

Analytic proof of Pecherskii-Rogozin identity and Wiener-Hopf factorization

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Abstract

We present an analytic proof of the Pecherskii-Rogozin identity and the Wiener-Hopf factorization. The proof is rather general and requires only one mild restriction on the tail of the Lévy measure. The starting point for the proof of the Pecherskii-Rogozin identity is a two dimensional integral equation satisfied by the joint distribution of the first passage time and the overshoot. This equation is reduced to a one-dimensional Wiener-Hopf integral equation, which is then solved using classic techniques from the theory of the Riemann boundary value problems. The Wiener-Hopf factorization is derived then as a corollary of the Pecherskii-Rogozin identity.

Keywords: Lévy process, Wiener-Hopf factorization, Pecherskii-Rogozin identity, integral equation, Riemann boundary value problem, Sokhotsky's formulas

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1 Introduction

In the last decade there has been a lot of interest in the Wiener-Hopf factorization and other fluctuation identities. During this time Lévy processes have found many applications in mathematical finance, risk management, insurance, etc. These applications have posed a multitude of new problems, whose solution requires some information about the distribution of various functionals of the process, such as extrema, first/last passage time, overshoot, etc. Fluctuation theory of Lévy processes gives us valuable insights into the properties of these functionals and at the same time provides us with the tools needed for the numerical computations.

It seems that the first general result on the distribution of the supremum of a Lévy process was obtained by Baxter and Donsker in [2]. The authors expressed the double Laplace transform of the distribution of the supremum (which is essentially the positive Wiener-Hopf factor) as a double integral involving the characteristic exponent of the process. The method that was used in this paper has become the standard approach for the next twenty years: first one should approximate a Lévy process by a discrete time random walk, then derive a formula for this random walk (Baxter and Donsker have used Spitzer's combinatorial identity, see [21] and [22]), and finally, prove that the result still holds true in the limit as the random walk converges to the Lévy process.

It is interesting to note that it took almost a decade before there have appeared any new general result. An important paper by Rogozin was published in 1966 (see [18]), where the author has used similar techniques to prove the Wiener-Hopf factorization, with the Wiener-Hopf factors expressed in terms of the distribution of the process. In [16] Pecherskii and Rogozin have obtained many other important factorization identities, including the Pecherskii-Rogozin identity, which describes the joint distribution of the first passage time and the overshoot. We should also mention that at around the same time there have appeared many important fluctuation results for Lévy processes with no positive or negative jumps, also known as spectrally one-sided processes, see [4], [20] and [23].

All of the above mentioned papers used essentially the same technique of approximating a Lévy process by a discrete time random walk. However there was also a very interesting alternative method, which was developed by Gusak and Korolyuk. Their idea was to approximate a Lévy process by a compound Poisson process with drift and to write down a stochastic (pathwise) relation for the functionals by using the Markov property and conditioning on the time and the location/size of the next jump of the process. The second step is to translate this stochastic relation into an integral equation satisfied by the distribution function, and finally, to solve this integral equation using the Wiener-Hopf techniques. This approach was used in [13] to obtain an identity for the distribution of the first passage time, which can be viewed as a particular case of the Wiener-Hopf factorization or the Pecherskii-Rogozin identity. This approach was later extended in [12] and was used to obtain an identity for the joint distribution of the first passage time and the overshoot, under the restriction that there is no Gaussian component and that the process has a finite mean.

We would also like to mention that the interested reader can find very useful surveys of the results and methods used at that time in [5] and [10].

After all these results were established at the end of 1960's, the next natural step was to develop a general approach, which would not require the initial approximation of the Lévy process by a random walk or a compound Poisson process. It turns out that both approaches discussed

above cannot be easily extended to the case of a general Lévy process. In the first approach the central idea was to apply the Markov property at the time when the random walk attains its maximum. It is impossible to translate this idea into the general setting for the simple reason that the ladder time set (the set where the process attains its maximum) may be not discrete. It is also not clear how one can extend the second approach by Gusak and Korolyuk to the case when a Lévy process has jumps of infinite activity, as in this case the notion of “the time of the next jump” loses its meaning. Therefore, only when all the necessary tools were developed in 1960’s and 1970’s, such as the theory of the local time process and the Poisson point process of excursions, it was possible to give a direct and completely general probabilistic proof of various factorization identities, which was first done in a seminal paper by Greenwood and Pitman [11].

The modern approach to the fluctuation theory mostly follows the path defined in [11] and relies on the excursion process, the local time, various path decompositions, etc. A good account of this theory can be found in [3], [8] and [14]. This probabilistic technique has provided a great benefit for the theory in terms of the generality and simplicity and has also led to a great number of new and important results. However, one could argue that there is also a downside to this development: even though the statements of various fluctuation identities are rather simple and can be easily understood by non-specialists, their proofs usually require a substantial background knowledge in the theory of the local time, the excursion process, etc. It is also true that some results are just easier to prove using analytic techniques, and these analytic proofs might also shed light on the corresponding probabilistic constructions.

In this paper we present a completely analytic and (almost) general proof of the two most famous results in fluctuation theory: the Wiener-Hopf factorization and the Pecherskii-Rogozin identity. The proof uses only the strong Markov property for the Lévy processes and some standard results from the Complex Analysis. We follow an approach which is similar to the one developed by Gusak and Korolyuk in [12] and [13]: we start with a certain integral equation, obtained by applying the Markov property at the first passage time, and then we solve this equation using the Wiener-Hopf techniques. However, there is one major difference: unlike Gusak and Korolyuk, we rely on the integral equation which is “global” in the sense that it describes an event which happens over a fixed time interval $[s, t]$, and not an event that happens “locally” – at the time of the next jump. This approach allows us to give a rather general proof of the Wiener-Hopf factorization: we only need to impose one mild restriction on the tail of the Lévy measure, which ensures that the characteristic exponent is Hölder continuous.

The paper is organized as follows: in Section 2 we introduce the notation and the main results while all the proofs are presented in Section 3.

2 Definitions, notations and statements of the results

Let X be a one-dimensional Lévy process started at zero, defined by the characteristic triplet (μ, σ, Π) , where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dx)$ is a positive measure on $\mathbb{R} \setminus \{0\}$ satisfying the usual integrability condition

$$\int_{\mathbb{R}} \min(1, x^2) \Pi(dx) < \infty.$$

The characteristic exponent of the process X is defined as $\Psi(z) = -\ln(\mathbb{E}[\exp(izX_1)])$ and it can be expressed by the Lévy-Khintchine formula (see [3]) as follows

$$\Psi(z) = \frac{1}{2}\sigma^2 z^2 - i\mu z - \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{I}(|x| < 1)) \Pi(dx). \quad (1)$$

Everywhere in this paper we will assume that the Lévy measure satisfies the following condition:

Assumption 1. There exists $\epsilon \in (0, 1)$ such that

$$\int_{\mathbb{R} \setminus [-1, 1]} |x|^\epsilon \Pi(dx) < \infty.$$

Remark 1. The integrability condition in the Assumption 1 is a very mild restriction: it is satisfied by most Lévy processes, including all stable processes and all known processes used in applications. This condition only excludes some pathological cases when the Lévy measure has extremely heavy tails, for example when $\Pi(\mathbb{R} \setminus (-x, x)) \sim \ln(|x|)^{-1}$ as $x \rightarrow +\infty$. We need this assumption only because we will use the Sokhotsky's formulas (in the equation (22) and in the proof of the Theorem 1(b) below), which require the integrand function to be Hölder continuous.

We are interested in the following functionals of the process X : the first passage time

$$\tau_h^+ = \inf\{t > 0 : X_t > h\}, \quad h \geq 0,$$

the overshoot $X_{\tau_h^+} - h$, and the supremum/infimum processes

$$S_t = \sup_{0 \leq s \leq t} X_s, \quad I_t = \inf_{0 \leq s \leq t} X_s.$$

We introduce the random variable $e(q)$, which is exponentially distributed with parameter $q > 0$ and is independent of the process X_t . Finally, we will use the following notations for the open/closed upper half plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}, \quad \bar{\mathbb{C}}^+ = \{z \in \mathbb{C} : \text{Im}(z) \geq 0\},$$

and similarly for the lower half plane and for the positive/negative real half line.

Theorem 1. Wiener-Hopf factorization:

Assume that $q > 0$. Define the Wiener-Hopf factors as

$$\begin{aligned} \phi_q^+(z) &= \exp \left[\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^+} (e^{izx} - 1) \mathbb{P}(X_t \in dx) t^{-1} e^{-qt} \right], \quad z \in \bar{\mathbb{C}}^+, \\ \phi_q^-(z) &= \exp \left[\int_{\mathbb{R}^+} dt \int_{\mathbb{R}^-} (e^{izx} - 1) \mathbb{P}(X_t \in dx) t^{-1} e^{-qt} \right], \quad z \in \bar{\mathbb{C}}^-. \end{aligned} \quad (2)$$

(a) For all $z \in \mathbb{R}$ we have

$$\frac{q}{q + \Psi(z)} = \phi_q^+(z)\phi_q^-(z). \quad (3)$$

(b) Wiener-Hopf factors $\phi_q^\pm(z)$ can also be expressed as

$$\phi_q^\pm(z) = \exp \left[\pm \frac{z}{2\pi i} \int_{\mathbb{R}} \ln \left(\frac{q}{q + \Psi(u)} \right) \frac{du}{u(u-z)} \right], \quad z \in \mathbb{C}^\pm. \quad (4)$$

(c) For all $z \in \bar{\mathbb{C}}^+ \{ z \in \bar{\mathbb{C}}^- \}$ we have $\phi_q^+(z) = \mathbb{E} [\exp(izS_{e(q)})]$ $\{ \phi_q^-(z) = \mathbb{E} [\exp(izI_{e(q)})] \}$.

(d) $S_{e(q)} \{ I_{e(q)} \}$ is a positive $\{ \text{negative} \}$ infinitely divisible random variable with zero linear drift.

(e) Assume that there exists a positive $\{ \text{negative} \}$ infinitely divisible random variable \tilde{S} $\{ \tilde{I} \}$ with zero linear drift, such that $q(q + \Psi(z))^{-1} = \mathbb{E} [\exp(iz\tilde{S})] \mathbb{E} [\exp(iz\tilde{I})]$ for all $z \in \mathbb{R}$. Then $\tilde{S} \stackrel{d}{=} S_{e(q)}$ and $\tilde{I} \stackrel{d}{=} I_{e(q)}$.

(f) Assume that there exist two functions $f^+(z)$ and $f^-(z)$, such that $f^\pm(0) = 1$, $f^\pm(z)$ is analytic in \mathbb{C}^\pm , continuous and has no roots in $\bar{\mathbb{C}}^\pm$ and $z^{-1} \ln(f^\pm(z)) \rightarrow 0$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^\pm$. If

$$\frac{q}{q + \Psi(z)} = f^+(z)f^-(z), \quad z \in \mathbb{R}, \quad (5)$$

then $f^\pm(z) = \phi_q^\pm(z)$ for all $z \in \bar{\mathbb{C}}^\pm$.

Theorem 2. Pecherskii-Rogozin identity: Assume that $q > 0$, $\text{Re}(w) > 0$ and $\text{Re}(z) > 0$ and $w \neq z$. Then

$$\int_{\mathbb{R}^+} e^{-wh} \mathbb{E} \left[e^{-q\tau_h^+ - z(X_{\tau_h^+} - h)} \right] dh = \frac{1}{w-z} \left(1 - \frac{\phi_q^+(iw)}{\phi_q^+(iz)} \right). \quad (6)$$

3 Proofs

We will adopt the following strategy to prove Theorems 1 and 2. First we will study the functions $\phi_q^\pm(z)$ defined by (2) and will derive the factorization identity (3) and some other technical results, such as upper/lower bounds for $\phi_q^\pm(z)$ as $z \rightarrow \infty$ and two identities for the Fourier transform of the q -potential measure. Next, armed with these results we will prove the Pecherskii-Rogozin identity (6) and at the same time establish Theorem 1(c), the proof of these results is a central contribution of this paper. Finally, we will establish the uniqueness results (Theorem 1(e,f)) and the alternative expression for the Wiener-Hopf factors (Theorem 1(b)).

Proposition 1. *Assumption 1 implies that $\Psi(z)$ is Hölder continuous of order ϵ .*

Proof. We only need to consider the “large jump” part of $\Psi(z)$, which is given by the integral over $\mathbb{R} \setminus [-1, 1]$ in (1), as the remaining part is an entire function, and therefore it is Hölder continuous of order 1. The rest of the proof follows from the following estimate

$$\begin{aligned} \left| \int_{\mathbb{R} \setminus [-1, 1]} (e^{ix(z+h)} - e^{ixz}) \Pi(dx) \right| &= |h|^\epsilon \left| \int_{\mathbb{R} \setminus [-1, 1]} e^{ixz} (e^{ixh} - 1) |xh|^{-\epsilon} |x|^\epsilon \Pi(dx) \right| \\ &\leq |h|^\epsilon \int_{\mathbb{R} \setminus [-1, 1]} \frac{|e^{ixh} - 1|}{|xh|^{-\epsilon}} |x|^\epsilon \Pi(dx) \leq C_1 |h|^\epsilon \int_{\mathbb{R} \setminus [-1, 1]} |x|^\epsilon \Pi(dx) = C_2 |h|^\epsilon, \end{aligned}$$

where in the last step we have used the fact that $|e^{iy} - 1|/|y|^\epsilon$ is a bounded function. \square

Next we will prove two simple results concerning the Fourier transform of the q -potential measure, which is defined for $q > 0$ as

$$U^{(q)}(dx) = \int_{\mathbb{R}^+} e^{-qt} \mathbb{P}(X_t \in dx) dt. \quad (7)$$

Lemma 1. *For $q > 0$, $z \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}^+$*

$$\int_{\mathbb{R}} e^{izx} U^{(q)}(dx) = \frac{1}{q + \Psi(z)}, \quad (8)$$

$$\int_{\mathbb{R}^+} e^{iz_1 h} dh \int_{\mathbb{R}^+} e^{iz_2 x} U^{(q)}(dx + h) = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{q + \Psi(u)} \frac{du}{(u - z_1)(u - z_2)}. \quad (9)$$

Proof. The identity (8) is a well-known result, its proof can be obtained by applying the Fubini’s Theorem to (7) and using the definition of the characteristic exponent $\Psi(z)$. To prove (9) we first assume that $\sigma > 0$. Then there exists the probability density function $p_t(x) = \frac{d}{dx} \mathbb{P}(X_t \leq x)$ and therefore the q -potential measure $U^{(q)}(dx)$ is absolutely continuous with respect to the Lebesgue measure and its density is given by

$$u^{(q)}(x) = \int_{\mathbb{R}^+} e^{-qt} p_t(x) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-iux}}{q + \Psi(u)} du.$$

Therefore we can rewrite the integral on the left-hand side of (9) as

$$\int_{\mathbb{R}^+} e^{iz_1 h} dh \int_{\mathbb{R}^+} e^{iz_2 x} u^{(q)}(x + h) dx = \frac{1}{2\pi} \int_{\mathbb{R}^+} e^{iz_1 h} dh \int_{\mathbb{R}^+} e^{iz_2 x} dx \int_{\mathbb{R}} \frac{e^{-iu(x+h)}}{q + \Psi(u)} du.$$

The right-hand side of (9) can now be obtained from the right-hand side of the above equation by applying the Fubini’s Theorem and computing the integrals in x and h variables. The proof of (9) in the remaining case $\sigma = 0$ follows by taking the limit of both sides of (9) as $\sigma \rightarrow 0^+$ and applying the Dominated Convergence Theorem (note that both sides of (9) are well defined for $\sigma = 0$). \square

Proof of Theorem 1(a): We will follow the proof of Lemma 6.15 in [14]. To prove the factorization identity (3) we use the Frullani integral

$$\int_{\mathbb{R}^+} (1 - e^{-at}) t^{-1} e^{-bt} dt = \ln \left(1 + \frac{a}{b} \right), \quad \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \quad (10)$$

which gives us

$$-\ln \left(1 + \frac{\Psi(z)}{q} \right) = \int_{\mathbb{R}^+} (e^{-\Psi(z)t} - 1) t^{-1} e^{-qt} dt = \int_{\mathbb{R}^+} dt \int_{\mathbb{R}} (e^{izx} - 1) \mathbb{P}(X_t \in dx) t^{-1} e^{-qt}.$$

The identity (3) now follows immediately from the above equation and the definition (2) of $\phi_q^\pm(z)$. \square

Lemma 2. $|\phi_q^\pm(z)| \leq 1$ for all $z \in \bar{\mathbb{C}}^\pm$ and $|\phi_q^\pm(z)|^{-1} = O(|z|^2)$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^\pm$.

Proof. First we will establish that $z^{-1} \ln(\phi_q^+(z)) \rightarrow 0$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^+$. In order to do this we use the Fubini's Theorem to rewrite (2) as

$$\ln(\phi_q^+(z)) = \int_{\mathbb{R}^+} (e^{izx} - 1) \nu(dx), \quad (11)$$

where

$$\nu(dx) = \int_{\mathbb{R}^+} t^{-1} e^{-qt} \mathbb{P}(X_t \in dx) dt.$$

Applying the integration by parts to (11) it is easy to prove that $z^{-1} \ln(\phi_q^+(z)) \rightarrow 0$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^+$. One could also establish this fact by noting that (11) is the characteristic exponent of a subordinator with no drift, and then applying the results of Proposition 2, (ii) in [3].

Now we are ready to prove Lemma 2. The statement $|\phi_q^+(z)| \leq 1$ for all $z \in \bar{\mathbb{C}}^+$ follows from the fact that the integrand in the right-hand side of (2) has negative real part. Next we use the factorization identity (3), the fact that $|\Psi(z)| = O(|z|^2)$ as $z \rightarrow \infty$ (see [3]) and the already established inequality $|\phi_q^-(z)| \leq 1$, $z \in \mathbb{R}$ to find that there exists some constant $A > 0$ such that $|\phi_q^+(z)|^{-1} < A(1 + z^2)$ for all $z \in \mathbb{R}$. Let us introduce the function $F(z) = \phi_q^+(z)^{-1}(z + i)^{-2}$. By construction, $F(z)$ is analytic in \mathbb{C}^+ , continuous in $\bar{\mathbb{C}}^+$ and bounded on the real line. As we have proved above, $F(z)$ also satisfies

$$z^{-1} \ln(F(z)) \rightarrow 0, \quad z \rightarrow \infty, \quad z \in \bar{\mathbb{C}}^+.$$

Thus, applying the Phragmen-Lindelöf Theorem (see [7], Corollary 4.4) we find that the function $F(z)$ must be bounded in $\bar{\mathbb{C}}^+$, which implies that $|\phi_q^+(z)|^{-1} = O(|z|^2)$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^+$. The case of $\phi_q^-(z)$ is identical. \square

The following result was proved in [1], we will reproduce the main steps of its proof here for the sake of completeness.

Lemma 3. For $q > 0$, $\operatorname{Re}(w) > 0$ and $\operatorname{Re}(z) \geq 0$

$$\mathbb{E} [e^{-zS_{e(q)}}] \times \int_{\mathbb{R}^+} e^{-wh} \mathbb{E} \left[e^{-q\tau_h^+ - z(X_{\tau_h^+} - h)} \right] dh = \frac{\mathbb{E} [e^{-zS_{e(q)}}] - \mathbb{E} [e^{-wS_{e(q)}}]}{w - z}. \quad (12)$$

Proof. The proof is based on the two facts: (i) $\{S_{e(q)} > h\} \equiv \{\tau_h^+ < e(q)\}$ and (ii) on the event $\{\tau_h^+ < e(q)\}$ the random variable $S_{e(q)} - X_{\tau_h^+}$ has the same distribution as $S_{e(q)}$ (this follows from the strong Markov property). Therefore we obtain

$$\begin{aligned} \mathbb{E} [\mathbb{I}(S_{e(q)} > h)e^{-zS_{e(q)}}] &= \mathbb{E} [\mathbb{I}(\tau_h^+ < e(q))e^{-zS_{e(q)}}] \\ &= \mathbb{E} \left[\mathbb{I}(\tau_h^+ < e(q))e^{-zX_{\tau_h^+}} \mathbb{E} \left[e^{-z(S_{e(q)} - X_{\tau_h^+})} \middle| \mathcal{F}_{\tau_h^+} \right] \right] \\ &= \mathbb{E} \left[\mathbb{I}(\tau_h^+ < e(q))e^{-zX_{\tau_h^+}} \right] \times \mathbb{E} [e^{-zS_{e(q)}}] \\ &= \mathbb{E} \left[e^{-q\tau_h^+ - zX_{\tau_h^+}} \right] \times \mathbb{E} [e^{-zS_{e(q)}}] \end{aligned}$$

Multiplying both sides of the above identity by $\exp(zh)$ and applying the Laplace transform in the h -variable we obtain (12). \square

Note that expressions (6) and (12) are not equivalent: in (12) the joint Laplace transform of the distribution of τ_h^+ and $X_{\tau_h^+}$ is given in terms of the law of the supremum, while in (6) the same quantity is expressed in terms of the law of the process X itself (via (2)). Thus we see that in order to prove Theorem 2 and Theorem 1(c) we need to establish that $\phi_q^+(z)$ defined by (2) is the characteristic function of $S_{e(q)}$, and this statement is a very nontrivial and hard result.

In order to proceed we need to introduce some additional notation. Let $\nu(dx_1, \dots, dx_n)$ be a finite measure on $(\mathbb{R}^+)^n$. We will denote its k -dimensional Laplace transform in variables $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ as $\nu_{i_1, i_2, \dots, i_k}$. The same notation will be used if ν also depends on some parameters. For example, for the measure

$$p(h; dt, dx) = \mathbb{P}(\tau_h^+ \in dt; X_{\tau_h^+} - h \in dx), \quad h > 0, t > 0, x \geq 0$$

we will define

$$p_2(h; q, dx) = \int_{\mathbb{R}^+} e^{-qt} p(h; dt, dx), \quad h > 0, q > 0, x \geq 0 \quad (13)$$

and similarly for all other indices.

Proof of Theorem 2 and Theorem 1(c): The starting point in the proof of the Pecherskii-Rogozin identity (6) is the following two-dimensional integral equation

$$\int_0^t \int_{\mathbb{R}^+} \mathbb{P}(\tau_h^+ \in ds; X_{\tau_h^+} - h \in dy) \mathbb{P}(X_{t-s} \in dx - y) = \mathbb{P}(X_t \in dx + h), \quad x \geq 0, h > 0, t > 0. \quad (14)$$

This equation is easy to derive by applying the strong Markov property at the first passage time τ_h^+ : assuming that $x \geq 0$, $h > 0$ and $t > 0$ we find

$$\begin{aligned}
\mathbb{P}(X_t \in dx + h) &= \mathbb{P}(X_t \in dx + h; \tau_h^+ < t) = \mathbb{E} \left[\mathbb{I}(\tau_h^+ \leq t) \mathbb{P}(X_t \in dx + h | \mathcal{F}_{\tau_h^+}) \right] \\
&= \int_0^t \int_{\mathbb{R}^+} \mathbb{P}(\tau_h^+ \in ds; X_{\tau_h^+} - h \in dy) \mathbb{P}(X_t \in dx + h | \tau_h^+ = s, X_{\tau_h^+} = h + y) \\
&= \int_0^t \int_{\mathbb{R}^+} \mathbb{P}(\tau_h^+ \in ds; X_{\tau_h^+} - h \in dy) \mathbb{P}(X_{t-s} \in dx - y).
\end{aligned}$$

Note that for some Lévy processes it is true that $\mathbb{P}(X_{\tau_h^+} = h) > 0$ (see [14] for the necessary and sufficient conditions). In this case the measure $\mathbb{P}(\tau_h^+ \in ds; X_{\tau_h^+} - h \in dy)$ will have an atom at $y = 0$; this explains why we have to integrate over the *closed* positive half-line in the left-hand side of (14).

The main idea of the proof is to solve the equation (14) by reducing it to a non-homogeneous Riemann boundary value problem (BVP). We note that the integral in the s -variable in (14) is a convolution, thus we can apply the Laplace transform to both sides of (14) and obtain

$$\int_{\mathbb{R}^+} p_2(h; q, dy) U^{(q)}(dx - y) = U^{(q)}(dx + h), \quad x \geq 0, \quad h > 0, \quad (15)$$

where the measure $p_2(h; q, dy)$ is defined by (13). Equation (15) is a Wiener-Hopf integral equation, i.e. a convolution equation over a restricted domain. In order to solve it we use the following classic technique (see [17]): we introduce the following three measures with the support on the whole real line

$$\begin{aligned}
g^+(h; dx) &= U^{(q)}(dx + h) \mathbb{I}(x \geq 0), \\
p_2^+(h; q, dx) &= p_2(h; q, dx) \mathbb{I}(x \geq 0), \\
p_2^-(h; q, dx) &= \int_{\mathbb{R}^+} p_2(h; q, dy) U^{(q)}(dx - y) \mathbb{I}(x < 0),
\end{aligned}$$

and rewrite (15) as a non-restricted convolution equation

$$\int_{\mathbb{R}} p_2^+(h; q, dy) U^{(q)}(dx - y) = g^+(h; dx) + p_2^-(h; q, dx), \quad x \in \mathbb{R}, \quad h > 0. \quad (16)$$

Note that the integral equation (16) has a very simple probabilistic interpretation. If we multiply both sides of (16) by q , use the identity $p_2(h; q, dx) = \mathbb{P}(\tau_h^+ < e(q); X_{\tau_h^+} - h \in dx)$ (which follows from (13)) and the fact that $\{\tau_h^+ < e(q)\} \equiv \{S_{e(q)} > h\}$ we can see that (16) is in fact equivalent to the following identity

$$\begin{aligned}
\mathbb{P}(S_{e(q)} > h; X_{e(q)} \in h + dx) &= \mathbb{P}(X_{e(q)} \in h + dx) \mathbb{I}(x \geq 0) \\
&+ \mathbb{P}(S_{e(q)} > h; X_{e(q)} \in h + dx) \mathbb{I}(x < 0),
\end{aligned}$$

however this fact is not used anywhere in our proof.

Next, in order to simplify (16) we apply the Fourier transform in the x -variable and use (8) to conclude that

$$p_{2,3}^+(h; q, z) \frac{1}{q + \Psi(z)} = g_2^+(h; z) + p_{2,3}^-(h; q, z), \quad z \in \mathbb{R}, \quad h > 0. \quad (17)$$

Equation (17) is a Riemann BVP, however in order to solve it we have to apply one additional Laplace transform in the h -variable. It is easy to see why we need this extra Laplace transform – it will simplify the free term in the above Riemann BVP (17), as the measure $g^+(h; dx)$ defined above depends only on $h + dx$. Therefore, applying the Laplace transform in the h -variable to both sides of (17) we finally arrive at the following non-homogeneous Riemann BVP

$$f^+(z) \frac{1}{q + \Psi(z)} = g^+(z) + f^-(z), \quad z \in \mathbb{R}, \quad (18)$$

where

$$f^+(z) = p_{1,2,3}^+(w; q, z) = \int_{\mathbb{R}^+} e^{-wh} p_{2,3}^+(h; q, z) dh = \int_{\mathbb{R}^+} e^{-wh} \mathbb{E} \left[e^{-q\tau_h^+ + iz(X_{\tau_h^+} - h)} \right] dh, \quad (19)$$

and similarly we have denoted $g^+(z) = g_{1,2}^+(w; z)$ and $f^-(z) = p_{1,2,3}^-(w; q, z)$.

Our goal now is to solve the Riemann BVP (18): the plan is to transform this equation into the form where the left { right } hand side is analytic in \mathbb{C}^+ { \mathbb{C}^- } and then to prove that both sides are identically equal to zero. Using the factorization identity (3) we rewrite (18) as the Riemann BVP of the following type

$$\tilde{f}^+(z) - \tilde{f}^-(z) = \frac{g^+(z)}{\phi_q^-(z)}, \quad z \in \mathbb{R}, \quad (20)$$

where

$$\tilde{f}^+(z) = q^{-1} f^+(z) \phi_q^+(z) \quad \text{for } z \in \bar{\mathbb{C}}^+ \quad \text{and} \quad \tilde{f}^-(z) = \frac{f^-(z)}{\phi_q^-(z)} \quad \text{for } z \in \bar{\mathbb{C}}^- \quad (21)$$

Equation (9) in Lemma 1 implies that

$$g^+(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \left[\frac{1}{(q + \Psi(u))(w + iu)} \right] \frac{du}{u - z}, \quad z \in \mathbb{C}^+.$$

Using Sokhotsky's formulas (see [9], [17]) to rewrite $g^+(z)$ as

$$g^+(z) = \frac{1}{(q + \Psi(z))(w + iz)} + g^-(z), \quad z \in \mathbb{R}, \quad (22)$$

where the function

$$g^-(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \left[\frac{1}{(q + \Psi(u))(w + iu)} \right] \frac{du}{u - z}, \quad z \in \mathbb{C}^-$$

is analytic in \mathbb{C}^- and is vanishing as $z \rightarrow \infty$. Note that the application of Sokhotsky's formulas is legitimate, since we have proved in Proposition 1 that $\Psi(u)$ is Hölder continuous. Next we use (22) and the factorization identity (3) in the form $(q + \Psi(z))^{-1} \phi_q^-(z)^{-1} = q^{-1} \phi_q^+(z)$ and rewrite (20) as

$$\tilde{f}^+(z) - \tilde{f}^-(z) = q^{-1} \frac{\phi_q^+(z)}{w + iz} + \frac{g^-(z)}{\phi_q^-(z)}, \quad z \in \mathbb{R}.$$

After rearranging the terms in the above equation we obtain

$$\tilde{f}^+(z) - q^{-1} \frac{\phi_q^+(z) - \phi_q^+(iw)}{w + iz} = \tilde{f}^-(z) + \frac{g^-(z)}{\phi_q^-(z)} + q^{-1} \frac{\phi_q^+(iw)}{w + iz}, \quad z \in \mathbb{R}, \quad w > 0. \quad (23)$$

Now, we can check that the left (right) hand side of (23) is analytic in \mathbb{C}^+ (\mathbb{C}^-). This follows from (21), the fact that $f^\pm(z)$ and $\phi_q^\pm(z)$ are analytic in \mathbb{C}^\pm and that $\phi_q^-(z) \neq 0$ when $z \in \mathbb{C}^-$. Therefore all that is left to do is to prove that the functions on both sides of (23) are identically equal to zero.

The left-hand side of (23) is bounded in $\bar{\mathbb{C}}^+$ and the right-hand side of (23) is $O(|z|^2)$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^-$. This can be established as follows. From (19) we find that $f^+(z)$ is bounded for $z \in \bar{\mathbb{C}}^+$, thus using Lemma 2 and the definition of $\tilde{f}^+(z)$ in (21) we establish that $\tilde{f}^+(z)$ is bounded for $z \in \bar{\mathbb{C}}^+$. Since $\phi_q^+(z)$ is bounded in $\bar{\mathbb{C}}^+$ (see Lemma 2) this implies that the left-hand side of (23) is also bounded for $z \in \bar{\mathbb{C}}^+$. In the same way we find that the right-hand side is $O(|z|^2)$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^-$.

Let us define the function $F(z)$, which is equal to the left-hand side of (23) when $z \in \bar{\mathbb{C}}^+$ and to the right-hand side of (23) when $z \in \bar{\mathbb{C}}^-$. This function is well defined for $z \in \mathbb{R}$ due to the boundary condition (23). This function is analytic in \mathbb{C}^+ and \mathbb{C}^- , continuous on \mathbb{R} , thus by Morera's theorem (see [7]) it must be analytic in the entire complex plane. Since $F(z)$ is $O(|z|^2)$ and analytic in the entire complex plane, we use Cauchy's estimates (see [7]) and conclude that $F(z)$ is a polynomial of degree not greater than two, and given the fact that $F(z)$ is bounded in \mathbb{C}^+ , this implies that $F(z) \equiv B$ for some constant $B \in \mathbb{C}$ (which may depend on q and w). Considering the case when $z \in \mathbb{C}^+$ we obtain

$$\tilde{f}^+(z) - q^{-1} \frac{\phi_q^+(z) - \phi_q^+(iw)}{w + iz} = B(q, w). \quad (24)$$

Recalling definitions (21), (19) and using (12) we find that for $\text{Re}(z) \geq 0$, $\text{Re}(w) > 0$, and $q > 0$

$$\frac{\phi_q^+(z)}{\mathbb{E}[e^{izS_{e(q)}}]} \frac{\mathbb{E}[e^{izS_{e(q)}}] - \mathbb{E}[e^{-wS_{e(q)}}]}{w + iz} - \frac{\phi_q^+(z) - \phi_q^+(iw)}{w + iz} = qB(q, w).$$

The above equation implies that $B(q, w) \equiv 0$ and $\phi_q^+(z) = \mathbb{E}[e^{izS_{e(q)}}]$, which ends the proof of the Pecherskii-Rogozin identity and of the Theorem 1(c). \square

The main idea for the next proof was borrowed from the proof of the Lemma 45.6 in [19].

Proof of Theorem 1(f): Define $F(z)$ as

$$F(z) = \begin{cases} \frac{\phi_q^-(z)}{f^-(z)}, & \text{if } z \in \bar{\mathbb{C}}^- \\ \frac{f^+(z)}{\phi_q^+(z)}, & \text{if } z \in \bar{\mathbb{C}}^+ \end{cases}$$

The function $F(z)$ is well defined for $z \in \mathbb{R}$ due to (3) and (5). From the definition of $\phi_q^\pm(z)$ we find that it is analytic in \mathbb{C}^\pm , thus using the assumptions on $f^\pm(z)$ we conclude that $F(z)$ is analytic in both \mathbb{C}^+ and \mathbb{C}^- and that it is continuous everywhere in \mathbb{C} . Therefore by Morera's theorem (see [7]) it must be analytic in the entire complex plane. Moreover, by construction function $F(z)$ has no zeros in \mathbb{C} , thus its logarithm is also an entire function.

As was established in the proof of the Lemma 2, the functions $\phi_q^\pm(z)$ satisfy $z^{-1} \ln(\phi_q^\pm(z)) \rightarrow 0$ as $z \rightarrow \infty$, $z \in \bar{\mathbb{C}}^\pm$. Since the functions $f^\pm(z)$ are assumed to satisfy the above condition, we find that $z^{-1} \ln(F(z)) \rightarrow 0$ as $z \rightarrow \infty$ in the entire complex plane, and using the Cauchy's estimates (see [7]) we conclude that $\ln(F(z))$ is constant. The value of this constant is easily seen to be zero, since $f^\pm(0) = \phi_q^\pm(0) = 1$. \square

Proof of Theorem 1(d): The fact that $S_{e(q)}$ is infinitely divisible with zero linear drift follows from the already established result $\mathbb{E}[\exp(izS_{e(q)})] = \phi_q^+(z)$ and (11). \square

Proof of Theorem 1(b): Define

$$f^\pm(z) = \pm \frac{z}{2\pi i} \int_{\mathbb{R}} \ln \left(\frac{q}{q + \Psi(u)} \right) \frac{du}{u(u-z)}, \quad z \in \mathbb{C}^\pm. \quad (25)$$

First let us prove that the integral in the right-hand side exists for all $z \in \mathbb{C} \setminus \mathbb{R}$. Using the fact that $\Psi(u)$ is Hölder continuous of order ϵ (see Proposition 1) and $\Psi(0) = 0$ we find that the integrand has an integrable singularity at zero of the form $|u|^{\epsilon-1}$. Next, Proposition 2, (i) in [3] tells us that $\Psi(z) = O(z^2)$ as $z \rightarrow \infty$, $z \in \mathbb{R}$, thus the integrand is $O(\ln(|u|)/u^2)$ as $u \rightarrow \infty$, and we can conclude that the integral converges absolutely.

Applying the Sokhotsky's formulas we find that the functions $\exp(f^\pm(z))$ satisfy the factorization identity (3) for $z \in \mathbb{R}$, $z \neq 0$. To check that the factorization identity is also true at $z = 0$ we need to show that $f^\pm(z) \rightarrow 0$ as $z \rightarrow 0$, and this requires some additional work. We assume that $z = r \exp(i\alpha)$ and perform a change of variable of integration $u \mapsto rw$ in (25)

$$f^+(z) = \frac{e^{i\alpha}}{2\pi i} \int_{\mathbb{R}} \ln \left(\frac{q}{q + \Psi(wr)} \right) \frac{dw}{w(w - e^{i\alpha})} = \frac{r^\epsilon e^{i\alpha}}{2\pi i} \int_{\mathbb{R}} \left[\frac{\ln \left(\frac{q}{q + \Psi(wr)} \right)}{r^\epsilon w^\epsilon} \right] \frac{dw}{w^{1-\epsilon}(w - e^{i\alpha})}.$$

The function inside the square brackets is bounded in w uniformly in r , thus the last integral represents a bounded function of r , therefore $f^+(z) \rightarrow 0$ as $r \rightarrow 0^+$. Next, using (25) and properties of Cauchy type integrals (see [9]) we check that $z^{-1} f^\pm(z) \rightarrow 0$ as $z \rightarrow \infty$, thus we can apply the uniqueness result stated in Theorem 1(f) to show that $\exp(f^\pm(z)) = \phi_q^\pm(z)$, which ends the proof. \square

Proof of Theorem 1(e): Follows from the Theorem 1(d,f). □

Remark 2. Expression (4) was first established in [15] (see Lemma 4.2) under the integrability condition which is similar to our Assumption 1. This result is also related to formulas (3.10, 3.12) in [6], which were derived for the class of processes for which the Lévy measure has exponentially decaying tails (and some additional assumptions).

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