Explicit formulas for Laplace transforms of stochastic integrals

T. R. Hurd [1] * and A. Kuznetsov [2] *
[1] Dept. of Mathematics and Statistics
McMaster University
Hamilton ON L8S 4K1
Canada

[2] Dept. of Mathematical Sciences
University of New Brunswick
Saint John NB E2L 4L5
Canada

February 9, 2006

Abstract

In this article we investigate some applications of a general measure change identity for expectations of the form

$$E\left[e^{-\int_0^T \phi(X_s)ds} g(X_T)\right]$$

for diffusion processes X_t and certain functions ϕ. In the case of Cox-Ingersoll-Ross (CIR) and Jacobi diffusions, two families of processes often applied in mathematical finance, this identity leads to explicit formulas for the Laplace transform of an important multidimensional family of random variables constructed from X_t and its integrals. Our results extend the range of applicability of these diffusions in finance.

Key words: Markov diffusions, Laplace transform, Girsanov theorem, hypergeometric functions, CIR process, Jacobi process, mathematical finance.

*Research supported by the Natural Sciences and Engineering Research Council of Canada and MITACS Mathematics of Information Technology and Complex Systems, Canada

1 Introduction

The modelling of financial time series such as stock prices, interest rates and foreign exchange rates has been important in the development of the theory of stochastic processes. The solution to practical problems such as option pricing, model calibration and portfolio selection relies to a great extent on the solvability properties of the underlying one-dimensional stochastic processes. Since financial mathematics is not a fundamental science like physics, the criteria for useful models are mostly pragmatic: models are of interest if they fit certain "stylized facts", and retain an adequate degree of computational tractability.

For example, the recent popularity of Lévy models for stock returns is due in large part to the possibility of using the fast Fourier transform to compute option prices. Similarly, the class of affine processes derives its popularity for interest rate theory, stock modelling and other financial applications in large part because explicit or close to explicit formulas are available for key functionals of the underlying processes. The so-called solvable Markov models studied by [2], constitute a further distinct family of models that includes geometric Brownian motion, the Ornstein-Uhlenbeck processes, the Cox-Ingersoll-Ross (CIR) model, and the Jacobi process: these processes have found hundreds of applications in finance and other areas of stochastic phenomena.

It is this last family of solvable diffusion models that provide the natural setting for the present paper. For a one-dimensional stationary diffusion process X_t on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\leq T}, \mathrm{P})$, we present a version of a proposition of [5] involving a measure change from P to Q that yields an identity of the form

$$E^{\mathrm{P}}\left[e^{-\int_0^T \phi(X_s)ds} g(X_T)\right] = f(X_0)E^{\mathrm{Q}}\left[\frac{g(X_T)}{f(X_T)}\right] \tag{1}$$

for certain functions ϕ, f, g. We are able to provide a number of interesting cases where the new expectation can be evaluated in terms of special functions.

The identity 1 is closely related to results contained in a recent article [3], where the authors present the complete classification scheme of diffusion processes (and also birth and death processes) X_t and functions $\phi(x)$, for which the following expectation

$$E\left[e^{-\int_0^T \phi(X_s)ds} g(X_T)\right] \tag{2}$$

can be computed in terms of hypergeometric functions. By using a PDE approach, non-singular transformations and spectral resolution, they were able not only to give a complete classification of these processes, but also to provide explicit series formulas in many important cases.

Our results, while of intrinsic interest in the theory of stochastic processes and partial differential equations, are motivated by financial mathematics, and they suggest a number of new applications in finance that we hope to explore in future papers. As just one example, we note here a connection between our new results and the theory of utility based pricing and hedging in incomplete markets. Utility theory, the proper economic foundation for financial decision making and contract valuation in general markets, has been under rapid development in the past decade. However, the list of explicitly solvable problems in utility theory is not long, despite the efforts of a large number of researchers, a fact which has hampered both the development of the theory and the adoption of utility methods by finance practitioners. A recent paper [9] shows how the utility based prices of important volatility derivatives in certain stochastic volatility models including the classic Heston model boil down to natural Feynman-Kac type expectations. Serendipitously, these expectations turn out to be of the precise form we address here and thus can be expressed explicitly in terms of our new formulas.

Our analysis starts with Proposition 2.1 of Section 2 that provides the desired measure change formula for general diffusion processes. We outline two justifications for the formula, one probabilistic involving the Girsanov theorem, the other by partial differential equations. In Section 3, we study the case when the underlying process X_t is a Cox-Ingersoll-Ross (CIR) process (a positive mean reverting diffusion). When we choose $f(x) = e^{-v_1 x} x^{v_2}, g(x) = e^{-w_1 x} x^{w_2}$, Theorem 3.1 provides the precise conditions under which

$$E^{\mathrm{P}}\left[e^{-\int_0^T \left(d_1 X_s + \frac{d_2}{X_s}\right) ds} e^{-w_1 X_T} X_T^{w_2} \right]$$

can be computed in closed form in terms of the confluent hypergeometric function. To facilitate computations of this formula over a wide range of parameter values, in Appendix 6 we provide a simple asymptotic expansion that complements the standard power series expansion of the hypergeometric function.

In Section 4, we investigate the lesser known Jacobi process (a mean reverting diffusion taking its values on $[0,1]$, see [7]). In this case, we take $f(x) = x^{v_1}(1-x)^{v_2}, g(x) = x^{w_1}(1-x)^{w_2}$, and are then able to show conditions under which the identity leads to a convergent and tractable representation of

$$E^{\mathrm{P}}\left[e^{-d_1 \int_0^T \left(\frac{1-X_s}{X_s}\right) ds - d_2 \int_0^T \left(\frac{X_s}{1-X_s}\right) ds} e^{-w_1 X_T} X_T^{w_2} \right]$$

in terms of a rapidly convergent series involving hypergeometric functions and Jacobi polynomials (see [11]). Appendix 7 examines the asymptotic properties of this expansion as $t \to 0$ thereby facilitating computations of our identity.

2 The general method

Let $X_t,\ t \in [0,T],\ T \geq 0$ be a one-dimensional stationary diffusion process on a closed (possibly infinite) interval $D = D_0 \cup \partial D$, defined by its initial condition $X_0 = x$ and its Markov generator under the measure P

$$\mathcal{L}^{\mathrm{P}} = \mu(x)\partial_x + \frac{1}{2}\sigma^2(x)\partial_x^2 \tag{3}$$

or equivalently by the stochastic differential equation (W^{P} is a P-Brownian motion):

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t^{\mathrm{P}}, X_0 = x\ .$$

We also assume that $\sigma(x) > 0$ in the open interval D_0 and that $\mu(x)$ and $\sigma(x)$ are bounded on the compact subsets of D.

The following proposition is a specialization of the main result of [5], and is based on measure changes of the form discussed in [12]:

Proposition 2.1. *Assume $f(x) \in C_2(D_0)$ and $f(x) \neq 0$ in D_0. Define a new measure $Q \ll P$ by the Radon-Nikodym derivative*

$$Z_T := \frac{d\mathrm{Q}}{d\mathrm{P}} = \frac{f(X_T)}{f(X_0)} \exp\left(-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds \right). \tag{4}$$

If the process X_t does not explode and the boundary ∂D is unattainable both under measures P *and* Q*, then* $\mathrm{P} \sim \mathrm{Q}$ *and*

$$E_x^{\mathrm{P}}\left[e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} g(X_T) \right] = f(x) E_x^{\mathrm{Q}}\left[\frac{g(X_T)}{f(X_T)} \right], \tag{5}$$

for any bounded measurable function $g(x)$. Under the measure Q *the Markov generator of X_t has a modified drift:*

$$\mathcal{L}^{\mathrm{Q}} = \left(\mu(x) + \sigma^2(x)\frac{f'(x)}{f(x)} \right)\partial_x + \frac{1}{2}\sigma^2(x)\partial_x^2. \tag{6}$$

This proposition is proved in greater generality in [5], but nonetheless we sketch the main ideas pertinent to our version. First, by the Itô formula, one finds that $Z = \mathcal{E}(L)$ is the stochastic exponential of the local martingale $L_t = \int_0^t \lambda_s dW_s$ where $\lambda_t = \sigma(X_t)f'(X_t)/f(X_t)$. Then, as explained in detail in the proof of Theorem 1 of [6], one can show that the boundary nonattainment and nonexplosive properties of X are sufficient to imply that $EZ_T = 1$, and hence that Z_t is a true martingale. The Girsanov measure change theorem then applies (see [10]) and justifies (5) and (6).

Equation (5) can also be derived by PDE methods. The main ingredients are the Feynman-Kac formula and the following (easily verified) operator identity

$$\mathcal{L}^{\mathrm{P}}\phi = f(\mathcal{L}^{\mathrm{Q}}\psi) + (\mathcal{L}^{\mathrm{P}} f)\psi, \tag{7}$$

where $\phi(t,x) = f(x)\psi(t,x)$. However for this argument to be made rigorous, the use of the Feynman-Kac formula requires careful attention to the boundary conditions of the PDE.

Here is our general approach for computing expectations of the form 2: first we find a function $f(x)$ that generates a measure change that satisfies the conditions of Proposition 2.1 and such that the transitional probability density $p_T^{\mathrm{Q}}(x,y)$ for the process X_t is known in closed form. Then we choose a function $g(x)$, such that $E_x^{\mathrm{Q}}\left[\frac{g(X_T)}{f(X_T)}\right] = \int p_T^{\mathrm{Q}}(x,y)\frac{g(y)}{f(y)}dy$ can be computed in closed form. Finally, we use equation (5) to obtain the expectation 2 for $\phi = \frac{\mathcal{L}^{\mathrm{P}} f}{f}$.

In the next two sections, we investigate the CIR diffusion and the Jacobi diffusion and show that Proposition 2.1 leads to new and useful formulas. The paper [3] gives a general classification of processes for which similar results may be expected to hold. Examples discussed there include geometric Brownian motion, Ornstein-Uhlenbeck process, CIR process, Jacobi process, and a number of discrete Markov processes, for which they prove a number of results related to our formulas.

3 The CIR process

Let $X_t \in [0,\infty)$ be the CIR process with generator

$$\mathcal{L}^{\mathrm{P}} = (a - bx)\partial_x + \frac{1}{2}c^2 x\partial_x^2. \tag{8}$$

Assume $\alpha = \frac{2a}{c^2} - 1 \geq 0$, $\beta = \frac{2b}{c^2} \geq 0$: it is known that the process X_t is nonexplosive, and has an unattainable boundary at 0 if and only if $\alpha \geq 0$.

In this section we will give a closed form expression for the function

$$G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2) = E_{0,x}^{\mathrm{P}}\left[e^{-\int_0^T\left(d_1X_s + \frac{d_2}{X_s}\right)ds} e^{-w_1X_T}X_T^{w_2}\right]. \tag{9}$$

Theorem 3.1. *Suppose $d_1 \geq -\frac{b^2}{2c^2}$ and $d_2 \geq -\frac{\alpha^2c^2}{8}$. Then the measure Q obtained from 4 by choosing $f(x) = e^{-v_1x}x^{v_2}$ with $v_1 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + \frac{8d_1}{c^2}}\right)$, $v_2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \frac{8d_2}{c^2}}\right)$ is well defined and equivalent to P. Define $\gamma_T = (\beta + 2v_1)\left(1 - e^{-(\beta/2+v_1)c^2T}\right)^{-1}$.*

1. *If $\alpha + v_2 + w_2 + 1 \leq 0$ or $\gamma_T + w_1 - v_1 \leq 0$ then $G^{\mathrm{CIR}}(T,x;d_1,d_2,w_1,w_2) = \infty$.*

2. *If* $\alpha + w_2 + v_2 + 1 > 0$ *and* $\gamma_T + w_1 - v_1 > 0$ *then*

$$G^{\text{CIR}}(T, x; d_1, d_2, w_1, w_2) = e^{-\left(av_1+bv_2+c^2v_1v_2\right)T} x^{v_2} \left(\gamma_T + w_1 - v_1\right)^{-\alpha-v_2-w_2-1} \gamma_T^{\alpha+2v_2+1}$$
$$\times \exp\left(-x\left(v_1 + \frac{\gamma_T(w_1 - v_1)}{\gamma_T + w_1 - v_1} e^{-(\beta/2+v_1)c^2T}\right)\right)$$
$$\times \frac{\Gamma(\alpha + v_2 + w_2 + 1)}{\Gamma(\alpha + 2v_2 + 1)} {}_1F_1\left(v_2 - w_2, \alpha + 2v_2 + 1; -\frac{\gamma_T^2 x e^{-(\beta/2+v_1)c^2T}}{\gamma_T + w_1 - v_1}\right). \qquad (10)$$

Remarks:

1. Formula 10 is convenient when the argument of the confluent hypergeometric function ${}_1F_1$ is small, that is when either x is small or T is large. One then uses the Taylor series for ${}_1F_1$

$$ {}_1F_1(a, b; z) = 1 + \frac{a}{b}z + \frac{1}{2!}\frac{a(a+1)}{b(b+1)}z^2 + \dots$$

When the argument is large (i.e. when x is large or T is small or both), one should instead use an asymptotic formula such as the one given in Appendix 6.

2. If $w_2 - v_2 = n$, where n is a nonnegative integer, then formula 10 can be simplified, since

$$\frac{\Gamma(\alpha + v_2 + w_2 + 1)}{\Gamma(\alpha + 2v_2 + 1)} {}_1F_1\left(v_2 - w_2, \alpha + 2v_2 + 1; -z\right) = n! L_n^{(\alpha+2v_2)}(-z),$$

where $L_n^{(a)}(z)$ is a *Laguerre polynomial* (see [11]), defined as

$$L_n^{(a)}(z) = \sum_{k=0}^{n} (-1)^k \binom{n+a}{n-k} \frac{z^k}{k!} = \frac{(a+1)_n}{n!} {}_1F_1\left(-n, a+1; z\right).$$

In particular, when $v_2 = w_2 = 0$, we obtain the well known exponential affine formula.

Proof: Let $f(x) = e^{-v_1 x} x^{v_2}$ and $g = e^{-w_1 x} x^{w_2}$. Since $d_1 = bv_1 + \frac{1}{2}c^2 v_1^2$ and $d_2 = av_2 + \frac{1}{2}c^2 v_2(v_2 - 1)$, we can compute

$$\begin{aligned}\frac{\mathcal{L}^{\text{P}} f}{f} &= -(av_1 + bv_2 + c^2 v_1 v_2) + x\left(bv_1 + \frac{1}{2}c^2 v_1^2\right) + \frac{1}{x}\left(av_2 + \frac{1}{2}c^2 v_2(v_2 - 1)\right) \\ &= -(av_1 + bv_2 + c^2 v_1 v_2) + d_1 x + \frac{d_2}{x}.\end{aligned}$$

Thus $G^{\text{CIR}}(T, x; d_1, d_2, w_1, w_2)$ equals

$$e^{-(av_1+bv_2+c^2v_1v_2)T} E_{0,x}^{\text{P}}\left[e^{-\int_0^T \frac{\mathcal{L}^{\text{P}} f(X_s)}{f(X_s)} ds} e^{-w_1 X_T} X_T^{w_2}\right]$$

By following the argument of Section 2, one computes from 6 that X_t under the measure Q has the generator

$$\mathcal{L}^{\mathrm{Q}} = (\tilde{a} - \tilde{b}x)\partial_x + \frac{1}{2}c^2 x \partial_x^2$$

where $\tilde{a} = a + c^2 v_2$ and $\tilde{b} = b + c^2 v_1$. By the assumptions, $\tilde{\alpha} = \frac{2\tilde{a}}{c^2} - 1 = \alpha + 2v_2 \geq 0$. Therefore X is also nonexplosive and has the boundary nonattainment property under Q so Proposition 2.1 applies and leads to the following identity:

$$\begin{aligned} E_{0,x}^{\mathrm{P}}\left[e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} e^{-w_1 X_T} X_T^{w_2}\right] &= x^{v_2} e^{-v_1 x} E_{0,x}^{\mathrm{Q}}\left[X_T^{w_2 - v_2} e^{(v_1 - w_1) X_T}\right] \\ &= x^{v_2} e^{-v_1 x} \int_0^\infty y^{w_2 - v_2} e^{(v_1 - w_1) y} p_T^{\mathrm{Q}}(x, y) dy. \end{aligned} \tag{11}$$

The transition probability density function $p_t^{\mathrm{Q}}(x, y)$ of the CIR process X_t is given by

$$p_t^{\mathrm{Q}}(x, y) = \gamma_t \left(\frac{y e^{\tilde{b}t}}{x}\right)^{\frac{\tilde{\alpha}}{2}} \exp\left[-\gamma_t (x e^{-\tilde{b}t} + y)\right] I_{\tilde{\alpha}}\left(2\gamma_t \sqrt{x y e^{-\tilde{b}t}}\right),$$

where $\gamma_t \equiv -2\tilde{b}/(c^2(e^{-\tilde{b}t} - 1))$ and I_ν is the modified Bessel function of the first kind. Moreover, the integral on the right side of (11) is finite if $\alpha + w_2 + v_2 + 1 > 0$ and $\gamma_T + w_1 - v_1 > 0$, and given in closed form by using the formula (see [8]):

$$\int_0^\infty y^{\mu - \frac{1}{2}} e^{-\alpha y} I_{2\nu}\left(2\beta\sqrt{y}\right) dy = e^{\frac{\beta^2}{\alpha}} \frac{\beta^{2\nu}}{\alpha^{\mu + \nu + \frac{1}{2}}} \frac{\Gamma(\mu + \nu + \frac{1}{2})}{\Gamma(2\nu + 1)} {}_1F_1\left(\nu - \mu + \frac{1}{2}, 2\nu + 1; -\frac{\beta^2}{\alpha}\right). \tag{12}$$

This proves (10). On the other hand, if either $\alpha + w_2 + v_2 + 1 > 0$ or $\gamma_T + w_1 - v_1 > 0$, then the right side of (11) is infinite.

□

4 The Jacobi process

Let $X_t \in [0, 1]$ be the Jacobi diffusion process with generator

$$\mathcal{L}^{\mathrm{P}} = (a - bx)\partial_x + \frac{1}{2}c^2 x(1 - x)\partial_x^2. \tag{13}$$

Define $\alpha = \frac{2a}{c^2} - 1$ and $\beta = \frac{2(b-a)}{c^2} - 1$. It is known that the process X_t has unattainable boundaries if and only if $\alpha \geq 0,\ \beta \geq 0$.

In this section we provide methods to compute the function

$$G^{\text{Jacobi}}(T,x;d_1,d_2,w_1,w_2)=E^{\mathrm{P}}_{0,x}\left[e^{-\int\limits_0^T d_1\left(\frac{1-X_s}{X_s}\right)+d_2\left(\frac{X_s}{1-X_s}\right)ds}X_T^{w_1}(1-X_T)^{w_2}\right]. \tag{14}$$

in terms of the hypergeometric function and the Jacobi polynomials $P_n(x)$.

Theorem 4.1. *Let the parameters of the Jacobi process satisfy $\alpha\geq 0,\beta\geq 0$, and assume $d_1\geq -\alpha^2c^2/8, d_2\geq -\beta^2c^2/8$. Then the measure* Q *obtained from 4 by choosing $f(x)=x^{v_1}(1-x)^{v_2}$ where $v_1=\frac{1}{2}\left(-\alpha+\sqrt{\alpha^2+\frac{8d_1}{c^2}}\right)$ and $v_2=\frac{1}{2}\left(-\beta+\sqrt{\beta^2+\frac{8d_2}{c^2}}\right)$ is well defined and equivalent to* P.

1. *If $\alpha+w_1+v_1\leq -1$ or $\beta+w_2+v_2\leq -1$ then $G^{\text{Jacobi}}(T,x;d_1,d_2,w_1,w_2)=\infty$.*
2. *If $\alpha+w_1+v_1>-1$ and $\beta+w_2+v_2>-1$ then*

$$\begin{aligned}G^{\text{Jacobi}}(T,x;d_1,d_2,w_1,w_2)&=e^{-\left((b-a)v_1+av_2+c^2v_1v_2\right)T}x^{v_1}(1-x)^{v_2}\\&\times\mathrm{B}(\alpha+w_1+v_1+1,\beta+w_2+v_2+1)\frac{\Gamma(\alpha+\beta+2v_1+2v_2+1)}{\Gamma(\alpha+2v_1+1)\Gamma(\beta+2v_2+1)}\\&\times\sum_{n=0}^{\infty}e^{-n(n+\alpha+\beta+2v_1+2v_2+1)\frac{c^2T}{2}}\ \frac{(\alpha+\beta+2v_1+2v_2+1)_n}{(\alpha+2v_1+1)_n}\\&\times(2n+\alpha+\beta+2v_1+2v_2+1)q_nP_n^{(\alpha+2v_1,\beta+2v_2)}(2x-1).\end{aligned} \tag{15}$$

The coefficients q_n are defined as

$$q_n\ =\ {}_3F_2\left(\begin{matrix}-n, & \alpha+\beta+2v_1+2v_2+n+1, & \beta+v_2+w_2+1\\ & \alpha+\beta+w_1+v_1+w_2+v_2+2, & \beta+2v_2+1\end{matrix};1\right) \tag{16}$$

and can be computed via a three term recurrence relation:

$$A_nq_{n+1}=(\beta+v_2+w_2+1+A_n+C_n)q_n-C_nq_{n-1},\quad q_{-1}=0,\quad q_0=1, \tag{17}$$

where A_n and C_n are given by the following formulas

$$A_n=-\frac{(n+\alpha+\beta+2v_1+2v_2+1)(n+\beta+2v_2+1)(n+\alpha+\beta+w_1+v_1+w_2+v_2+2)}{(2n+\alpha+\beta+2v_1+2v_2+1)(2n+\alpha+\beta+2v_1+2v_2+2)}$$

$$C_n=\frac{n(n+v_1+v_2-w_1-w_2-1)(n+\alpha+2v_1)}{(2n+\alpha+\beta+2v_1+2v_2+1)(2n+\alpha+\beta+2v_1+2v_2)}.$$

Proof: Let $f(x) = x^{v_1}(1-x)^{v_2}$, and $g(x) = x^{w_1}(1-x)^{w_2}$. Since $d_1 = av_1 + \frac{1}{2}c^2 v_1(v_1-1)$ and $d_2 = (b-a)v_2 + \frac{1}{2}c^2 v_2(v_2-1)$, we can compute

$$\begin{aligned}
\frac{\mathcal{L}^{\mathrm{P}} f}{f} &= -\left(b(v_1+v_2) + c^2 v_1 v_2 + \frac{1}{2}c^2(v_1(v_1-1) + v_2(v_2-1))\right) + \\
&+ \frac{1}{x}\left(av_1 + \frac{1}{2}c^2 v_1(v_1-1)\right) + \frac{1}{1-x}\left((b-a)v_2 + \frac{1}{2}c^2 v_2(v_2-1)\right) \\
&= d_1\left(\frac{1}{x}-1\right) + d_2\left(\frac{1}{1-x}-1\right) - ((b-a)v_1 + av_2 + c^2 v_1 v_2).
\end{aligned}$$

Thus $G^{\mathrm{Jacobi}}(T, x; d_1, d_2, w_1, w_2)$ equals

$$e^{-((b-a)v_1+av_2+c^2v_1v_2)T} E^{\mathrm{P}}_{0,x}\left[e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} X_T^{w_1}(1-X_T)^{w_2}\right]$$

By following the argument of Section 2, one computes from 6 that X_t under the measure Q has the generator

$$\mathcal{L}^{\mathrm{Q}} = (\tilde{a} - \tilde{b}x)\partial_x + \frac{1}{2}c^2 x(1-x)\partial_x^2 \tag{18}$$

where $\tilde{a} = a + c^2 v_1$ and $\tilde{b} = b + c^2(v_1+v_2)$. By the assumptions, $\tilde{\alpha} = \frac{2\tilde{a}}{c^2} - 1 = \alpha + 2v_1 \geq 0$ and $\tilde{\beta} = \frac{2(\tilde{b}-\tilde{a})}{c^2} - 1 = \beta + 2v_2 \geq 0$. Therefore X also has the boundary nonattainment property under Q so Proposition 2.1 applies and leads the following identity:

$$\begin{aligned}
E^{\mathrm{P}}_{0,x}\left[e^{-\int_0^T \frac{\mathcal{L}^{\mathrm{P}} f(X_s)}{f(X_s)} ds} X_T^{w_1}(1-X_T)^{w_2}\right] &= x^{v_1}(1-x)^{v_2} E^{\mathrm{Q}}_{0,x}\left[X_T^{w_1-v_1}(1-X_T)^{w_2-v_2}\right] \\
&= x^{v_1}(1-x)^{v_2}\int_0^1 y^{w_1-v_1}(1-y)^{w_2-v_2} p_T^{\mathrm{Q}}(x,y)dy. \qquad (19)
\end{aligned}$$

The transition probability density function $p_t^{\mathrm{Q}}(x,y)$ of the Jacobi process is given by

$$p_t^{\mathrm{Q}}(x,y) = \frac{y^{\tilde{\alpha}}(1-y)^{\tilde{\beta}}}{B(\tilde{\alpha}+1, \tilde{\beta}+1)} \sum_{n=0}^{\infty} \frac{e^{-n(n+\tilde{\alpha}+\tilde{\beta}+1)\frac{c^2 t}{2}}}{p_n^2} P_n^{(\tilde{\alpha},\tilde{\beta})}(2x-1) P_n^{(\tilde{\alpha},\tilde{\beta})}(2y-1),$$

where

$$p_n^2 = \frac{(\tilde{\alpha}+1)_n(\tilde{\beta}+1)_n}{(\tilde{\alpha}+\tilde{\beta}+2)_{n-1}(2n+\tilde{\alpha}+\tilde{\beta}+1)n!}.$$

The integral on the right side of 19 is finite if and only if $\alpha+w_1+v_1 > -1, \beta+w_2+v_2 > -1$ and can be computed explicitly using the following formula (see [4] and [8])

$$\int_0^1 y^{\alpha+w_1+v_1}(1-y)^{\beta+w_2+v_2}P_n^{(\alpha+2v_1,\beta+2v_2)}(2y-1)dy$$
$$= \mathrm{B}(\alpha+w_1+v_1+1, \beta+w_2+v_2+1)\frac{(\beta+2v_2+1)_n}{n!}\ q_n$$

The three term recurrence relation 17 for the coefficients q_n follows from the fact that q_n are related to the continuous Hahn polynomials

$$p_n(0; \beta+v_2+w_2+1, v_1-w_1, \alpha+v_1+w_1+1, v_2-w_2)$$

(see [11], page 31).

□

In the case $d_2 = w_2 = 0$ 15 can be simplified to give the following

Theorem 4.2.

$$\begin{aligned} G^{\mathrm{Jacobi}}(T, x; d_1, 0, w_1, 0) &= x^{v_1}e^{((a-b)v_1T)}\frac{\Gamma(\alpha+w_1+v_1+1)\Gamma(\alpha+\beta+2v_1+1)}{\Gamma(\alpha+\beta+w_1+v_1+2)\Gamma(\alpha+2v_1+1)} \\ &\times\sum_{n=0}^{\infty}(-1)^n e^{-n(n+\alpha+\beta+2v_1+1)\frac{c^2T}{2}}\frac{(v_1-w_1)_n(\alpha+\beta+2v_1+1)_n}{(\alpha+\beta+w_1+v_1+2)_n(\alpha+2v_1+1)_n} \\ &\times(2n+\alpha+\beta+2v_1+1)P_n^{(\alpha+2v_1,\beta)}(2x-1). \end{aligned} \tag{20}$$

Proof: If $d_2 = 0$ then $v_2 = 0$ and the ${}_3F_2$ function collapses to ${}_2F_1$:

$$q_n = {}_2F_1(-n, \alpha+\beta+2v_1+n+1; \alpha+\beta+w_1+v_1+2; 1),$$

which can be computed explicitly (see [8])

$$q_n = (-1)^n\frac{(v_1-w_1)_n(\alpha+\beta+2v_1+1)_n}{(\alpha+\beta+w_1+v_1+2)_n(\alpha+2v_1+1)_n}.$$

□

5 Conclusion

We have presented two families of diffusion processes for which explicit formulas for important expectations are proved. These results are consequences of a simple expectation identity derived for Markov diffusions. In the case where X_t is a CIR process, we have derived an explicit formula for the Laplace transform of the four dimensional random variable $(X_T, \log X_T, \int_0^T X_s ds, \int_0^T X_s^{-1} ds)$. The closest related result we are aware of is the well known closed formula for the Laplace transform of $(X_T, \int_0^T X_s ds)$. Our more general formula 10 has uses in interest rate theory and credit risk that we will explore elsewhere.

For the less known Jacobi process, we have derived an explicit formula for the Laplace transform of the four dimensional random variable $(\log X_T, \log(1-X_T), \int_0^T \frac{1-X_s}{X_s} ds, \int_0^T \frac{X_s}{1-X_s} ds)$. This process has properties that may also prove useful in finance. For example, the related process $Y_t = \frac{1-X_s}{X_s}$ has the character of the spot interest rate or a default hazard rate. X_t itself could be taken as a stochastic recovery rate in credit risk modelling. In such approaches, 15 will no doubt prove very useful.

All of our results are explicit realizations of certain integral formulas given in [3]. Their classification suggests that in the diffusion case, geometric Brownian motion and the Ornstein-Uhlenbeck process are the only further processes which admit identities of this type.

References

[1] Dong-Hyun Ahn and Bin Gao. A parametric nonlinear model of term structure dynamics. *The Review of Financial Studies.*, 12(4):721-762, 1999.

[2] C. Albanese and A. Kuznetsov. Unifying the three volatility models. *Risk Magazine.*, March 2004.

[3] C. Albanese and S. Lawi. Laplace transforms for integrals of Markov processes. *Markov processes and related fields.*, Vol. 11, Issue 4,677–724, 2005.

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, (1953).

[5] P. Cheridito, D. Filipović, and M. Yor. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Annals of applied probability.*, 15(3):1713-1732, 2005.

[6] P. Cheridito, D. Filipović, and R.L. Kimmel. Market price of risk specifications for affine models: theory and evidence. *Journal of Financial Economics*, 83(1):123-170, 2007.

[7] F. Delbaen and H. Shirakawa. An interest rate model with upper and lower bounds. *Financial engineering and the Japanese markets.*, 9(3-4):191-209, 2002.

[8] I.S. Gradshteyn, and I.M. Ryzhik Tables of integrals,series and products, Academic Press, 6-th edition, 2000.

[9] M.R. Grasselli and T.R. Hurd. Indifference pricing and hedging for volatility derivatives To appear in *Applied Mathematical Finance*, 2005.

[10] J. Jacod and A.N. Shiryaev Limit Theorems for Stochastic Processes, Berlin Heidelberg, Springer, 1987.

[11] R. Koekoek, and R.F. Swarttouw. The Askey-scheme of orthogonal polynomials and its q-analog. Delft University of Technology, Faculty of Information Technology and Systems, Dep-t of Technical Mathematics and Informatics, Report no. 98-17, 1998.

[12] Z. Palmowski and T. Rolski. A technique for exponential change of measure for Markov processes. *Bernoulli*, 8(6), 767-785, 2002.

6 CIR process: asymptotic expansion for small T

The following asymptotic formula [4] is useful when the argument of ${}_1F_1$ is large and negative (i.e. as $Re(z) \to -\infty$):

$$ {}_1F_1(a,b;z) = (-z)^{-a}\frac{\Gamma(b)}{\Gamma(b-a)}\sum_{k=0}^{N}\frac{(a)_n(a-b+1)_n}{n!}(-z)^{-n} + O\left(|z|^{-a-N-1}\right). $$

This leads to an approximation to 10:

$$ \begin{aligned} G^{\text{CIR}}(t,x;d_1,d_2,w_1,w_2) &\approx x^{w_2}\exp\left(x\left(v_1-(w_1+v_1)\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-1}e^{-\left(\frac{\beta}{2}-v_1\right)c^2t}\right)\right)\times \\ &\times \exp\left(\frac{1}{2}\left((\alpha+1)v_1-w_2(\beta-2v_1)\right)c^2t\right)\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-(\alpha+2w_2+1)}\times \\ &\times\left[\sum_{n=0}^{N}\frac{(v_2-w_2)_n(-\alpha-v_2-w_2)_n}{n!}(-z)^{-n}+O\left(|z|^{w_2-v_2-N-1}\right)\right], \end{aligned} \tag{21} $$

where $z = -\gamma_t x e^{-\left(\frac{\beta}{2}-v_1\right)c^2t}\left(1+\frac{w_1+v_1}{\gamma_t}\right)^{-1}$ is the argument of ${}_1F_1$ function. We find that this formula is very convenient for computations when t is small and/or when x is large, all other parameters being bounded.

7 Jacobi process: asymptotic expansion for small t

In this section we will derive an asymptotic expansion for $G^{\text{Jacobi}}(t,x;d_1,d_2,w_1,w_2)$ for small t. For fixed $x \in (0,1)$, we start with 19 and write

$$ E^{\text{Q}}_{0,x}\left[X_t^{w_1-v_1}(1-X_t)^{w_2-v_2}\right] = E^{\text{Q}}_{0,x}\left[\sum_{n\geq 0}c_n(x)(X_t-x)^n\right] \approx \sum_{n\geq 0}c_n(x)M_n(t,x), \tag{22} $$

where $c_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^{w_1-v_1}(1-x)^{w_2-v_2}\right]$ and $M_n(t) := M_n(t,x) = E^{\mathrm{Q}}_{0,x}\left[(X_t - x)^n\right]$. Coefficients $c_n(x)$ can be computed explicitly as $c_n(x) = x^{w_1-v_1}(1-x)^{w_2-v_2}\hat{c}_n(x)$, where

$$\hat{c}_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(v_1-w_1)_k(v_2-w_2)_{n-k}}{x^k(1-x)^{n-k}}.$$

By applying Ito's lemma to the process $(X_t - x)^n$ we find that functions $M_n(t)$ can be found by recursively solving differential equations

$$\frac{d}{dt}M_n(t) = \left(\frac{c^2}{2}n(n-1) - n\tilde{b}\right)M_n(t) + \left(n(\tilde{a}-\tilde{b}x) + \frac{c^2}{2}n(n-1)(1-2x)\right)M_{n-1}(t) + \\ + \frac{c^2}{2}n(n-1)x(1-x)M_{n-2}(t), \tag{23}$$

with initial conditions $M_n(0) = \delta_{n0}$. It follows at once from this equation that $M_n(t) = O(t^{[\frac{n+1}{2}]})$ as $t \to 0$.

Using this method one could obtain approximations of any order $O(t^N)$. Here as an example we provide the first four functions $M_n(t,x)$ which combined with 22 will give us a useful approximation to $G^{\mathrm{Jacobi}}(t,x;d_1,d_2,w_1,w2)$ of order $O(t^3)$.

$$\begin{cases} M_1(t) = (\tilde{a}-\tilde{b}x)\left(t - \tilde{b}\frac{t^2}{2}\right) + O(t^3) \\ M_2(t) = c^2x(1-x)t + \left(\left(2(\tilde{a}-\tilde{b}x) + c^2(1-2x)\right)(\tilde{a}-\tilde{b}x) - c^2(2\tilde{b}+c^2)x(1-x)\right)\frac{t^2}{2} + O(t^3) \\ M_3(t) = 3\left(2(\tilde{a}-\tilde{b}x) + c^2(1-2x)\right)c^2x(1-x)\frac{t^2}{2} + O(t^3) \\ M_4(t) = 6c^4x^2(1-x)^2\frac{t^2}{2} + O(t^3) \end{cases}$$

Thus we have the following approximation

$$G^{\mathrm{Jacobi}}(t,x;d_1,d_2,w_1,w_2) \approx x^{w_1}(1-x)^{w_2}e^{-((b-a)v_1+av_2+c^2v_1v_2)t}\left(1 + \sum_{n=1}^{4}\frac{\hat{c}_n(x)}{n!}M_n(t)\right) + O(t^3).$$