévy subordinated Brownian motions* (LSBM) have been studied in the text by [9] and in [12]. The general LSBM is constructed as follows:

1. For an initial value $x_0 > 0$ and drift β, let $\tilde{W}_T = x_0 + W_T + \beta T$ be a drifting BM;

2. For a Lévy characteristic triple $(b, 0, \mu)$ with $b \geq 0$ and $\text{supp}(\mu) \subset \mathbb{R}^+$, let the time change process T_t be the associated nondecreasing Lévy process (a subordinator), taken to be independent of W;

3. The time changed process $X_t = \tilde{W}_{T_t}$ is defined to be a LSBM.

So constructed, it is known that X_t is itself a Lévy process. X_t will allow creeping if and only if $b > 0$: we henceforth assume for simplicity that $b = 0$. [9][Theorem 4.3] provides a characterization of which Lévy processes are LSBMs. While, unlike the class of phase type Lévy processes, the class of LSBMs is not dense in the class of Lévy processes, their analytic properties make them a useful and flexible class. It was observed in [12] that for any LSBM X_t, one can define an alternative notion of first passage time, which we denote here by $\tilde{t}$.

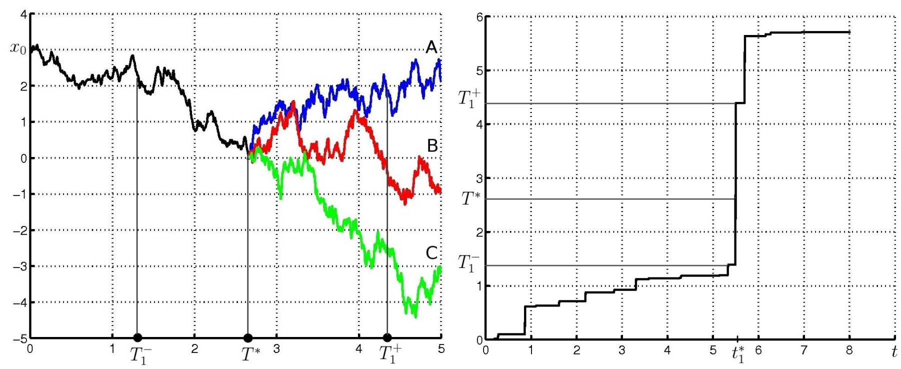

FIGURE 1: These three trajectories of the Brownian motion X_t with the same T^* and the sample path of the time change T_t illustrate that in general $\tilde{t} \le t^*$. On the paths B and C, $X_{t_1^*} = \tilde{W}_{T_1^+} \le 0$ and on these paths $t_1^* = \tilde{t} = t^*$. On path A, $X_{t_1^*} = \tilde{W}_{T_1^+} > 0$ and so $t_1^* = \tilde{t} < t^*$.

Definition 2. For any LSBM $X_t = \tilde{W}_{T_t}$ we define the *first passage time of* $\tilde{W}$ to be $T^* = T^*(x_0) = \inf\{T : x_0 + W_T + \beta T \le 0\}$. Note $T^*(x_0) = 0$ when $x_0 \le 0$. The *first passage time of the second kind* of X_t is defined to $\tilde{t} = \tilde{t}(x_0) = \inf\{t : T_t \ge T^*(x_0)\}$

This definition of $\tilde{t}$, and its relation to t^*, is illustrated in Figure 1.

We now show that $\tilde{t}(x_0) := t_1^*(x_0)$ is the first of a sequence of approximations $\{t_i^*(x_0)\}_{i=1,2,...}$ to the stopping time t^*. Fig 1 illustrates the first excursion overjump t_1^*, which may be either t^* or not. The construction of $t_i^*(x_0, \pi)$ is pathwise: We introduce the second argument $\pi \in \Omega$ which denotes a *sample path*, that is, a pair (ω, τ) where ω is a continuous drifting Brownian path $\tilde{W}$ and τ is a cádlág sample time change path T. Thus $\pi : (S, s) \to (\omega(S), \tau(s))_{S,s\ge 0}$. The natural "big filtration" $(\mathcal{F}_t)_{t\ge 0}$ for time-changed Brownian motion has $\mathcal{F}_t = \sigma\{\omega(S), \tau(s), S \le \tau(t), s \le t\}$. For any $t \ge 0$ there is a natural "time translation" operation on paths $\rho_t : (\omega, \tau) \to (\omega', \tau')$ where $\omega'(S) = \omega(S+\tau(t)), \tau'(s) = \tau(s+t) - \tau(t)$.

The construction of $\{t_i^*(x_0, \pi)\}_{i=1,2,...}$ for a given sample path π is as follows. Inductively, for $i \ge 2$, we define the *time of the i-th excursion overjump*

$$t_i^*(x_0, \pi) = \inf\{t \ge t_{i-1}^*(x_0, \pi) : T_t - T_{t_{i-1}^*} \ge T^*(X_{t_{i-1}^*}, \pi')\} \quad (6)$$

where $\pi' = \rho_{t_{i-1}^*}(\pi)$ denotes a time shifted sample path. Note that $t_i^*(x_0, \pi) = t_{i-1}^*(x_0, \pi)$ if and only if $X_{t_{i-1}^*(x_0,\pi)} \le 0$ or $t_{i-1}^*(x_0, \pi) = \infty$. At any excursion overjump event t_i^*, the time interval which covers the event has left and right endpoints $T_i^- = T_{t_i^*-}$ and $T_i^+ = T_{t_i^*}$.

Let

$$p_i^*(x_0; s, x) = E_{x_0}[\delta(t_i^* - s)\delta(X_{t_i^*} - x)] \tag{7}$$

denote the joint distribution of $t_i^*(x_0), X_{t_i^*(x_0)}$.

The definition of this sequence of stopping times is summarized by the pathwise equation

$$t_i^*(x_0, \pi) = t_{i-1}^*(x_0, \pi)\mathbf{1}_{\{X_{t_{i-1}^*} \le 0\}} + \Big(t_1^*(X_{t_{i-1}^*}, \pi') + t_{i-1}^*(x_0, \pi)\Big)\mathbf{1}_{\{X_{t_{i-1}^*} > 0\}}, \quad i \ge 2 \tag{8}$$

where $\pi' = \rho_{t_{i-1}^*}(\pi)$. The identical increments property of the LSBM implies the joint probability densities satisfy the recursive relation

$$p_i^*(x_0; s, x) = p_1^*(x_0; s, x)I(x \le 0) + \int_0^\infty dy \int_0^s du\, p_1^*(x_0; u, y)p_{i-1}^*(y; s-u, x), \quad i \ge 2 \tag{9}$$

Similarly, the PDF of the first passage time t^* satisfies the relation

$$p_i^*(x_0; s) = \int_{-\infty}^0 p_1^*(x_0; s, x)dx + \int_0^\infty dy \int_0^s du\, p_1^*(x_0; u, y)p_{i-1}^*(y; s-u), \quad i \ge 2. \tag{10}$$

We note in passing that when downward creeping is included, the probability function includes an atom at $x = 0$, and that when interpreted in that light, (9) and (10) are still correct.

Examples of LSBMs: We note here three classes of Lévy processes that can be written as LSBMs and have been used extensively in financial modeling.

1. The exponential model with parameters (a, b, c) arises by taking T_t to be the increasing process with drift $b \ge 0$ and jump measure $\mu(z) = ce^{-az}, c, a > 0$ on $(0, \infty)$. The Laplace exponent of T is

   $$\psi_T(u) := -\log E[e^{-uT_1}] = bu + uc/(a+u).$$

   One can show using [9][equation 4.14] that the resulting time-changed process $X_t := \tilde{W}_{T_t}$ has triple $(\beta b, b, \rho)$ with

   $$\rho(y) = \frac{c}{\sqrt{\beta^2 + 2a}} e^{-(\sqrt{\beta^2+2a}-\beta)(y)^+ - (\sqrt{\beta^2+2a}+\beta)(y)^-},$$

   where $(y)^+ = \max(0, y), (y)^- = (-y)^+$. This forms a four dimensional subclass of the six-dimensional family of exponential jump diffusions studied by [14].

2. The variance gamma (VG) model [17] arises by taking T_t to be a gamma process with drift defined by the characteristic triple $(b, 0, \mu)$ with $b \geq 0$ (usually b is taken to be 0) and jump measure $\mu(z) = (\nu z)^{-1} \exp(-z/\nu), \nu > 0$ on $(0, \infty)$. The Laplace exponent of $T_t, t = 1$ is

$$\psi_T(u) := -\log E[e^{-uT_1}] = bu + \frac{1}{\nu} \log(1 + \nu u).$$

The resulting time-changed process has triple $(\beta b, b, \rho)$ with

$$\rho(y) = \frac{1}{\nu |y|} \exp\left(\beta x - \sqrt{\frac{2}{\nu} + \beta^2} |x| \right).$$

3. The normal inverse Gaussian model (NIG) with parameters $\tilde{\beta}, \tilde{\gamma}$ [5] arises when T_t is the first passage time for a second independent Brownian motion with drift $\tilde{\beta} > 0$ to exceed the level γt. Then

$$\psi_T(u) = \tilde{\gamma}(\tilde{\beta} + \sqrt{\tilde{\beta}^2 + 2u})$$

and the resulting time-changed process has Laplace exponent

$$\psi_X(u) = x\mu + \tilde{\gamma}[\tilde{\beta} + \sqrt{\tilde{\beta}^2 - u^2 + 2\tilde{\beta} u}].$$

3. Computing First Passage of the Second Kind

We have just seen that first passage for LSBMs admits an expansion as a sum of first passage times of the second kind. In this section, we show that this expansion can be useful, by proving several equivalent integral formulas for computing the structure function $p_1^*(x_0; s, x_1)$ for general LSBMs. While the equivalence of these formulas can be demonstrated analytically, their numerical implementations will perform differently: which formula will be superior in practice is not a priori clear, but will likely depend on the range of parameters involved. For a complete picture, we provide independent proofs of the two given formulas.

Theorem 1. *Let the time change T_t have $b = 0$ and Laplace exponent $\psi(u)$, and let $\tilde{W}$ have drift $\beta \neq 0$. Then*

$$\begin{aligned} p_1^*(x_0; s, x_1) \\ = \frac{e^{\beta(x_1 - x_0)}}{4\pi^2} \iint_{\mathbb{R}^2} \frac{\psi(iz_1) - \psi(iz_2)}{i(z_1 - z_2)} \frac{e^{-s\psi(iz_1)}}{\sqrt{\beta^2 - 2iz_2}} e^{-x_0\sqrt{\beta^2 - 2iz_1} - |x_1|\sqrt{\beta^2 - 2iz_2}} dz_1 dz_2 \end{aligned} \quad (11)$$

Provided the time change is not a compound Poisson process, then

$$\begin{aligned}
&p_1^*(x_0; s, x_1) \\
&\quad = -\frac{2e^{\beta(x_1-x_0)}}{\pi^2} PV \iint_{(\mathbb{R}^+)^2} dk_1 dk_2 \frac{k_2 \cos|x_1|k_1 \sin x_0 k_2}{k_1^2 - k_2^2} e^{-s\psi((k_1^2+\beta^2)/2)} \psi((k_2^2+\beta^2)/2) \\
&\qquad -\frac{e^{\beta(x_1-x_0)}}{\pi} \int_{\mathbb{R}^+} \sin|x_1|k \ \sin x_0 k \ e^{-s\psi((k^2+\beta^2)/2)} \psi((k^2+\beta^2)/2) dk \qquad (12)
\end{aligned}$$

Here PV denotes that the principal value contour is taken.

Remark 2. The equivalence of these two formulas can be demonstrated directly by performing a change of variables $k_j = i\sqrt{\beta^2 - 2iz_j^2}$, $j = 1, 2$, followed by a deformation of the contours. Justification of the contour deformation (from the branch of a left-right symmetric hyperbola in the upper half k_j-plane to the real axis) depends on the decay of the integrand, and the computation of certain residues.

3.1. First proof of Theorem 1

For a fixed level $h > 0$, the first passage time and the overshoot of the process T_t above the level h are defined to be $\tilde{t}(h) = \inf\{t > 0 \mid T_t > h\}$ and $\tilde{\delta}(h) = T_{\tilde{t}(h)} - h$. The Pecherskii-Rogozin identity [20] applied to the nondecreasing process T says that

$$\int_0^\infty e^{-z_1 h} E\left[e^{-z_2\tilde{\delta}(h) - z_3\tilde{t}(h)}\right] dh = \frac{\psi(z_1) - \psi(z_2)}{z_1 - z_2} \left(z_3 + \psi(z_1)\right)^{-1},$$

Inversion of the Laplace transform in the above equation then leads to

$$E\left[e^{-z_2\tilde{\delta}(h) - z_3\tilde{t}(h)}\right] = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} \left(z_3 + \psi(iz_1)\right)^{-1} \ e^{iz_1 h} \ dz_1 \qquad (13)$$

The first passage time of the BM with drift is defined as $T^* = T^*(x_0) = \inf\{T > 0 \mid x_0 + W_T + \beta T < 0\}$. Next, we need to find the joint Laplace transform of $t_1^* = \inf\{t \mid T_t > T^*\} = \tilde{t}(T^*)$ and the overshoot $\delta^* = \tilde{\delta}(T^*)$. Since T_t is independent of W_T we find that

$$\begin{aligned}
E\left[e^{-z_2\delta^* - z_3 t_1^*}\right] &= E\left[E\left[e^{-z_2\tilde{\delta}(T^*) - z_3\tilde{t}(T^*)} \mid T^*\right]\right] \qquad (14) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} \left(z_3 + \psi(iz_1)\right)^{-1} E\left[e^{iz_1 T^*}\right] dz_1 \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} \left(z_3 + \psi(iz_1)\right)^{-1} e^{-x_0\left(\beta + \sqrt{\beta^2 - 2iz_1}\right)} dz_1,
\end{aligned}$$

where in the last equality we have used the following well-known result for the characteristic function of the first passage time of BM with drift:

$$E\left[e^{iz_1 T^*(x_0)}\right] = e^{-x_0\left(\beta+\sqrt{\beta^2-2iz_1}\right)}.$$

Next we use the Fourier transform of the PDF of the BM with drift in time variable to obtain

$$E[\delta(\tilde{W}_t - x_1)] = \frac{e^{-\frac{(x_1-\beta t)^2}{2t}}}{\sqrt{2\pi t}} = \frac{e^{\beta x_1}}{2\pi}\int\limits_{\mathbb{R}} e^{-iz_2 t}\frac{e^{-|x_1|\sqrt{\beta^2-2iz_2}}}{\sqrt{\beta^2-2iz_2}}dz_2. \tag{15}$$

Thus, using the fact that $\tilde{W}$ is independent of t_1^* and δ^* we obtain

$$\begin{aligned}
&E\left[e^{-z_3 t_1^*}\delta\left(\tilde{W}_{\delta^*} - x_1\right)\right] = E\left[E\left[e^{-z_3 t_1^*}\delta\left(\tilde{W}_{\delta^*} - x_1\right) \mid \delta^*\right]\right] \\
= \; & \frac{e^{\beta x_1}}{2\pi}\int\limits_{\mathbb{R}} E\left[e^{-z_3 t_1^* - iz_2\delta^*}\right]\frac{e^{-|x_1|\sqrt{\beta^2-2iz_2}}}{\sqrt{\beta^2-2iz_2}}dz_2 \\
= \; & \frac{e^{\beta(x_1-x_0)}}{4\pi^2}\iint\limits_{\mathbb{R}^2}\frac{\psi(iz_1)-\psi(iz_2)}{i(z_1-z_2)}\frac{(z_3+\psi(iz_1))^{-1}}{\sqrt{\beta^2-2iz_2}}e^{-x_0\sqrt{\beta^2-2iz_1}-|x_1|\sqrt{\beta^2-2iz_2}}dz_1dz_2.
\end{aligned}$$

Now, the statement of the Theorem follows after one additional Fourier inversion:

$$\begin{aligned}
&p_1^*(x_0;s,x_1) = E\left[\delta\left(t_1^*-s\right)\left(\tilde{W}_{\delta^*}-x_1\right)\right] \\
= \; & \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{iz_3 s}E\left[e^{-iz_3 t_1^*}\delta\left(\tilde{W}_{\delta^*}-x_1\right)\right]dz_3 \\
= \; & \frac{e^{\beta(x_1-x_0)}}{4\pi^2}\iint\limits_{\mathbb{R}^2}\frac{\psi(iz_1)-\psi(iz_2)}{i(z_1-z_2)}\frac{e^{-s\psi(iz_1)}}{\sqrt{\beta^2-2iz_2}}e^{-x_0\sqrt{\beta^2-2iz_1}-|x_1|\sqrt{\beta^2-2iz_2}}dz_1dz_2,
\end{aligned}$$

where we have also used the following Fourier integral:

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}}\frac{e^{iz_3 s}}{iz_3+\psi(iz_1)}dz_3 = e^{-s\psi(iz_1)}.$$

□

3.2. Second proof of Theorem 1

The strategy of the proof is to compute

$$I(u) = E_{0,x_0}[\mathbf{1}_{\{s<t_1^*\le s+u\}}\delta(X_{s+u}-x_1)] \tag{16}$$

and then take the limit of $I(u)/u$ as $u \to 0+$. The key idea is to note that $X_{s+u} = X_{s-} + \tilde{W}'_{T'_u}$ where $\tilde{W}', T'$ are copies of $\tilde{W}, T$, independent of the filtration $\mathcal{F}_{s-}$. We can then perform the above expectation via an intermediate conditioning on $\mathcal{F}_{s-}$:

$$E[\delta(X_{s+u} - x_1)\mathbf{1}_{\{s<t_1^* \le s+u\}}|\mathcal{F}_{s-}] \tag{17}$$
$$= \mathbf{1}_{\{s<t_1^*\}} E[\delta(\ell + \tilde{W}'_{T_u} - x_1)\mathbf{1}_{\{u \ge t'^*_1\}}|X_{s-} = \ell]. \tag{18}$$

To evaluate the expectations that arise, we will need the second and third of the following results that were stated and proved in [12]:

Lemma 1. *1. For any $s > 0$*

$$E_{0,x}[\mathbf{1}_{\{s<t_1^*\}}\delta(X_s - y)] = \mathbf{1}_{\{y>0\}} \frac{e^{\beta(y-x)}}{2\pi} \int_{\mathbb{R}} \left[e^{iz(x-y)} - e^{iz(x+y)}\right] e^{-s\psi((z^2+\beta^2)/2)} dz. \tag{19}$$

2. For any $s > 0$ and $\epsilon \in \mathbb{R}$

$$E_{0,x}[\mathbf{1}_{\{s \ge t_1^*\}}\delta(X_s - y)] = \frac{e^{\beta(y-x)}}{2\pi} \int_{\mathbb{R}+i\epsilon} e^{iz(x+|y|)} e^{-s\psi((z^2+\beta^2)/2)} dz. \tag{20}$$

3. For any k in the upper half plane,

$$E_{0,x}[\mathbf{1}_{\{s<t_1^*\}} e^{-\beta X_s + ikX_s}] = \frac{e^{-\beta x}}{2\pi} \int_{\mathbb{R}} \left[\frac{i}{k-z} - \frac{i}{k+z}\right] e^{izx} e^{-s\psi((z^2+\beta^2)/2)} dz. \tag{21}$$

First, using (20), we find

$$E[\delta(X_{s+u} - x_1)\mathbf{1}_{\{s<t_1^* \le s+u\}}|X_{s-} = \ell] \tag{22}$$
$$= \mathbf{1}_{\{s<t_1^*\}} \frac{e^{\beta(x_1-\ell)}}{2\pi} \int_{\mathbb{R}+i\epsilon} dk e^{ik(\ell+|x_1|)} e^{-u\psi((k^2+\beta^2)/2)}. \tag{23}$$

When we paste this expression into the final expectation over X_{s-} we can use Fubini to interchange the expection and integral providing we choose $\epsilon > 0$. Then we find

$$I = \frac{e^{\beta x_1}}{2\pi} \int_{\mathbb{R}+i\epsilon} dk e^{ik|x_1|} e^{-u\psi((k^2+\beta^2)/2)} E_{0,x_0}[e^{ikX_s - \beta X_s}\mathbf{1}_{\{s<t_1^*\}}]. \tag{24}$$

We can now use (21) from Lemma 1 obtain

$$I = \frac{e^{\beta(x_1-x_0)}}{(2\pi)^2} \iint_{(\mathbb{R}+i\epsilon)\times\mathbb{R}} e^{ik|x_1|+izx_0} \left[\frac{i}{k-z} - \frac{i}{k+z}\right] e^{-u\psi((k^2+\beta^2)/2) - s\psi((z^2+\beta^2)/2)} dz dk. \tag{25}$$

Noting that $I(0) = 0$ and taking $\lim_{u\to 0} I(u)/u$ now gives

$$p_1^*(x_0; s, x_1) = \frac{e^{\beta(x_1-x_0)}}{2\pi^2} \iint_{(\mathbb{R}+i\epsilon)\times\mathbb{R}} dkdz \frac{iz}{k^2-z^2} e^{ik|x_1|+izx_0} e^{-s\psi((z^2+\beta^2)/2)} \psi((k^2+\beta^2)/2). \tag{26}$$

Here the arbitrary parameter $\epsilon > 0$ can be seen to ensure the correct prescription for dealing with the pole at $k^2 = z^2$.

Finally, the complex integration in (27) can be expressed in the following manifestly real form:

$$\begin{aligned} p_1^*(x_0; s, x_1) &= -\frac{2e^{\beta(x_1-x_0)}}{\pi^2} \mathrm{PV} \iint_{(\mathbb{R}^+)^2} dkdz \frac{z\cos|x_1|k \sin x_0 z}{k^2-z^2} e^{-s\psi((k^2+\beta^2)/2)} \psi((z^2+\beta^2)/2) \\ &\quad -\frac{e^{\beta(x_1-x_0)}}{\pi} \int_{\mathbb{R}^+} \sin|x_1|z \ \sin x_0 z \ e^{-s\psi((z^2+\beta^2)/2)} \psi((z^2+\beta^2)/2) dz. \end{aligned} \tag{27}$$

involving a principle value integral plus explicit half residue terms for the poles $k = \pm z$. □

4. The iteration scheme and its convergence

The next theorem shows that (9) can be used to compute $p^*(x_0; s, x)$. We define a suitable L^∞ norm for functions $f(x_0; u, x)$:

$$\|f\|_\infty = \sup_{x_0\geq 0} \left[\int_0^\infty \int_0^\infty |f(x_0; u, x)| dudx \right]. \tag{28}$$

Theorem 2. *The sequence $(p_n^*)_{n\geq 1}$ converges exponentially in the L^∞ norm.*

Proof. First we find from (9)

$$p_{n+1}^*(x_0; s, x_1) - p_n^*(x_0; s, x_1) = \int_0^\infty dy \int_0^s du \, p_1^*(x_0; s-u, y) \left[p_n^*(y; u, x_1) - p_{n-1}^*(y; u, x_1) \right],$$

thus

$$\|p_{n+1}^* - p_n^*\|_\infty \leq C \|p_n^* - p_{n-1}^*\|_\infty$$

where

$$C = \sup_{x_0\geq 0} \left[\int_0^\infty \int_0^\infty p_1^*(x_0; u, x) dudx \right].$$

The proof is based on the probabilistic interpretation of the constant C: by definition $p_1^*(x_0;u,x)$ is the joint density of t_1^* and $X_{t_1^*}$, thus we obtain:

$$C = \sup_{x_0 \geq 0} P\left(t_1^* < +\infty,\ X_{t_1^*} > 0 \mid X_0 = x_0\right).$$

Next, using the fact that $\tilde{W}_{T^*} = 0$ (T^* is the first passage time of X_T and X is a continuous process) and the strong Markov property of the Brownian motion we find that

$$\begin{aligned} C &= P\left(t_1^* < +\infty,\ \tilde{W}_{T_{t_1^*}} - \tilde{W}_{T^*} > 0 \mid \tilde{W}_0 = x_0\right) \\ &= P\left(t_1^* < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right), \end{aligned}$$

where the Brownian motion W_t is independent of T_t and $\delta^* = \delta^*(x_0) = T_{t_1^*} - T^*$ is the overshoot of the time change above T^*.

Thus we need to prove that

$$C = \sup_{x_0 \geq 0} P\left(t_1^* < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) < 1, \tag{29}$$

where $t_1^* = t_1^*(x_0)$, $\delta^* = \delta^*(x_0)$ and the Brownian motion W is independent of t_1^* and δ^*.

First we will consider the case when $\beta < 0$. In this case we obtain

$$\begin{aligned} P\left(t_1^* < +\infty,\ W_{\Delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) &\leq P\left(W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) \\ = \int_0^\infty P(W_t + \beta t > 0 \mid W_0 = 0) P(\delta^* \in dt) &< \int_0^\infty \frac{1}{2} P(\delta^* \in dt) = \frac{1}{2}, \end{aligned}$$

where we have used the fact that W_t is independent of the overshoot δ^* and that $P(W_t + \beta t > 0 \mid W_0 = 0) < \frac{1}{2}$ for any t and any $\beta < 0$. Thus in the case when the drift β is negative we obtain an estimate $C < \frac{1}{2}$.

The case when the drift β is positive is more complicated. We can not use the same techniques as before, since the bound $P(W_t + \beta t > 0 \mid W_0 = 0) < \frac{1}{2}$ is no longer true: in fact $P(W_t + \beta t > 0 \mid W_0 = 0)$ monotonically increases to 1 as $t \to \infty$.

First we will consider the case when x_0 is bounded away from 0: $x_0 \geq c > 0$. Then $x_0 + W_t + \beta t$ has a positive probability of escaping to $+\infty$ and never crossing the barrier at 0, thus

$$\begin{aligned} P\left(t_1^* < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) &\leq P\left(t_1^*(x_0) < +\infty\right) \\ &< P\left(t_1^*(c) < +\infty\right) = 1 - \epsilon_1(c). \end{aligned}$$

Now we need to consider the case when $x_0 \to 0^+$. The proof in this case is based on the following sequence of inequalities:

$$
\begin{aligned}
& P\left(t_1^* < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) \qquad (30)\\
\le \quad & P\left(W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right)\\
= \quad & 1 - P\left(W_{\delta^*} + \beta\delta^* < 0 \mid W_0 = 0\right)\\
= \quad & 1 - \int_0^\infty P(W_t + \beta t < 0 \mid W_0 = 0) P(\delta^* \in dt)\\
< \quad & 1 - \int_0^\tau P(W_t + \beta t < 0 \mid W_0 = 0) P(\delta^* \in dt)\\
< \quad & 1 - \int_0^\tau P(W_\tau + \beta \tau < 0 \mid W_0 = 0) P(\delta^* \in dt)\\
= \quad & 1 - P(W_\tau + \beta\tau < 0 \mid W_0 = 0) P(\delta^* < \tau),
\end{aligned}
$$

where τ is any positive number and the last inequality is true since $P(W_t + \beta t < 0 \mid W_0 = 0)$ is a decreasing function of t.

Since $x_0 \to 0^+$, we also have $T^*(x_0) \to 0^+$ with probability 1. Since δ^* is the overshoot of T^*, and $T^* \to 0^+$ as $x_0 \to 0^+$, we see that the distribution of the overshoot $\delta^*(x_0)$ converges either to the distribution of the jumps of T_t if the time change process T_t is a compound Poisson process or to the Dirac delta distribution at 0 if T_t has infinite activity of jumps. Therefore in the case when T_t is a compound Poisson process with the jump measure $\nu(dx)$ we choose τ such that $\nu([0,\tau]) > 0$, and if T_t has infinite activity of jumps we can take any $\tau > 0$. Then we obtain $\lim_{x_0 \to 0^+} P(\delta^*(x_0) < \tau) = \xi$, where $\xi = \nu([0,\tau])$ in the case of compound Poisson process, and $\xi = 1$ in the case of the process with infinite activity of jumps. Using (30) we find that as $x_0 \to 0^+$

$$
\begin{aligned}
P\left(t_1^*(x_0) < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right) \quad & < \quad 1 - P(W_\tau + \beta\tau < 0 \mid W_0 = 0)\xi\\
& < \quad 1 - \epsilon_2.
\end{aligned}
$$

To summarize, we have proved that the function

$$
P(x_0) = P\left(t_1^*(x_0) < +\infty,\ W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0\right),
$$

satisfies the following properties:

- for any $c > 0$ there exists $\epsilon_1 = \epsilon_1(c) > 0$ such that $P(x_0) < 1 - \epsilon_1(c)$ for all $x_0 > c$
- there exists $\epsilon_2 > 0$ such that $\lim_{x_0 \to 0^+} P(x_0) < 1 - \epsilon_2$.

Therefore we conclude that there exists $\epsilon > 0$, such that $P(x_0) < 1 - \epsilon$ for all $x_0 \geq 0$, thus $C < 1 - \epsilon$. This ends the proof in the second case $\beta > 0$.

For a complementary point of view, the next result shows that the sequence $(t_i^*(x_0)_{i \geq 1}$ converges pathwise.

Theorem 3. *For any TCBM with Lévy subordinator T_t and Brownian motion with drift β the sequence of stopping times $(t_i^*(x_0)_{i \geq 1}$ converges a.s to t^*.*

Proof: If $t^* = \infty$, then certainly $t_i^* \to \infty$, so we suppose $t^* < \infty$. In this case, if $t_i^* = t_{i+1}^*$ for some i the sequence converges, and thus the only interesting case to analyze is if $t^* \neq t_i^*$ for all $i < \infty$. Then we have $t_1^* < t_2^* < \cdots < t_i^* < \ldots$. Correspondingly, we have an infinite sequence of excursion overjump intervals which do not overlap: let their endpoints be $T_{i-}^* := T_{t_i^*-} < T_{i+}^* := T_{t_i^*}$. The following observations lead to the conclusion:

1. by monotonicity and boundness of the sequences (T_{i-}^*) and (T_{i+}^*), $\lim_{i \to \infty} T_{i-}^* = \lim_{n \to \infty} T_{i+}^* = T_\infty$ exists;
2. $x_0 + W_{T_\infty} + \beta T_\infty = 0$ by the continuity of Brownian motion;
3. $\lim_{i \to \infty} t_i^* = t_\infty$ exists, and $t_\infty \leq t^*$;
4. Jump times are totally inaccessible, so there is no time jump at time t_∞ almost surely, hence $T_{t_\infty} = T_\infty$;
5. Therefore $X_{t_\infty} = 0$ and so $t_\infty \geq t^*$: hence $t_\infty = t^*$.

$\square$

5. The Variance Gamma model

The Variance Gamma (VG) process described in Section 2 is the LSBM where the time change process T_t is the Lévy process with jump measure $\mu(z) = (\nu z)^{-1} \exp(-z/\nu)$ on

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The authors would like to thank Professor Martin Barlow for providing ths proof.

$(0,\infty)$ and the Laplace exponent $\psi_T(u) = \frac{1}{\nu}\log(1+\nu u)$. In this section we take the parameter $b = 0$. This model has been widely used for option pricing where it has been found to provide a better fit to market data than the Black-Scholes model, while preserving a degree of analytical tractability. The main result in this section reduces the 2D integral representation (11) for $p_1^*(x_0; s, x_1)$ to a 1D integral and leads to greatly simplified numerical computations.

Theorem 4. *Define* $\alpha = \sqrt{\frac{2}{\nu} + \beta^2}$. *Then*

$$p_1^*(x_0; s, x_1) = \frac{e^{\beta(x_1 - x_0)}}{2\pi\nu} \int\limits_{\mathbb{R}} \frac{(1 + i\nu z)^{-\frac{s}{\nu}}}{\sqrt{\beta^2 - 2iz}} e^{-x_0\sqrt{\beta^2 - 2iz}} \times \Big[e^{|x_1|\sqrt{\beta^2-2iz}} Ei\left(- \mid x_1 \mid \left(\alpha + \sqrt{\beta^2 - 2iz}\right)\right) - e^{-|x_1|\sqrt{\beta^2-2iz}} Ei\left(- \mid x_1 \mid \left(\alpha - \sqrt{\beta^2 - 2iz}\right)\right)\Big] dz, \tag{31}$$

where $Ei(x)$ *is the exponential integral function (see [11]).*

Proof. Consider the function $I(z_1)$ which represents the outer integral in (11)

$$I(z_1) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \frac{\log(1 + i\nu z_1) - \log(1 + i\nu z_2)}{i(z_1 - z_2)} \frac{e^{-|x_1|\sqrt{\beta^2-2iz_2}}}{\sqrt{\beta^2 - 2iz_2}} dz_2$$

First we perform the change of variables $u = i\sqrt{\beta^2 - 2iz_2}$ and obtain

$$I(z_1) = \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{\log\left(1 + \frac{\nu}{2}(u^2 + \beta^2)\right) - \log(1 + i\nu z_1)}{u^2 + \beta^2 - 2iz_1} e^{i|x_1|u} du, \tag{32}$$

where the contour L obtained from $\mathbb{R}$ under map $z_2 \to u = i\sqrt{\beta^2 - 2iz_2}$ is transformed into contour $\mathbb{R}$ (this is justified since the integrand is an analytic function in this region for any z_1). To finish the proof we separate the logarithms

$$\log\left(1 + \frac{\nu}{2}(u^2 + \beta^2)\right) - \log(1 + i\nu z_1) = \log(u + i\alpha) + \log(u - i\alpha) - \log\left(\frac{2}{\nu} + 2iz_1\right),$$

use the partial fractions decomposition

$$\frac{1}{u^2 + \beta^2 - 2iz_1} = \frac{1}{2i\sqrt{\beta^2 - 2iz_1}} \left[\frac{1}{u - i\sqrt{\beta^2 - 2iz_1}} - \frac{1}{u + i\sqrt{\beta^2 - 2iz_1}}\right]$$

and we obtain the six integrals, which can be computed by shifting the contours of integration and using the following Fourier transform formulas (see Gradshteyn-Ryjik...)

$$\begin{aligned}\int\limits_{i\epsilon+\mathbb{R}} \log\left(1+\frac{iy}{b}\right)e^{ixy}\frac{dy}{y} &= -2\pi i Ei(-bx), \quad b>0\\ \int\limits_{i(b+\epsilon)+\mathbb{R}} \log\left(\frac{iy}{b}-1\right)e^{ixy}\frac{dy}{y} &= -2\pi i Ei(bx), \quad b>0\end{aligned}$$

Remark: Using the change of variables $u = i\sqrt{\beta^2 - 2iz}$ and simplifying the expression we can obtain a simpler formula for $p_1^*(x_0; s, x_1)$:

$$p_1^*(x_0; s, x_1) = e^{\beta(x_1-x_0)}\frac{\left(\frac{\nu}{2}\right)^{-\frac{s}{\nu}-1}}{2\pi i}\int\limits_{\mathbb{R}}\left(\alpha^2+y^2\right)^{-\frac{s}{\nu}}\sin\left(x_0 y\right)e^{i|x_1|y}Ei\left(-\mid x_1\mid(\alpha+iy)\right)dy.$$

Applying the Plancherel formula to the above expression gives us the following representation for p_1^*:

$$\begin{aligned}p_1^*(x_0; s, x_1) &= e^{\beta(x_1-x_0)-\alpha(x_0+|x_1|)}\frac{\left(\frac{\nu\alpha^2}{2}\right)^{-\frac{s}{\nu}}}{\sqrt{\pi}\nu(x_0+\mid x_1\mid)}\frac{\Gamma\left(\frac{s}{\nu}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{\nu}+1\right)} \qquad (33)\\ &+ \sqrt{\frac{2\alpha^3}{\pi}}e^{\beta(x_1-x_0)-\alpha|x_1|}\frac{(\alpha\nu)^{-\frac{s}{\nu}-1}}{\Gamma\left(\frac{s}{\nu}\right)}\int\limits_0^\infty u^{\frac{s}{\nu}-\frac{1}{2}}K_{\frac{s}{\nu}-\frac{1}{2}}\left(\alpha u\right)f(x_0,x_1;u)du,\end{aligned}$$

where

$$f(x_0,x_1;u) = \frac{e^{-\alpha(u+x_0)}}{u+x_0+\mid x_1\mid} - \mathrm{sign}\left(u-x_0\right)\frac{e^{-\alpha\left|u-x_0\right|}}{\left|u-x_0\right|+\mid x_1\mid} - 2\frac{e^{-\alpha(u+x_0)}}{x_0+\mid x_1\mid}.$$

The above expression is useful for computations when s is small. In particular, when $s = 0$ we find

$$p_1^*(x_0; 0, x_1) = \frac{e^{\beta(x_1-x_0)-\alpha(x_0+|x_1|)}}{\nu(x_0+\mid x_1\mid)}. \qquad (34)$$

6. Numerical implementation for VG model

The algorithm for computing the functions $p^*(x_0; s, x)$ and $p^*(x_0; s)$ can be summarized as follows:

1. Choose the discretization step sizes δ_x, δ_t and discretization intervals $[-X, X]$, $[0, T]$. The grid points are $t_i = i\delta_t$, $1 \le i \le N_s$ and $x_j = (j + 1/2)\delta_x$, $-N_x \le j \le N_x$.

2. Compute the 3D array $p_1^*(x_i; t_j, x_k)$. For $j > 0$ use equation (31) and for $j = 0$ use explicit formula (34).

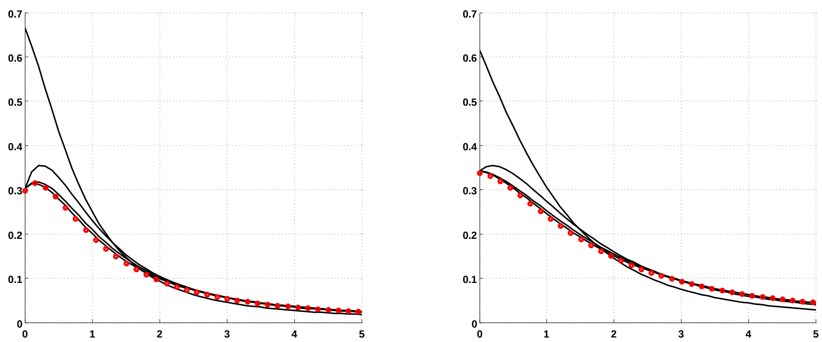

FIGURE 2: The density of the first passage time for the two set of parameters. The circles show the "exact" result; the three solid lines show the first three approximations.

3. Iterate equation (9) or (10). This step can be considerably accelerated if the convolution in u-variable is done using Fast Fourier Transform methods. We used the midpoint rule for integration in the y and u variables.

Theorem 2 implies that Step 3 in the above algorithm has to be repeated only a few times: In practice we found that 3-4 iterations is usually enough. An important empirical fact is that the above algorithm works quite well with just a few discretization points in the x-variable. We found that if one uses a non-linear grid (which places more points x_i near $x = 0$) then the above algorithm produces reasonable results with values of N_x as small as 10 or 20.

We compared our algorithm for the PDF $p^*(x_0; s)$ to a finite-difference method that was implemented as follows. First we approximated the first passage time by its discrete counterpart:

$$\hat{t}^* = \hat{t}^*(x) = \min\{t_i : X_{t_i} < 0 | X_0 = x\}$$

where $t_i = i\delta_t$, $0 \le i \le n_t$ is the discretization of the interval $[0,T]$. The probabilities $f_i(x) = P(\hat{t}^* > t_i | X_0 = x)$ satisfy the iteration:

$$f_{i+1}(x) = \mathbf{1}_{x>0} \int_{\mathbb{R}} p(\delta_t, x-y) f_i(y) dy, i \ge 1 \tag{35}$$

with $f_0(x) = \mathbf{1}_{x>0}$ and can be computed numerically with the following steps:

1. discretize the space variables $x = i\delta_x$, $y = j\delta_x$, $0 < i, j < n_x$;
2. compute the array of transitional probabilities $\hat{p}_i = p(\delta_t, x_i)$, and normalize $\hat{p}_0$ so that $\sum_i \hat{p}_i = 1$;
3. use the convolution (based on FFT) to iterate equation (35) n_t times;
4. compute the approximation of the first passage time density $\hat{p}^*(x, t_i + \delta_t/2) = (f_{i+1}(x) - f_i(x))/\delta_t$.

The big advantage of this method is that it is explicit and unconditionally stable: we can choose the number of discretization points in x-space and t-space independently. This is not true in general explicit finite difference methods, where one would solve the Fokker-Plank equation by discretizing the Markov generator and derivative in time, since δ_t and δ_x have to lie in a certain subset in order for the methods to be stable.

Figure 6 summarizes the numerical results for the PDF $p^*(x_0; s)$ over the time interval $[0, 5]$ for the VG model with the following two sets of parameters:

Set I: $x_0 = 0.5, \quad \beta = 0.2, \quad \nu = 1$

Set II: $x_0 = 0.5, \quad \beta = -0.2, \quad \nu = 2.$

The number of grid points used was $N_t = 50$ and $N_x = 10$. The red circles correspond to the solution obtained by a high resolution finite difference PIDE method as described above (with $n_t = 1000$ and $n_x = 10000$), and the black lines show successive iterations $p_i^*(x_0, t)$ converging to $p^*(x_0, t)$. As we see, 3 iterations of equation (10) provide a visually acceptable accuracy in a running time of less than 0.1 second (on a 2.5Ghz laptop).

Figure 3 illustrates the convergence of our method and Table 1 shows the computation times (on the same 2.5Ghz laptop). We used Set II of parameters for the VG process, and the PIDE

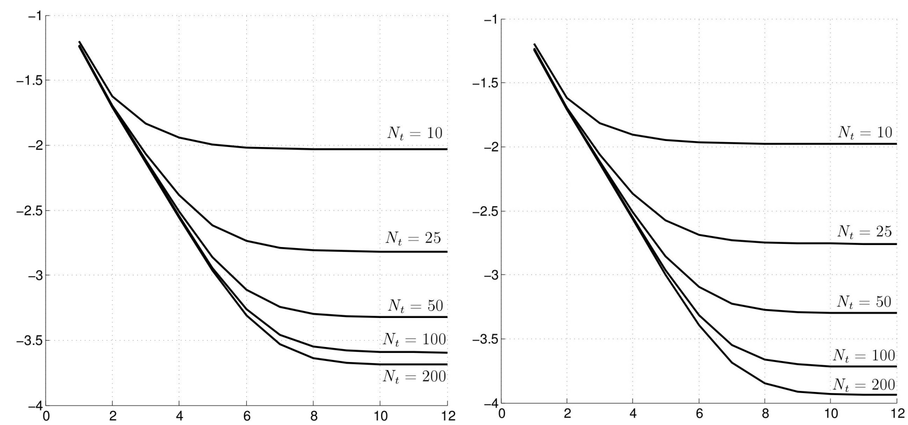

FIGURE 3: The error $\log_{10}(||p^* - p_i^*||_{L_1})$ for the new approach plotted against the number of iterations. The left plot has $N_x = 10$, the right plot has $N_x = 20$ (right), and $N_t \in \{10, 25, 50, 100, 200\}$.

method with $n_t = 1000$ and $n_x = 10000$ to compute the "exact" solution $p^*(x_0, t)$. Figure 3 shows the $\log_{10}$ of the error

$$||p^* - p_i^*||_{L_1} = \int_0^T |p^*(x_0, t) - p_i^*(x_0, t)| dt$$

on the vertical axis and the number of iterations on the horizontal axis; different curves correspond to different number of discretization points in t-space. The number of discretization points in x-space is fixed at $N_x = 10$ for the left picture and $N_x = 20$ for the right picture. We see that initially the error decreases exponentially and then flattens out. The flattening indicates that our method converges to the wrong target (which is to be expected since there is always a discretization error coming from N_x and N_t being finite). However, increasing N_t and N_x brings us closer to the "target". In the table 1 we show precomputing time needed to compute the 3D array $p_1^*(x_i; t_j, x_k)$ and the time needed to perform each iteration (10).

To put these results into perspective, Figure 4 and Table 2 show similar results for finite difference method. In Figure 4 we show the same logarithm of the error on the vertical axis, and the number of discretization points n_x on the horizontal axis. Different curves correspond to $n_t \in \{50, 100, 200\}$. The running time presented in Table 2 includes only the time needed to perform n_t convolutions (35) using the FFT. As one can see by comparing Figures 3 and 4, even with a relatively large number of discretization points $n_x = 5750$ and $n_t = 200$

		$N_t = 10$	$N_t = 25$	$N_t = 50$	$N_t = 100$	$N_t = 200$
precomputing time	$N_x = 10$	0.0313	0.0259	0.0324	0.0461	0.0687
each iteration	$N_x = 10$	0.0006	0.0008	0.0011	0.0021	0.0046
precomputing time	$N_x = 20$	0.0645	0.0612	0.0745	0.0868	0.1298
each iteration	$N_x = 20$	0.0037	0.0045	0.0066	0.0120	0.0269

TABLE 1: Computation time (sec) for the new approach.

the accuracy produced by a finite difference method is an order of magnitude worse than the accuracy produced by our method (with much fewer discretization points). Moreover, one can see that consistently the running times of the PIDE method are orders of magnitude larger.

	$n_x = 1150$	$n_x = 2300$	$n_x = 3450$	$n_x = 4600$	$n_x = 5750$
$n_t = 50$	0.0756	0.2935	0.9582	1.8757	2.9967
$n_t = 100$	0.1456	0.5821	1.9397	3.7409	6.0026
$n_t = 200$	0.2870	1.1478	3.8833	7.4768	11.9935

TABLE 2: Computation time (sec) for the finite difference approach.

7. Conclusions

First passage times are an important modeling tool in finance and other areas of applied mathematics. The main result of this paper is the theoretical connection between two distinct notions of first passage time that arise for Lévy subordinated Brownian motions. This relation leads to a new way to compute true first passage for these processes that is apparently less expensive than finite difference methods for a given level of accuracy. Our paper opens up many avenues for further theoretical and numerical work. For example, the methods we describe are certainly applicable for a much broader class of time changed Brownian motions and time changed diffusions. Finally, it will be worthwhile to explore the use of the first passage of the second kind as a modeling alternative to the usual first passage time.

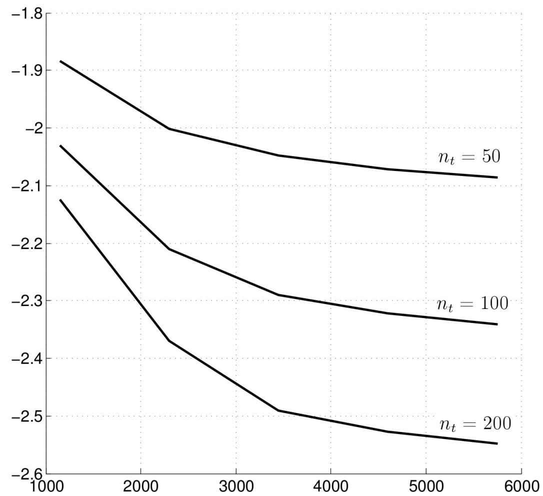

FIGURE 4: The error $\log_{10}(||p^* - \hat{p}^*||_{L_1})$ for the finite difference method is plotted against the number of grid points n_x, for the values $n_t = 50, 100, 200$.

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