Fractional Laplace operator and Meijer G-function

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Abstract

We significantly expand the number of functions whose image under the fractional Laplace operator can be computed explicitly. In particular, we show that the fractional Laplace operator maps Meijer G-functions of $|x|^2$, or generalized hypergeometric functions of $-|x|^2$, multiplied by a solid harmonic polynomial, into the same class of functions. As one important application of this result, we produce a complete system of eigenfunctions of the operator $(1-|x|^2)_+^{\alpha/2}(-\Delta)^{\alpha/2}$ with the Dirichlet boundary conditions outside of the unit ball. The latter result will be used to estimate the eigenvalues of the fractional Laplace operator in the unit ball in a companion paper [9].

Keywords: Fractional Laplace operator, Riesz potential, Meijer G-function, hypergeometric function, Jacobi polynomial, harmonic polynomial, radial function

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1 Introduction

The fractional Laplace operator $(-\Delta)^{\alpha/2}$ is one of the most well-studied pseudo-differential operators. One reason for this lies in the multitude of important applications. This operator arises as a Markov generator of an isotropic stable process in $\mathbb{R}^d$, it is of increasing interest for the partial differential equations community and it is used in various areas of applied mathematics, see [6, 23]. Another reason is that the fractional Laplace operator is also one of the simplest pseudo-differential operators, having symbol which is just a power function. In view of the simplicity and importance of this operator, it is surprising that there exist only a handful of functions for which the action of this operator can be computed explicitly. To the best of our knowledge, the following list provides a complete catalogue of known examples, up to translations, dilations and Kelvin transform (as usual, we write a_+ for $\max(0, a)$):

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(a) Expressions for the *harmonic measure* and the *Green function* for a ball, as well as for the complement of the ball, were essentially established by M. Riesz in [21, 22] using Kelvin transform. For a formal derivation of the expression for the harmonic measure, see Section IV.5 in [17]. The expression for the Green function in its most common form was obtained in [3].

(b) If h is harmonic (that is, $\Delta h = 0$) in the unit ball and $f(x) = (1 - |x|^2)_+^{\alpha/2-1} h(x)$, then $(-\Delta)^{\alpha/2} f = 0$ in the unit ball, as it was proved in [14], see also [4].

(c) The formula for $(-\Delta)^{\alpha/2} f$ for $f(x) = (1 - |x|^2)_+^{\sigma}$ with $\sigma = \frac{\alpha}{2}$ is essentially contained in [21] and stated explicitly in [10], while $\sigma = \frac{\alpha}{2} - 1$ is covered by the previous item. The case of general σ, as well as the function $g(x) = x_1 f(x)$, was studied in [2, 8].

(d) Similar results are easily found for half-space and for the complement of the unit ball by means of Kelvin transformation, see [5, 17, 21]

(e) The results for functions supported in the full space $\mathbb{R}^d$ are more rare: a formula for $(-\Delta)^{\alpha/2} f(x)$ is known when $f(x) = e^{iy\cdot x}$ (Fourier transform), $f(x) = |x|^{-a}$ with $a \in (0, d)$ (composition with Riesz kernels, [17, 26]), $f(x) = e^{-|x|^2}$ or $f(x) = (1+|x|^2)^{-a}$ with $a = \frac{d+1}{2}$ or $a = \frac{d-\alpha}{2} + n$, $n = 0, 1, \ldots$ (see [25]).

Our goal in this paper is to extend the above list: We want to find more functions f for which $(-\Delta)^{\alpha/2} f$ can be computed explicitly. Our main motivation for doing this comes from the study of spectral properties of the fractional Laplace operator acting on functions supported in the unit ball – we consider this problem in a companion paper [9].

Our results are stated in terms of certain special functions: Meijer G-function, hypergeometric function and Jacobi polynomials. Before we introduce these notions formally, we summarise our findings. Our most general result (Theorems 1 and 2) states that, under a number of conditions on the parameters α, l, m, n, p, q, $\mathbf{a} = (a_1, \ldots, a_p)$ $\mathbf{b} = (b_1, \ldots, b_q)$ and the variable $x \in \mathbb{R}^d$, if

$$f(x) := V(x) G_{pq}^{mn}\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| \, |x|^2 \right), \tag{1}$$

then

$$(-\Delta)^{\alpha/2} f(x) = 2^\alpha V(x) G_{p+2,q+2}^{m+1,n+1}\left(\begin{matrix} 1 - \frac{d+2l+\alpha}{2}, & \mathbf{a} - \frac{\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \mathbf{b} - \frac{\alpha}{2}, & 1 - \frac{d+2l}{2} \end{matrix} \middle| \, |x|^2 \right). \tag{2}$$

Here V is a solid harmonic polynomial of degree l (one can take, for example, $V(x) = 1$ and $l = 0$, or $V(x) = x_1$ and $l = 1$) and $G_{pq}^{mn}(\mathbf{a}; \mathbf{b}|r)$ is the Meijer G-function. Essentially our assumptions only require that $(-\Delta)^{\alpha/2} f(x)$ is well defined as a singular integral, and we allow for any $\alpha > -d$.

By specifying some or all of the parameters, we obtain expressions for $(-\Delta)^{\alpha/2} f(x)$ for a wide collection of functions $f(x) = V(x)\phi(|x|^2)$. For example, we can let $\phi(r) = r^\rho (1 + r)^\sigma$, $\phi(r) = r^\rho (1 - r)_+^\sigma$ or $\phi(r) = r^\rho (r - 1)_+^\sigma$ (Corollaries 1 and 3). We also obtain an elegant expression for the generalised hypergeometric function: if $\mathbf{b}' = (b_1, \ldots, b_{q-1})$ and

$$f(x) := V(x)\, {}_pF_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b}', \; \frac{d+2l}{2} \end{matrix} \middle| -|x|^2 \right), \tag{3}$$

then

$$(-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \prod_{j=1}^{p} \Gamma(a_j + \frac{\alpha}{2}) \prod_{j=1}^{q-1} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j) \prod_{j=1}^{q-1} \Gamma(b_j + \frac{\alpha}{2})} V(x)\, {}_pF_q\left(\begin{matrix} \mathbf{a} + \frac{\alpha}{2} \\ \mathbf{b}' + \frac{\alpha}{2}, \; \frac{d+2l}{2} \end{matrix} \middle| -|x|^2 \right) \tag{4}$$

(Corollary 2). Finally, if $P_n^{(\alpha,\beta)}(r)$ is the Jacobi polynomial and

$$f(x) := (1 - |x|^2)_+^{\alpha/2} V(x) P_n^{(\alpha/2, d/2+l-1)}(2|x|^2 - 1), \tag{5}$$

then

$$(-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \Gamma(1 + \frac{\alpha}{2} + n)\Gamma(\frac{d+2l+\alpha}{2} + n)}{n!\, \Gamma(\frac{d+2l}{2} + n)} V(x) P_n^{(\alpha/2, d/2+l-1)}(2|x|^2 - 1) \tag{6}$$

in the unit ball, that is, $(1 - |x|^2)_+^{\alpha/2} (-\Delta)^{\alpha/2} f(x)$ is a multiple of $f(x)$ (Theorem 3).

1.1 Riesz potential operator and the fractional Laplace operator

Everywhere in this paper we assume that $d \geq 1$ is the dimension. For $\alpha \in (0, d)$, the *Riesz potential operator* is defined by

$$(-\Delta)^{-\alpha/2} f(x) := \frac{1}{\gamma_d(\alpha)} \int_{\mathbb{R}^d} \frac{f(x-y)}{|y|^{d-\alpha}} \mathrm{d}y, \tag{7}$$

where

$$\gamma_d(\alpha) := \frac{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d-\alpha}{2})}. \tag{8}$$

When $0 < \alpha < 2$ we define the *fractional Laplace operator* (also known as the *fractional Riesz derivative*) as

$$(-\Delta)^{\alpha/2} f(x) := \frac{1}{|\gamma_d(-\alpha)|} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \frac{f(x) - f(x-y)}{|y|^{d+\alpha}} \mathrm{d}y, \tag{9}$$

where $B(0, \varepsilon)$ denotes the ball of radius ε centered at the origin. This definition can be extended to $\alpha \geq 2$, though it requires more work. Following [25], we define the centered difference

$$\Delta_y^k f(x) := \sum_{j=0}^{k} (-1)^j \tbinom{k}{j} f(x + (\tfrac{k}{2} - j)y)$$

and introduce

$$\chi_{d,k}(\alpha) := -\gamma_d(-\alpha) \sum_{j=0}^{k} (-1)^j \tbinom{k}{j} |\tfrac{k}{2} - j|^\alpha.$$

Choose k to be an even integer. It is known that $\chi_{d,k}(\alpha)$ is analytic and zero-free in the region $0 < \mathrm{Re}(\alpha) < k$ (the poles of $\gamma_d(-\alpha)$ when α is an even integer are canceled by the corresponding zeros of the sum, see Theorem 3.4 in [25]). We define the fractional Laplace operator as

$$(-\Delta)^{\alpha/2} f(x) := \frac{1}{\chi_{d,k}(\alpha)} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \frac{-\Delta_y^k f(x)}{|y|^{d+\alpha}} \mathrm{d}y, \tag{10}$$

where k is an even integer strictly greater than $\mathrm{Re}(\alpha)$. When $0 < \alpha < 2$, this definition is easily seen to be equivalent to (9) (provided that $(1 + |y|)^{-d-\alpha} f(y)$ is in integrable). For a different representation of $(-\Delta)^{\alpha/2} f(x)$, based on the Pizzetti's formula, see formula (1.1.10) in [17].

For Schwartz functions f the Riesz potential and the fractional Laplacian can be defined in an alternative way

$$(-\Delta)^{\alpha/2} f = \mathcal{F}_d^{-1}(\hat{k}_\alpha \mathcal{F}_d f), \tag{11}$$

where $\hat{k}_\alpha(x) = |x|^\alpha$ and $\mathcal{F}_d$ denotes the Fourier transform in $\mathbb{R}^d$,

$$\mathcal{F}_d f(x) := \int_{\mathbb{R}^d} e^{\mathrm{i}x \cdot y} f(y) \mathrm{d}y. \tag{12}$$

Formula (11) shows that when $0 < \mathrm{Re}(\alpha) < d$, the fractional Laplace operator is the inverse of the Riesz potential operator for nice enough functions, which explains the notation $(-\Delta)^{-\alpha/2}$. This is made precise in Theorem 3.24 in [25] in the setting of L^p spaces, see also [16, 26]. We will need the following pointwise version of this result.

Proposition 1 (Proposition 7.2 in [16])**.** *Assume that $0 < \alpha < d$, $(1 + |y|)^{\alpha-d} g(y)$ is integrable and $f(x) = (-\Delta)^{-\alpha/2} g(x)$. If g is continuous at some x, then $g(x) = (-\Delta)^{\alpha/2} f(x)$.*

1.2 Hypergeometric function

Let p, q be nonnegative integers such that $p \le q + 1$, and let $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{C}^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in \mathbb{C}^q$. The generalised hypergeometric function is defined as the hypergeometric series

$${}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} a_j^{(n)}}{\prod_{j=1}^{q} b_j^{(n)}} \frac{r^n}{n!},$$

as long as none of the parameters b_j is a non-positive integer; here $c^{(n)} = c(c+1)\ldots(c+n-1)$ denotes the rising factorial. If $p \le q$, then the above series is convergent for all $r \in \mathbb{C}$ and ${}_pF_q$ is an entire function. In the important case $p = q+1$, the series converges when $|r| < 1$, but ${}_pF_q$ extends to an analytic function in $\mathbb{C} \setminus [1, \infty)$.

The regularised hypergeometric function ${}_p\mathbf{F}_q$ (note that many authors use a different notation ${}_p\tilde{F}_q$) is defined as

$${}_p\mathbf{F}_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) := \frac{1}{\prod_{j=1}^{q} \Gamma(b_j)} \, {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right).$$

Here none of b_j is a non-positive integer, but ${}_p\mathbf{F}_q$ extends analytically to arbitrary values of b_j. For more information on hypergeometric functions, see [12, 20].

1.3 Meijer G-function

In this section we define the Meijer G-function and discuss some of its properties. We begin with four non-negative integers m, n, p and q and two vectors $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{C}^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in \mathbb{C}^q$, and define

$$\mathcal{G}_{pq}^{mn}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| s\right) := \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod\limits_{j=m+1}^{q} \Gamma(1 - b_j - s) \prod\limits_{j=n+1}^{p} \Gamma(a_j + s)}. \tag{13}$$

This will serve as the Mellin transform of the Meijer G-function. We denote

$$\nu = \sum_{j=1}^{p} \operatorname{Re}(a_j) - \sum_{j=1}^{q} \operatorname{Re}(b_j) \tag{14}$$

and

$$\underline{b} := \min_{1 \le j \le m} \operatorname{Re}(b_j), \qquad \overline{a} := \max_{1 \le j \le n} \operatorname{Re}(a_j); \tag{15}$$

we set $\underline{b} = +\infty$ if $m = 0$ and $\overline{a} = -\infty$ if $n = 0$.

Throughout the entire article, when speaking about the Meijer G-function, we will tacitly assume that $b_i - a_j$ is not a positive integer for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, so that no pole of $\Gamma(b_i + s)$ coincides with a pole of $\Gamma(1 - a_j - s)$; otherwise, the Meijer G-function is not defined. We will also always assume that $p + q \le 2m + 2n$, so that there are at least as many gamma functions in the numerator of (13) as there are in the denominator. We also introduce the following five conditions on parameters m, n, p, q, $\mathbf{a}$ and $\mathbf{b}$ that will be required for some statements:

$$\begin{aligned}
&\text{Condition S:} && 1 - \overline{a} > -\underline{b},\\
&\text{Condition A:} && p + q < 2m + 2n,\\
&\text{Condition B:} && p + q = 2m + 2n,\ p = q\\
&\text{Condition C:} && p + q = 2m + 2n,\ p < q\\
&\text{Condition D:} && p + q = 2m + 2n,\ p > q.
\end{aligned}$$

Note that Conditions A through D are mutually exclusive. Before we define the Meijer G-function, we discuss the role of the above conditions.

Condition S is needed because it separates the poles of $\Gamma(b_j + s)$ from the poles of $\Gamma(1 - a_j - s)$ in the numerator in (13), thus the function $\mathcal{G}_{pq}^{mn}(\mathbf{a}; \mathbf{b}|s)$ is analytic in s in the strip $-\underline{b} < \operatorname{Re}(s) < 1 - \overline{a}$. By Stirling's asymptotic formula for the gamma function, for $\lambda \in \mathbb{R}$,

$$\lim_{t \to \pm\infty} \frac{e^{|t|\pi/2} |\Gamma(\lambda + \mathrm{i}t)|}{|t|^{\lambda - 1/2}} = \sqrt{2\pi}$$

(see formula 8.328.1 in [12]). Therefore, Condition A ensures that $\mathcal{G}_{pq}^{mn}(\mathbf{a}; \mathbf{b}|s)$ converges to zero exponentially fast as $|\operatorname{Im} s| \to \infty$ within the strip $-\underline{b} < \operatorname{Re}(s) < 1 - \overline{a}$. When Condition B, C or D is satisfied, then the exponential parts of the gamma functions cancel out, and one can check that for every $\varepsilon > 0$ the function $|\mathcal{G}^{mn}_{pq}(\mathbf{a};\mathbf{b}|s)|$ is of smaller order than $|\operatorname{Im} s|^{-\nu-(p-q)(\lambda-1/2)+\varepsilon}$ as $|\operatorname{Im} s| \to \infty$ along the line $\lambda + \mathrm{i}\mathbb{R}$. In particular, this function is integrable if $\nu - (p-q)(\lambda - \frac{1}{2}) > 1$.

Definition 1. Assume that parameters m, n, p, q, $\mathbf{a}$ and $\mathbf{b}$ satisfy Condition S and either of Conditions A through D. Suppose in addition that: $\nu > 1$ if Condition B is satisfied; $\nu > 1 + (q-p)(-\underline{b} - \frac{1}{2})$ if Condition C holds; and $\nu > 1 - (p-q)(\frac{1}{2} - \overline{a})$ if Condition D holds. We define the *Meijer G-function* as the inverse Mellin transform

$$G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) := \frac{1}{2\pi\mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} \mathcal{G}^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| s\right) r^{-s} \mathrm{d}s, \tag{16}$$

where $r > 0$ and $\lambda \in (-\underline{b}, 1 - \overline{a})$; if Condition C or D is satisfied, we also require that $\nu + (p-q)(\lambda - \frac{1}{2}) > 1$.

For any λ as in the above definition, the Meijer G-function is defined as the inverse Mellin transform of an absolutely integrable function along the line $\lambda + \mathrm{i}\mathbb{R}$. In particular, the Meijer G-function is then bounded by $Cr^{-\lambda}$ for some $C > 0$. Therefore, if we denote

$$\begin{aligned} \underline{\lambda} &= \begin{cases} -\underline{b} & \text{under Conditions A, B or C} \\ \max(-\underline{b}, \frac{1}{2} - \frac{\nu-1}{p-q}) & \text{under Condition D,} \end{cases} \\ \overline{\lambda} &= \begin{cases} 1 - \overline{a} & \text{under Conditions A, B or D} \\ \min(1 - \overline{a}, \frac{1}{2} + \frac{\nu-1}{q-p}) & \text{under Condition C,} \end{cases} \end{aligned} \tag{17}$$

then we immediately have that for any $\varepsilon > 0$,

$$G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) = \begin{cases} O(r^{-\underline{\lambda}-\varepsilon}) & \text{as } r \to 0^+, \\ O(r^{-\overline{\lambda}+\varepsilon}) & \text{as } r \to +\infty. \end{cases} \tag{18}$$

Intuitively, the parameter ν describes the regularity of the Meijer G-function near 1 when Condition B is satisfied, its oscillations at ∞ when Condition C holds, and its oscillations near 0 when Condition D holds. Note that in each case the corresponding regularity improves as ν increases.

The following transformation rules also follow easily from the definition as the inverse Mellin transform:

$$r^c\, G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) = G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}+c\\ \mathbf{b}+c\end{matrix}\middle| r\right), \tag{19}$$

$$G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r^{-1}\right) = G^{nm}_{qp}\left(\begin{matrix}1-\mathbf{b}\\ 1-\mathbf{a}\end{matrix}\middle| r\right), \tag{20}$$

$$G^{mn}_{pq}\left(\begin{matrix}c, & \mathbf{a}'\\ \mathbf{b}', & c\end{matrix}\middle| r\right) = G^{m,n-1}_{p-1,q-1}\left(\begin{matrix}\mathbf{a}'\\ \mathbf{b}'\end{matrix}\middle| r\right), \tag{21}$$

$$G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}', & c\\ c, & \mathbf{b}'\end{matrix}\middle| r\right) = G^{m-1,n}_{p-1,q-1}\left(\begin{matrix}\mathbf{a}'\\ \mathbf{b}'\end{matrix}\middle| r\right). \tag{22}$$

where $\mathbf{a}' = (a_1, \ldots, a_{p-1})$ and $\mathbf{b}' = (b_1, \ldots, b_{q-1})$.

The above definition of Meijer G-function is not the most general possible. One can relax Condition S and restrictions on ν by appropriately deforming the contour of integration in (16), see Chapter 8.2 in [20] for more details. Another way of extending the definition of Meijer G-function is provided by expansion in terms of generalised hypergeometric functions, described briefly below.

Assume that $b_j - b_k$ is not an integer for $1 \le j < k \le m$. If $p < q$ or $p = q$ and $|r| < 1$, then we have

$$G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| r\right) = \sum_{k=1}^{m} \frac{\pi^{m-1}\prod_{j=1}^{n}\Gamma(1+b_k-a_j)}{\prod_{\substack{1\le j\le m\\ j\ne k}} \sin((b_j-b_k)\pi) \prod_{j=n+1}^{p}\Gamma(a_j-b_k)} \times \\ \times r^{b_k}\,{}_p\mathbf{F}_{q-1}\left(\begin{matrix}1+b_k-\mathbf{a}\\ 1+b_k-\mathbf{b}'_k\end{matrix}\middle| (-1)^{p-m-n}r\right), \tag{23}$$

where $\mathbf{b}'_k = (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_q)$, see [12, 19]. If $p > q$ or $p = q$ and $|r| > 1$, the corresponding representation of Meijer G-function in terms of ${}_q\mathbf{F}_{p-1}$ functions can be obtained using (20) and (23). Furthermore, if $b_j = b_k$ for some $j \ne k$, a similar, but much more complicated expansion holds true, see Chapter V in [19].

The Meijer G-function, whenever defined, is analytic in all parameters a_j and b_j, and it has the properties (18) through (22). Under Condition A, formula (16) in fact defines the Meijer G-function as an analytic function in a sector $|\arg r| < (m+n-(p+q)/2)\pi$. When condition B holds true (with arbitrary ν), the Meijer G-function extends to an analytic function in the unit disk $|r| < 1$ and in the complement of the unit disk $|r| > 1$. Under Conditions C and D (or more generally, when $p \ne q$), the Meijer G-function is analytic in $|\arg r| < \infty$, the Riemann surface of the logarithm. For a comprehensive description of the Meijer G-function, we refer to [18], [19] and [20].

1.4 Main results

We define *solid harmonic polynomials* of order l to be polynomials $V(x)$ in $\mathbb{R}^d$ homogeneous of degree l, which satisfy $\Delta V = 0$. In dimension $d = 1$ there are only two solid harmonic polynomials, $V(x) \equiv 1$ and $V(x) \equiv x$. When $d = 2$, solid harmonic polynomials can be obtained as real or imaginary parts of monomials (if $\mathbb{R}^2$ is identified with $\mathbb{C}$). The main idea behind our results is stated in the following proposition, which is a pointwise version of an L^2 result given as Lemma 24.8 in [24].

Proposition 2. *Let $V(x)$ be a solid harmonic polynomial of degree $l \ge 0$ and $\delta = d + 2l$. Suppose that*

$$f(x) = V(x)\phi(|x|^2),$$

where ϕ is given as the (absolutely convergent) inverse Mellin transform

$$\phi(r) := \frac{1}{2\pi\mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} \mathcal{M}\phi(s) r^{-s} \mathrm{d}s \tag{24}$$

for some $\lambda \in \mathbb{R}$. If

$$0 < \alpha < 2\lambda - l < d, \tag{25}$$

then the Riesz potential $(-\Delta)^{-\alpha/2} f(x)$ *is well-defined for* $x \neq 0$, *and*

$$(-\Delta)^{-\alpha/2} f(x) = V(x)\psi(|x|^2),$$

where

$$\psi(r) := \frac{1}{2\pi \mathrm{i}} \int_{\lambda-\alpha/2+\mathrm{i}\mathbb{R}} \frac{\Gamma(s)\Gamma(\frac{\delta-\alpha}{2}-s)}{2^\alpha \Gamma(\frac{\alpha}{2}+s)\Gamma(\frac{\delta}{2}-s)} \mathcal{M}\phi(s+\tfrac{\alpha}{2}) r^{-s} \mathrm{d}s. \tag{26}$$

In other words, the Mellin transform of ψ satisfies

$$\mathcal{M}\psi(s) = \frac{\Gamma(s)\Gamma(\frac{\delta-\alpha}{2}-s)}{2^\alpha \Gamma(\frac{\alpha}{2}+s)\Gamma(\frac{\delta}{2}-s)} \mathcal{M}\phi(s+\tfrac{\alpha}{2})\,.$$

The above formula fits perfectly into the world of Meijer G-function: if ϕ is a Meijer G-function, then so is ψ. We recall that $\overline{a}$ and $\underline{b}$ are defined in (15), while $\overline{\lambda}$ and $\underline{\lambda}$ are defined in (17). The following two theorems are our main results about Meijer G-function. For technical reasons, we consider the Riesz potential and the fractional Laplace operators separately.

Theorem 1. *Let* $V(x)$ *be a solid harmonic polynomial of degree* $l \geq 0$ *and* $\delta = d + 2l$. *Assume that* $0 < \alpha < d$ *and parameters* m, n, p, q, $\mathbf{a}$ *and* $\mathbf{b}$ *satisfy Condition A, as well as*

$$2(1-\overline{a}) > \alpha + l, \qquad -2\underline{b} < d + l. \tag{27}$$

Define $f(x) := V(x) G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|\, |x|^2\right)$. *Then*

$$(-\Delta)^{-\alpha/2} f(x) = 2^{-\alpha} V(x) G^{m+1,n+1}_{p+2,q+2}\left(\begin{matrix} 1-\frac{\delta-\alpha}{2}, & \mathbf{a}+\frac{\alpha}{2}, & \frac{\alpha}{2} \\ 0, & \mathbf{b}+\frac{\alpha}{2}, & 1-\frac{\delta}{2}\end{matrix}\middle|\, |x|^2\right) \tag{28}$$

for all $x \neq 0$. *The same statement holds under Conditions B, C and D, provided that*

$$2\overline{\lambda} > \alpha + l, \qquad 2\underline{\lambda} < d + l; \tag{29}$$

if Condition B is satisfied, we additionally require that $\nu > 0$ *and either* $|x| \neq 1$ *or* $\nu + \alpha > 1$ *(see* (14) *for the definition of* ν*).*

Theorem 2. *Let* $V(x)$ *be a solid harmonic polynomial of degree* $l \geq 0$ *and* $\delta = d + 2l$. *Assume that* $\alpha > 0$ *and parameters* m, n, p, q, $\mathbf{a}$ *and* $\mathbf{b}$ *satisfy Condition A, and in addition*

$$2(1-\overline{a}) > -\alpha + l, \quad -2\underline{b} < d + l. \tag{30}$$

Define $f(x) := V(x) G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|\, |x|^2\right)$. *Then*

$$(-\Delta)^{\alpha/2} f(x) = 2^{\alpha} V(x) G^{m+1,n+1}_{p+2,q+2}\left(\begin{matrix} 1-\frac{\delta+\alpha}{2}, & \mathbf{a}-\frac{\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \mathbf{b}-\frac{\alpha}{2}, & 1-\frac{\delta}{2}\end{matrix}\middle|\, |x|^2\right) \tag{31}$$

for all $x \neq 0$. *The same statement holds under Conditions B, C and D, provided that*

$$2\overline{\lambda} > -\alpha + l, \qquad 2\underline{\lambda} < d + l; \tag{32}$$

if Condition B is satisfied, we additionally require that $\nu > 0$ *and either* $|x| \neq 1$ *or* $\nu > 1 + \alpha$. *Finally, the result extends to* $x = 0$ *whenever both* f *and* $(-\Delta)^{\alpha/2} f$ *are continuous at* 0.

By considering special cases of Meijer G-functions and using Theorems 1 and 2 one can obtain a multitude of new explicit examples and give a short proof of many known results. The most useful resource for deriving this results is the large collection of formulas, expressing known elementary and special functions in terms of Meijer G-function, which can be found in Table 8.4 in [20]. In the next two sections we provide several examples with different level of specialisation. Whenever possible, we state our results in terms of the regularised hypergeometric function ${}_p\mathbf{F}_q$, a somewhat simpler object than the Meijer G-function. Note that if none of the a_j is a non-positive integer, we have, by (23),

$$ {}_p\mathbf{F}_q\biggl(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\biggm| -r\biggr) = \frac{1}{\prod_{j=1}^{p}\Gamma(a_j)}\, G^{1,p}_{p,q+1}\biggl(\begin{matrix} & 1-\mathbf{a}\\ 0, & 1-\mathbf{b}\end{matrix}\biggm| r\biggr). \tag{33}$$

1.5 Full space

By formula 8.4.2.5 in [20] and the property (19), we have

$$ r^\rho(1+r)^\sigma = \frac{1}{\Gamma(-\sigma)}\, G^{11}_{11}\biggl(\begin{matrix}1+\rho+\sigma\\ \rho\end{matrix}\biggm| r\biggr). \tag{34}$$

This implies the following extension of expressions given in [25]. We remark that when $\rho = 0$ or $2\rho + 2\sigma = \alpha - \delta$, then the expression for $(-\Delta)^{\alpha/2} f(x)$ can be written in terms of the Gauss's hypergeometric function ${}_2F_1$.

Corollary 1. *Let $V(x)$ be a solid harmonic polynomial of degree $l \ge 0$ and $\delta = d + 2l$. Assume that $-d < \alpha < 0$ or $\alpha > 0$, $2\rho > -d - l$ and $2\rho + 2\sigma < \alpha - l$. Define*

$$ f(x) := V(x)|x|^{2\rho}(1+|x|^2)^\sigma. $$

Then

$$ (-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha}{\Gamma(-\sigma)}\, V(x) G^{2,2}_{3,3}\biggl(\begin{matrix}1-\frac{\delta+\alpha}{2}, & 1+\rho+\sigma-\frac{\alpha}{2}, & -\frac{\alpha}{2}\\ 0, & \rho-\frac{\alpha}{2}, & 1-\frac{\delta}{2}\end{matrix}\biggm| |x|^2\biggr) $$

for all $x \ne 0$, and also for $x = 0$ if $2\rho > \alpha - l$ or 2ρ is an even integer not less than $-l$.

Due to (33) (and the reduction formulas (21) and (22)), the operator $(-\Delta)^{\alpha/2}$ acts nicely on generalised hypergeometric functions. In the following statement the last parameter $b_q = \frac{\delta}{2}$ has a special role; to avoid ambiguities, we denote $\mathbf{b}' = (b_1, \ldots, b_{q-1})$.

Corollary 2. *Let $V(x)$ be a solid harmonic polynomial of degree $l \ge 0$ and $\delta = d + 2l$. Assume that $\alpha > -d$ and parameters p, q, $\mathbf{a} \in \mathbb{C}^p$ and $\mathbf{b}' \in \mathbb{C}^{q-1}$ satisfy*

$$ p \in \{q-1, q, q+1\}, \qquad 2\underline{a} > -\alpha + l, $$

where $\underline{a} = \min_{1\le j\le p} \operatorname{Re}(a_j)$. Define $f(x) := V(x)\,{}_p\mathbf{F}_q\biggl(\begin{matrix}\mathbf{a} & \\ \mathbf{b}', & \frac{\delta}{2}\end{matrix}\biggm| -|x|^2\biggr)$. Then

$$ (-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \prod_{j=1}^{p}\Gamma(a_j+\frac{\alpha}{2})}{\prod_{j=1}^{p}\Gamma(a_j)}\, V(x)\,{}_p\mathbf{F}_q\biggl(\begin{matrix}\mathbf{a}+\frac{\alpha}{2} & \\ \mathbf{b}'+\frac{\alpha}{2}, & \frac{\delta}{2}\end{matrix}\biggm| -|x|^2\biggr) $$

for all $x \in \mathbb{R}^d$.

The requirement that $b_q = \frac{\delta}{2}$ is of purely notational nature. In order to use Corollary 2 in the general case, simply write

$$ {}_p\mathbf{F}_q\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| \, r \right) = \Gamma(\tfrac{\delta}{2}) \, {}_{p+1}\mathbf{F}_{q+1}\left(\begin{matrix} \mathbf{a}, & \frac{\delta}{2} \\ \mathbf{b}, & \frac{\delta}{2} \end{matrix} \middle| \, r \right). $$

We remark that ${}_0\mathbf{F}_1(a \,|\, -|x|^2) = |x|^{2-2a} J_{a-1}(2|x|)$, where J_{a-1} is the Bessel function. In particular, ${}_0\mathbf{F}_1(\frac{1}{2} \,|\, -|x|^2) = \pi^{-1/2} \cos(2|x|)$, which gives a very surprising expression: if $f(x) := \cos(|x|)$, then

$$ (-\Delta)^{\alpha/2} f(x) = \frac{\sqrt{\pi}\, \Gamma(\frac{d+\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{d}{2})} \, {}_1\mathbf{F}_2\left(\begin{matrix} \frac{d+\alpha}{2} \\ \frac{1+\alpha}{2}, \quad \frac{d}{2} \end{matrix} \middle| -\tfrac{1}{2}|x|^2 \right) $$

for all $\alpha > 0$ and all $x \in \mathbb{R}^d$.

1.6 Unit ball and its complement

The function f in Theorems 1 and 2 is supported in the unit ball when $p = q = m$ and $n = 0$, and in the complement of the unit ball when $p = q = n$ and $m = 0$, see (23). Therefore, we obtain explicit expressions for $(-\Delta)^{\alpha/2} f_1(x)$ and $(-\Delta)^{\alpha/2} f_2(x)$ (with $|x| \neq 1$), where

$$ f_1(x) := V(x) G^{p0}_{pp}\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| \, |x|^2 \right), \qquad\qquad f_2(x) := V(x) G^{0p}_{pp}\left(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| \, |x|^2 \right) $$

are supported in the unit ball and its complement, respectively. Here we assume that Condition B holds, $\nu > 0$ and $-2\underline{b} < d + l$ in the expression for $(-\Delta)^{\alpha/2} f_1(x)$ with $x \neq 0$ (a more restrictive condition is needed when $x = 0$), and that Condition B holds, $\nu > 0$ and $2(1 - \overline{a}) > -\alpha + l$ in the expression for $(-\Delta)^{\alpha/2} f_2(x)$.

For example, formulas 8.4.2.3–4 in [20] combined with the property (19) tell us that

$$ r^\rho (1 - r)^\sigma_+ = \Gamma(1 + \sigma) G^{10}_{11}\left(\begin{matrix} 1 + \rho + \sigma \\ \rho \end{matrix} \middle| \, r \right), \tag{35} $$

$$ r^\rho (r - 1)^\sigma_+ = \Gamma(1 + \sigma) G^{01}_{11}\left(\begin{matrix} 1 + \rho + \sigma \\ \rho \end{matrix} \middle| \, r \right). \tag{36} $$

Theorems 1 and 2 readily imply the following explicit expressions, which extend those previously derived in [2, 8] (see also [15]).

Corollary 3. *Let $V(x)$ be a solid harmonic polynomial of degree $l \geq 0$ and $\delta = d + 2l$. Assume that $-d < \alpha < 0$ or $\alpha > 0$. Define*

$$ f_1(x) := V(x) |x|^{2\rho} (1 - |x|^2)^\sigma_+, $$
$$ f_2(x) := V(x) |x|^{2\rho} (|x|^2 - 1)^\sigma_+. $$

(i) *If $2\rho > -d - l$ and $\sigma > -1$, then*

$$ (-\Delta)^{\alpha/2} f_1(x) = 2^\alpha \Gamma(1 + \sigma) V(x) G^{2,1}_{3,3}\left(\begin{matrix} 1 - \frac{\delta+\alpha}{2}, & 1 + \rho + \sigma - \frac{\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \rho - \frac{\alpha}{2}, & 1 - \frac{\delta}{2} \end{matrix} \middle| \, |x|^2 \right) $$

for all x such that $x \neq 0$ and $|x| \neq 1$, and also for $x = 0$ if $2\rho > \alpha - l$ or 2ρ is an even integer not less than $-l$.

(ii) *If $2\rho + 2\sigma < \alpha - l$ and $\sigma > -1$, then*

$$(-\Delta)^{\alpha/2} f_2(x) = 2^\alpha \Gamma(1+\sigma) V(x) G^{1,2}_{3,3}\biggl(\begin{matrix} 1 - \frac{\delta+\alpha}{2}, & 1 + \rho + \sigma - \frac{\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \rho - \frac{\alpha}{2}, & 1 - \frac{\delta}{2} \end{matrix} \biggm| |x|^2 \biggr)$$

for all x such that $|x| \neq 1$.

The above result resembles Corollary 1. There are, however, only partial analogues of Corollary 2 for functions supported in the unit ball. We provide two results of this kind: Corollary 4 and Theorem 3.

Corollary 4. *Let $V(x)$ be a solid harmonic polynomial of degree $l \geq 0$ and $\delta = d + 2l$. Assume that $-d < \alpha < 0$ or $\alpha > 0$, as well as $2\rho > -d - l$ and $\sigma > -1$. Define*

$$f(x) := V(x)(1 - |x|^2)_+^\sigma \, {}_2\mathbf{F}_1\biggl(\begin{matrix} 1 + \sigma - \frac{\alpha}{2}, & \frac{\alpha}{2} - \rho \\ 1 + \sigma & \end{matrix} \biggm| 1 - |x|^2 \biggr). \tag{37}$$

Then

$$(-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \Gamma(\rho + \frac{\delta}{2})}{\Gamma(1 + \sigma - \frac{\alpha}{2})} V(x) |x|^{\alpha - 2\rho} {}_2\mathbf{F}_1\biggl(\begin{matrix} \rho + \frac{\delta}{2}, & \frac{\alpha}{2} - \sigma \\ \rho + \frac{\delta - \alpha}{2} & \end{matrix} \biggm| |x|^2 \biggr) \tag{38}$$

for all x such that $0 < |x| < 1$, and also for $x = 0$ if $2\rho > \alpha - l$.

With $\sigma = \frac{\alpha}{2}$, we simply have

$$(-\Delta)^{\alpha/2} f(x) = \frac{2^\alpha \Gamma(\rho + \frac{\delta}{2})}{\Gamma(\rho + \frac{\delta - \alpha}{2})} V(x) |x|^{\alpha - 2\rho}.$$

In particular, setting $\alpha > 0$, $\sigma = \frac{\alpha}{2}$ and $2\rho = \alpha - \delta$ in the above proposition we obtain a function

$$f(x) := V(x)(1 - |x|^2)_+^{\alpha/2} \, {}_2\mathbf{F}_1\biggl(\begin{matrix} 1, & \frac{\delta}{2} \\ 1 + \frac{\alpha}{2} & \end{matrix} \biggm| 1 - |x|^2 \biggr), \tag{39}$$

which is continuous except at $x = 0$, is equal to zero in the complement of the unit ball and satisfies $(-\Delta)^{\alpha/2} f(x) = 0$ for all x in the unit ball except at $x = 0$.

Remark 1. The Green function $G(x, y)$ for $(-\Delta)^{\alpha/2}$ in the unit ball was found in [21]. Traditionally, it is written in the form that was first stated in [3],

$$G(x, y) = \frac{\Gamma(\frac{d}{2})}{2^\alpha \pi^{d/2} (\Gamma(\frac{\alpha}{2}))^2} \frac{1}{|x - y|^{d - \alpha}} \int_0^{\frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}} \frac{s^{\alpha/2 - 1}}{(1 + s)^{d/2}} \, \mathrm{d}s.$$

Using formula 3.194.1 in [12], we obtain

$$G(x, y) = \frac{\Gamma(\frac{d}{2})(1 - |x|^2)^{\alpha/2}(1 - |y|^2)^{\alpha/2}}{2^\alpha \pi^{d/2} \Gamma(\frac{\alpha}{2}) |x - y|^d} \, {}_2\mathbf{F}_1\biggl(\begin{matrix} \frac{d}{2}, & \frac{\alpha}{2} \\ 1 + \frac{\alpha}{2} & \end{matrix} \biggm| -\frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \biggr).$$

For $y = 0$, by formula 9.131.1 in [12],

$$G(x,0) = \frac{\Gamma(\frac{d}{2})(1-|x|^2)^{\alpha/2}}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})|x|^d}\, {}_2\mathbf{F}_1\Big(\begin{matrix}\frac{d}{2}, & \frac{\alpha}{2} \\ 1+\frac{\alpha}{2} & \end{matrix}\Big|\, 1-\frac{1}{|x|^2}\Big)$$
$$= \frac{\Gamma(\frac{d}{2})(1-|x|^2)^{\alpha/2}}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})}\, {}_2\mathbf{F}_1\Big(\begin{matrix}\frac{d}{2}, & 1 \\ 1+\frac{\alpha}{2} & \end{matrix}\Big|\, 1-|x|^2\Big) = \frac{\Gamma(\frac{d}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})}\, f(x),$$

where f is given by (39) with $l = 0$ and $V(x) \equiv 1$. Thus Corollary 4 can be viewed as an alternative derivation of the expression for $G(x,0) = Cf(x)$ (the value of the constant C can be found by comparing the asymptotic expansion of f and the Riesz potential near 0, we omit the details). We remark that the expression for $G(x,y)$ can be obtained from that for $G(x,0)$ by means of the Kelvin transformation, see [17, 21].

For our last main result we need some additional notation. Solid harmonic polynomials of degree $l \geq 0$ form a finite-dimensional space, having dimension

$$M_{d,l} := \frac{d+2l-2}{d+l-2}\binom{d+l-2}{l}.$$

For each l we fix a linear basis of this space, denoted by $V_{l,m}$ with $m = 1, \ldots, M_{d,l}$, which is orthonormal with respect to the surface measure μ on the unit sphere. Since the space $L^2(\mu)$ is a direct sum of the above linear spaces, the collection $V_{l,m}$, with $l = 0, 1, \ldots$ and $m = 1, \ldots, M_{d,l}$, is an orthonormal basis of $L^2(\mu)$. For further properties of harmonic polynomials, see [1, 7].

Recall that the Jacobi polynomials are defined as

$$P_n^{(\alpha,\beta)}(z) := \frac{\Gamma(\alpha+1+n)}{n!}\, {}_2\mathbf{F}_1\Big(\begin{matrix}-n, & 1+\alpha+\beta+n \\ & \alpha+1\end{matrix}\Big|\, \frac{1-z}{2}\Big)$$
$$= \frac{(-1)^n\Gamma(\beta+1+n)}{n!}\, {}_2\mathbf{F}_1\Big(\begin{matrix}-n, & 1+\alpha+\beta+n \\ & \beta+1\end{matrix}\Big|\, \frac{1+z}{2}\Big). \tag{40}$$

Given $d \geq 1$ and $\alpha > 0$, we denote

$$P_{l,m,n}(x) := V_{l,m}(x) P_n^{(\alpha/2, d/2+l-1)}(2|x|^2 - 1), \tag{41}$$

where $l, n \geq 0$ and $1 \leq m \leq M_{d,l}$. This system of polynomials forms a complete orthogonal system in $L^2(w)$, the weighted L^2 space with weight function $w(x) = (1-|x|^2)_+^{\alpha/2}$, see Proposition 2.3.1 in [7]. Finally, we define

$$p_{l,m,n}(x) := (1-|x|^2)_+^{\alpha/2} P_{l,m,n}(x).$$

The following proposition will play an important role in the study of eigenvalues of the fractional Laplace operator in [9]. For the special case of this result when $d = \alpha = 2$, see [11].

Theorem 3. *Assume that $\alpha > 0$, $l, n \geq 0$, $1 \leq m \leq M_{d,l}$. Then*

$$(-\Delta)^{\alpha/2} p_{l,m,n}(x) = \frac{2^\alpha \Gamma(1+\frac{\alpha}{2}+n)\Gamma(\frac{\delta+\alpha}{2}+n)}{n!\,\Gamma(\frac{\delta}{2}+n)}\, P_{l,m,n}(x) \tag{42}$$

for all x such that $|x| < 1$. In other words, the polynomials $P_{l,m,n}$ form a complete orthogonal system of eigenfunctions of the operator $f \mapsto (-\Delta)^\alpha(wf)$ in $L^2(w)$, the weighted L^2 space with weight function $w(x) = (1-|x|^2)_+^{\alpha/2}$.

2 Proofs

2.1 Technical results

The following result essentially reduces the case of a general solid harmonic polynomial V of degree $l \geq 0$ to $V(x) \equiv 1$ and $l = 0$.

Proposition 3. *Let $V(x)$ be a solid harmonic polynomial of degree $l \geq 0$. Let f and $\tilde{f}$ be two radial functions in $\mathbb{R}^d$ and $\mathbb{R}^{d+2l}$ with the same profile function.*

(i) *If $0 < \alpha < d$, $x \in \mathbb{R}^d \setminus \{0\}$ and the function $y \mapsto V(y)f(y)|x-y|^{\alpha-d}$ is in $L^1(\mathbb{R}^d)$, then*

$$(-\Delta)^{-\alpha/2}(Vf)(x) = V(x)(-\Delta)^{-\alpha/2}\tilde{f}(\tilde{x}), \tag{43}$$

for any $\tilde{x} \in \mathbb{R}^{d+2l}$ such that $|\tilde{x}| = |x|$. Similar statement also holds for $x = 0$ if the function $y \mapsto (1+|V(y)|)f(y)|y|^{\alpha-d}$ is in $L^1(\mathbb{R}^d)$.

(ii) *If $\alpha > 0$, $x \in \mathbb{R}^d$, $f(y)$ is smooth in a neighbourhood of x and the function $y \mapsto V(y)f(y)(1+|y|)^{-\alpha-d}$ is in $L^1(\mathbb{R}^d)$, then*

$$(-\Delta)^{\alpha/2}(Vf)(x) = V(x)(-\Delta)^{\alpha/2}\tilde{f}(\tilde{x}), \tag{44}$$

for any $\tilde{x} \in \mathbb{R}^{d+2l}$ such that $|\tilde{x}| = |x|$.

The proof is based on the following result.

Bochner's relation (Corollary on page 72 in [26])**.** *Let f and $\tilde{f}$ be two radial Schwartz functions in $\mathbb{R}^d$ and $\mathbb{R}^{d+2l}$ with the same profile function, and $x \in \mathbb{R}^d$, $\tilde{x} \in \mathbb{R}^{d+2l}$, $|x| = |\tilde{x}|$. Then*

$$\mathcal{F}_d(Vf)(x) = \mathrm{i}^l V(x)\mathcal{F}_{d+2l}\tilde{f}(\tilde{x}). \tag{45}$$

Proof of Proposition 3. Everywhere in this proof we will assume that $x \in \mathbb{R}^d$ and $\tilde{x} \in \mathbb{R}^{d+2l}$, and $|x| = |\tilde{x}|$. Our first goal is to establish the following result: For any radial Schwartz functions ϕ and $\tilde{\phi}$ on $\mathbb{R}^d$ and $\mathbb{R}^{d+2l}$ with the same profile it is true that

$$(\mathcal{F}_d^{-1}\phi) * (Vf)(x) = V(x)(\mathcal{F}_{d+2l}^{-1}\tilde{\phi}) * \tilde{f}(\tilde{x}). \tag{46}$$

To prove this result, for $\tilde{x} \in \mathbb{R}^{d+2l}$ we define

$$\tilde{h}(\tilde{x}) := \tilde{\phi}(\tilde{x})\mathcal{F}_{d+2l}\tilde{f}(\tilde{x}),$$

and denote by $h(x)$ the radial function in $\mathbb{R}^d$ having the same profile as $\tilde{h}$. By Bochner's relation (45) we have

$$\phi(x)\mathcal{F}_d(Vf)(x) = \mathrm{i}^l\phi(x)V(x)\mathcal{F}_{d+2l}\tilde{f}(\tilde{x}) = \mathrm{i}^l V(x)h(x),$$

and so

$$(\mathcal{F}_d^{-1}\phi) * (Vf) = \mathcal{F}_d^{-1}\big[\phi\mathcal{F}_d(Vf)\big] = \mathrm{i}^l\mathcal{F}_d^{-1}\big[Vh\big].$$

On the other hand,

$$(\mathcal{F}_{d+2l}^{-1}\tilde{\phi}) * \tilde{f} = \mathcal{F}_{d+2l}^{-1}\big[\tilde{\phi}\mathcal{F}_{d+2l}\tilde{f}\big] = \mathcal{F}_{d+2l}^{-1}\tilde{h}.$$

Identity (46) follows from connection between Fourier and inverse Fourier transforms, and another application of Bochner's relation (45),

$$\mathrm{i}^l\mathcal{F}_d^{-1}\big[Vh\big](x) = (2\pi)^{-d}\mathrm{i}^l\overline{\mathcal{F}_d\big[Vh\big](x)} = (2\pi)^{-d}V(x)\overline{\mathcal{F}_{d+2l}\tilde{h}(\tilde{x})} = V(x)\mathcal{F}_{d+2l}^{-1}\tilde{h}(\tilde{x}).$$

Now we are ready to prove part (i) of Proposition 3. Let us consider first the case when f is a Schwartz function. We set ϕ_ε and $\tilde{\phi}_\varepsilon$ to be functions on $\mathbb{R}^d$ and $\mathbb{R}^{d+2l}$ with the same profile function

$$\phi_\varepsilon(x) = \tilde{\phi}_\varepsilon(\tilde{x}) = (\varepsilon^2 + r^2)^{-\alpha/2}e^{-\varepsilon r^2}, \qquad |x| = |\tilde{x}| = r.$$

Note that both ϕ_ε and $\tilde{\phi}_\varepsilon$ are radial Schwartz functions and their profile increases to $r^{-\alpha}$ as $\varepsilon \to 0^+$. In particular, ϕ_ε and $\tilde{\phi}_\varepsilon$ converge in the space of Schwartz distributions to $\phi(x) = |x|^{-\alpha}$ and $\tilde{\phi}(x) = |\tilde{x}|^{-\alpha}$, respectively. It follows that $\mathcal{F}_d^{-1}\phi$ and $\mathcal{F}_{d+2l}^{-1}\tilde{\phi}$ also converge as Schwartz distributions to kernel functions of Riesz potential operators $(-\Delta)^{-\alpha/2}$ in $\mathbb{R}^d$ and $\mathbb{R}^{d+2l}$, respectively (see (7) and (11)). Using (46) for ϕ_ε and $\tilde{\phi}_\varepsilon$ and passing to the limit $\varepsilon \to 0^+$, we conclude that (43) holds true when f is a Schwartz function.

The general case (when f is not a Schwartz function, but satisfies the integrability condition stated in part (i)) follows by approximation. Indeed, let f_n be a sequence of radial Schwartz functions on $\mathbb{R}^d$ which converges to f in the weighted $L^1(\mathbb{R}^d)$ norm, with weight function $y \mapsto |y|^l|x-y|^{\alpha-d}$ (or $y \mapsto 1 + |y|^l$ when $x = 0$). Then it is easy to see that the corresponding radial functions $\tilde{f}_n$ on $\mathbb{R}^{d+2l}$ converge to $\tilde{f}$ in the weighted $L^1(\mathbb{R}^{d+2l})$ norm with weight function $\tilde{y} \mapsto |\tilde{x} - \tilde{y}|^{\alpha-d}$, and (43) follows.

The proof of part (ii) is obtained in the same way, except that we use the functions

$$\phi_\varepsilon(x) = \tilde{\phi}_\varepsilon(\tilde{x}) = (\varepsilon^2 + r^2)^{\alpha/2}e^{-\varepsilon r^2}, \qquad |x| = |\tilde{x}| = r.$$

The details are left to the reader. □

We often prove our results for a restricted range of parameters, and then extend them by analytic continuation. This is possible thanks to our next proposition.

Proposition 4. *Let U be an open set in $\mathbb{C}^n$, $n \geq 1$. Assume that $z \mapsto f_z(x)$ is an analytic function on U for almost all $x \in \mathbb{R}^d$.*

(i) *Let us fix $x \in \mathbb{R}^d$ and $0 < \alpha < d$. If*

$$y \mapsto |y-x|^{-d+\alpha} \sup_{z\in U} |f_z(y)| \quad \text{is in } L^1(\mathbb{R}^d),$$

then $(-\Delta)^{-\alpha/2} f_z(x)$ is an analytic function of $z \in U$.

(ii) *Let us fix $x \in \mathbb{R}^d$ and $\alpha_0 > 0$. If*

$$y \mapsto (1+|y|)^{-d-\alpha_0} \sup_{z\in U} |f_z(y)| \quad \text{is in } L^1(\mathbb{R}^d) \tag{47}$$

and the function $y \mapsto f_z(y)$ is smooth in some small ball $B(x,r)$ with partial derivatives (of arbitrary order) bounded uniformly in (y,z) on $B(x,r)\times U$, then $(-\Delta)^{\alpha/2} f_z(x)$ is an analytic function of $z \in U$ and of α in the half-plane $\mathrm{Re}(\alpha) > \alpha_0$.

The proof is based on the following result.

Lemma 1 (Lemma 1.31 in [25]). *Let D be an open set in $\mathbb{C}$, $\Omega \subseteq \mathbb{R}^d$ and $g(y,z)$ be an analytic function in $z \in D$ for almost all $y \in \Omega$. If $y \mapsto \sup_{z\in D} |g(y,z)|$ is in $L^1(\Omega)$ then the function $z \mapsto \int_\Omega g(y,z)\mathrm{d}y$ is analytic on D.*

Proof of Proposition 4. Let us prove part (i). Let $z = (z_1, z_2, \ldots, z_n) \in U$. Lemma 1 guarantees that the function $(-\Delta)^{-\alpha/2} f_z(x)$ is analytic in each variable z_i, $1 \le i \le n$. Applying Hartog's Theorem we obtain joint analyticity in $z \in U$.

For the proof of part (ii), we observe that the boundedness of partial derivatives of the functions f_z implies that for $y \in B(0,r)$

$$|\Delta_y^k f_z(x)| \le C|y|^k,$$

for some constant $C = C(k,r)$ which does not depend on z. Therefore, the function

$$y \mapsto \sup_{\substack{z\in U \\ \alpha_0 < \mathrm{Re}\,\alpha < k}} \big| |y|^{-d-\alpha} \Delta_y^k f_z(x) \big|$$

is integrable in $B(0,r)$. On the other hand, the same function is integrable in $\mathbb{R}^d \setminus B(0,r)$ by condition (47). The desired result under additional condition $\mathrm{Re}\,\alpha < k$ follows now from (10) and Lemma 1 (and Hartog's Theorem). Since k was arbitrary, the proof is complete. □

2.2 Main results

Proof of Proposition 2. When $l = 0$ and $V(x) \equiv 1$, the result follows rather easily from Fubini's theorem and the semigroup property of Riesz potential operators $(-\Delta)^{-(\alpha+\beta)/2} = (-\Delta)^{-\alpha/2}(-\Delta)^{-\beta/2}$. More precisely, by formula (1.1.12) in [17] (see also formula (8) on page 118 in [26]),

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\alpha}} \frac{1}{|y|^{d-\beta}} \mathrm{d}y = \frac{\gamma_d(\alpha)\gamma_d(\beta)}{\gamma_d(\alpha+\beta)} \frac{1}{|x|^{d-\alpha-\beta}} \tag{48}$$

for all $x \in \mathbb{R}^d$ and all $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$ and $\mathrm{Re}\,\alpha + \mathrm{Re}\,\beta < d$. Applying Fubini's Theorem, using (48) for $\alpha \in (0,d)$ and $\beta = d - 2s$, and substituting $\tilde{s} + \frac{\alpha}{2}$ for s, we obtain

$$\begin{aligned}
(-\Delta)^{-\alpha/2} f(x) &= \frac{1}{2\pi\mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} \mathcal{M}\phi(s) \left(\frac{1}{\gamma_d(\alpha)} \int_{\mathbb{R}^d} \frac{1}{|x-y|^{2s}} \frac{1}{|y|^{d-\alpha}} \mathrm{d}y \right) \mathrm{d}s \\
&= \frac{1}{2\pi\mathrm{i}} \int_{\lambda+\mathrm{i}\mathbb{R}} \mathcal{M}\phi(s) \frac{\gamma_d(d-2s)}{\gamma_d(d+\alpha-2s)} \frac{1}{|x|^{2s-\alpha}} \mathrm{d}s \\
&= \frac{1}{2\pi\mathrm{i}} \int_{\lambda-\alpha/2+\mathrm{i}\mathbb{R}} \mathcal{M}\phi(\tilde{s} + \tfrac{\alpha}{2}) \frac{\gamma_d(d-\alpha-2\tilde{s})}{\gamma_d(d-2\tilde{s})} \frac{1}{|x|^{2\tilde{s}}} \mathrm{d}\tilde{s},
\end{aligned}$$

which is equivalent to the desired result (26). The use of Fubini's Theorem is justified since for $s = \lambda + \mathrm{i}t$, the integral

$$\frac{1}{\gamma_d(\alpha)} \int_{\mathbb{R}^d} \left| \frac{1}{|x-y|^{2s}} \, \frac{1}{|y|^{d-\alpha}} \right| \mathrm{d}y = \frac{1}{\gamma_d(\alpha)} \int_{\mathbb{R}^d} \frac{1}{|x-y|^{2\lambda}} \, \frac{1}{|y|^{d-\alpha}} \, \mathrm{d}y$$

is finite and does not depend on s, while the integral in (24) is absolutely convergent.

The case of general V (that is, $l \geq 0$) reduces to $V(x) \equiv 1$ in dimension $\delta = d + 2l$ by Proposition 3. $\square$

Proof of Theorem 1. We provide a detailed argument when Condition A is satisfied. Suppose first that the parameters p, q, m, n, $\mathbf{a}$ and $\mathbf{b}$ satisfy Conditions S and A. In this case the desired result is a simple consequence of Proposition 2. Indeed, for $\lambda \in (-\underline{b}, 1-\overline{a}) \cap (\frac{\alpha+l}{2}, \frac{d+l}{2})$ (such a number λ exists by Condition S and assumption (27)) we have

$$\begin{aligned} (-\Delta)^{-\alpha/2} f(x) &= V(x) \, \frac{1}{2\pi \mathrm{i}} \int_{\lambda-\alpha/2+\mathrm{i}\mathbb{R}} \mathcal{G}^{mn}_{pq}\biggl(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \biggm| s + \tfrac{\alpha}{2} \biggr) \frac{\Gamma(s)\Gamma(\frac{\delta-\alpha}{2} - s)}{2^\alpha \Gamma(\frac{\alpha}{2} + s)\Gamma(\frac{\delta}{2} - s)} \, |x|^{-2s} \mathrm{d}s \\ &= 2^{-\alpha} V(x) G^{m+1,n+1}_{p+2,q+2}\biggl(\begin{matrix} 1 - \frac{\delta-\alpha}{2}, & \mathbf{a} + \frac{\alpha}{2}, & 0 \\ 0, & \mathbf{b} + \frac{\alpha}{2}, & 1 - \frac{\delta}{2} \end{matrix} \biggm| |x|^2 \biggr), \end{aligned}$$

where the second equality is a consequence of the definition (13) of $\mathcal{G}^{mn}_{pq}$ and the definition (16) of the Meijer G-function.

Relaxing Condition S is possible by analytic continuation and Proposition 4. Indeed, let U be a region with compact closure contained in the set of admissible parameters of f under Condition A. Using the definition (16), with an appropriately modified contour of integration in the general case, one can prove that the constants in the asymptotic estimates (18) for the Meijer G-function can be chosen continuously with respect to the parameters; we omit the details. It follows that the supremum of $|f(y)|$ taken over all parameters from U has, for some $\varepsilon > 0$, the following properties: (i) it is locally bounded in $y \neq 0$; (ii) it is $O(|y|^{-d+\varepsilon})$ as $y \to 0$; (iii) it is $O(|y|^{-\alpha-\varepsilon})$ as $|y| \to \infty$. Hence, the assumptions of Proposition 4(i) are satisfied, and the desired result follows by analyticity of both sides of (28).

We now sketch the proof in the remaining cases. When Conditions S and B are satisfied and $\nu > 1$, the argument is very similar, based on Proposition 2. In order to relax the condition on ν in the analytic continuation argument, one also needs the following regularity of the Meijer G-function near 1: if $0 \leq \nu \leq 1$ and $\varepsilon > 0$, then

$$G^{mn}_{pq}\biggl(\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \biggm| r \biggr) = O(|r-1|^{\nu-\varepsilon-1}) \quad \text{as } r \to 1, \tag{49}$$

again with constant in the asymptotic notation depending continuously on the parameters. This can be proved using the estimates for the hypergeometric function ${}_p\mathbf{F}_{q-1}$ and (23); we omit the details.

Similarly, when Conditions S and either C or D are satisfied, and in addition $-\underline{b} < \nu < 1 - \overline{a}$, then $\underline{\lambda} < \overline{\lambda}$ and Proposition 2 applies. The conditions on $\mathbf{a}$ and $\mathbf{b}$ again can be relaxed by analytic continuation. $\square$

Proof of Theorem 2. Assume first that $x \neq 0$ and the parameters α, p, q, m, n, $\mathbf{a}$ and $\mathbf{b}$ satisfy Condition A and in addition

$$0 < \alpha < d, \qquad 2(1 - \overline{a}) > l, \qquad -2\underline{b} < d + l - \alpha. \tag{50}$$

Let us denote the function in the right-hand side of (31) by $g(x)$. Then $g(y)$ is continuous in $y \neq 0$, and by (50) and (18), for some $\varepsilon > 0$ we have $|g(y)| = O(|y|^{-d+\varepsilon})$ as $y \to 0^+$ and $|g(y)| = O(|y|^{-\alpha-\varepsilon})$ as $y \to \infty$ (because $l + 2\underline{b} - \alpha > -d$ and $l - 2(1 - \overline{a}) - \alpha < -\alpha$). Conditions (50) ensure also that $2(1-\overline{a}+\frac{\alpha}{2}) > \alpha+l$ and $-2(\underline{b}-\frac{\alpha}{2}) < d+l$. It follows that we can compute $(-\Delta)^{-\alpha/2}g$ via Theorem 1: performing this computation and using reduction formulas (21) and (22) we check that:

$$\begin{aligned}(-\Delta)^{-\alpha/2}g(x) &= V(x)G^{m+2,n+2}_{p+4,q+4}\left(\begin{matrix}1-\frac{\delta-\alpha}{2}, & 1-\frac{\delta}{2}, & \mathbf{a}, & 0, & \frac{\alpha}{2} \\ 0, & \frac{\alpha}{2}, & \mathbf{b}, & 1-\frac{\delta-\alpha}{2}, & 1-\frac{\delta}{2}\end{matrix}\,\middle|\,|x|^2\right) \\ &= V(x)G^{mn}_{pq}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,|x|^2\right) = f(x).\end{aligned}$$

To finish the proof we only need to note that g is continuous at x and apply Proposition 1. If g is continuous at 0, we also have (31) at $x = 0$.

Relaxing the additional conditions is done using Proposition 4 in a very similar way as in the proof of Theorem 1. Both f and g are analytic functions of α, $\mathbf{a}$ and $\mathbf{b}$ in the admissible region specified by condition (30). From the estimates (18) of the Meijer G-function it follows that f satisfies the assumptions of Proposition 4(ii) for every domain U with compact closure in the admissible region.

The remaining cases, when Conditions B, C or D are satisfied, are very similar, and we only sketch the argument. Under Condition B, one initially assumes that $\nu > 1+\alpha$ and then uses also (49) for the analytic continuation. If Conditions C or D hold, the analytic continuation part requires no modifications, but in order to use Theorem 1, one needs to initially assume that either $1 + 2\frac{\nu-1-\alpha}{q-p} > l$ when Condition C is satisfied, or $1 - 2\frac{\nu-1-\alpha}{p-q} < d + l - \alpha$ when Condition D holds. We omit the details. □

2.3 Full space

Proof of Corollary 1. The result follows immediately from Theorem 1 (when $-d < \alpha < 0$) or Theorem 2 (when $\alpha > 0$) by using (34). □

Proof of Corollary 2. Denote $\mathbf{b} = (b_1, \ldots, b_{q-1}, \frac{\delta}{2})$. Suppose first that none of a_j is a non-positive integer. Using (33), Theorem 1 (when $-d < \alpha < 0$) or Theorem 2 (when $\alpha > 0$) and then the reduction formulas (21) and (22), we obtain the desired result. The restriction that none of a_j is a non-positive integer can be relaxed by analytic continuation using Proposition 4. □

2.4 Unit ball and its complement

Proof of Corollary 3. The result follows immediately from Theorem 1 (when $-d < \alpha < 0$) or Theorem 2 (when $\alpha > 0$) by using (35) and (36). $\square$

Proof of Corollary 4. From formula 8.4.49.22 in [20] we find that

$$f(x) = G^{20}_{22}\left(\begin{matrix} \frac{\alpha}{2}, & 1+\rho+\sigma-\frac{\alpha}{2} \\ 0, & \rho \end{matrix} \,\middle|\, |x|^2 \right).$$

Applying Theorem 1 (when $-d < \alpha < 0$) or Theorem 2 (when $\alpha > 0$), and simplifying the result we obtain

$$\begin{aligned} (-\Delta)^{\alpha/2} f(x) &= 2^{\alpha} V(x) G^{31}_{44}\left(\begin{matrix} 1-\frac{\delta+\alpha}{2}, & 0, & 1+\rho+\sigma-\alpha, & -\frac{\alpha}{2} \\ 0, & -\frac{\alpha}{2}, & \rho-\frac{\alpha}{2}, & 1-\frac{\delta}{2} \end{matrix} \,\middle|\, |x|^2 \right) \\ &= 2^{\alpha} V(x) G^{11}_{22}\left(\begin{matrix} 1-\frac{\delta+\alpha}{2}, & 1+\rho+\sigma-\alpha \\ \rho-\frac{\alpha}{2}, & 1-\frac{\delta}{2} \end{matrix} \,\middle|\, |x|^2 \right), \end{aligned}$$

where in the second step we have used reduction formula (22). Formula (38) follows from the above expression and (23). $\square$

In the next result, we will use the following property of Meijer G-function, which follows easily from the definition (16) as the inverse Mellin transform: if k is an integer, $\mathbf{a}' = (a_2, \ldots, a_p)$ and $\mathbf{b}' = (b_1, \ldots, b_{q-1})$, then

$$G^{mn}_{pq}\left(\begin{matrix} c, & \mathbf{a}' \\ \mathbf{b}', & c+k \end{matrix} \,\middle|\, r \right) = (-1)^k G^{m+1,n-1}_{p,q}\left(\begin{matrix} \mathbf{a}', & c \\ c+k, & \mathbf{b}' \end{matrix} \,\middle|\, r \right). \tag{51}$$

Proof of Theorem 3. Define $\delta := d+2l$, $V(x) := V_{l,m}(x)$,

$$f(x) := \frac{n!\, p_{l,m,n}(x)}{\Gamma(1+\frac{\alpha}{2}+n)} = V(x)(1-|x|^2)_+^{\alpha/2} {}_2\mathbf{F}_1\left(\begin{matrix} -n, & \frac{\delta+\alpha}{2}+n \\ & 1+\frac{\alpha}{2} \end{matrix} \,\middle|\, 1-|x|^2 \right)$$

(the equality follows from the definitions of $p_{l,m,n}$ and $P_{l,m,n}$ and the first expression for the Jacobi polynomial in (40)) and

$$g(x) := \frac{n!\, P_{l,m,n}(x)}{\Gamma(\frac{\delta}{2}+n)} = (-1)^n V(x) {}_2\mathbf{F}_1\left(\begin{matrix} \frac{\delta+\alpha}{2}+n, & -n \\ & \frac{\delta}{2} \end{matrix} \,\middle|\, |x|^2 \right)$$

(here we used the second expression for the Jacobi polynomial in (40)). By formula 8.4.49.22 in [20] and (51), we have

$$\begin{aligned} f(x) &= V(x) G^{20}_{22}\left(\begin{matrix} 1+\frac{\alpha}{2}+n, & 1-\frac{\delta}{2}-n \\ 1-\frac{\delta}{2}, & 0 \end{matrix} \,\middle|\, |x|^2 \right) \\ &= (-1)^n V(x) G^{11}_{22}\left(\begin{matrix} 1-\frac{\delta}{2}-n, & 1+\frac{\alpha}{2}+n \\ 0, & 1-\frac{\delta}{2} \end{matrix} \,\middle|\, |x|^2 \right). \end{aligned}$$

Using Theorem 2 and the reduction formula (22) we find that when $|x| \neq 1$,

$$(-\Delta)^{\alpha/2} f(x) = (-1)^n 2^\alpha V(x) G^{11}_{22}\left(\begin{matrix} 1 - \frac{\delta+\alpha}{2} - n, & 1+n, \\ 0, & 1 - \frac{\delta}{2} \end{matrix} \,\middle|\, |x|^2 \right).$$

By (23), when $|x| < 1$,

$$\begin{aligned} (-\Delta)^{\alpha/2} f(x) &= \frac{(-1)^n 2^\alpha \Gamma(\frac{\delta+\alpha}{2} + n)}{n!} V(x) {}_2\mathbf{F}_1\left(\begin{matrix} \frac{\delta+\alpha}{2} + n, & -n \\ \frac{\delta}{2} & \end{matrix} \,\middle|\, |x|^2 \right) \\ &= \frac{2^\alpha \Gamma(\frac{\delta+\alpha}{2} + n)}{n!} g(x), \end{aligned}$$

as desired. □

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