UNIFYING THE THREE VOLATILITY MODELS

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ABSTRACT. This article describes a method for building analytically tractable option pricing models which combine state dependent volatility, stochastic volatility and jumps. Starting from a Laguerre representation of the pricing kernel, we show how to account for jumps and stochastic volatility by altering the time dependent coefficients of a series expansion. This operation is easy to implement analytically and gives rise to numerically efficient formulas for the pricing kernel. This technique stemmed from a line of research on barrier models for credit derivatives, but the method is of broader relevance to option pricing theory.

1. INTRODUCTION

The success of pricing models is often measured by the extent to which closed form solutions of the Black-Scholes type are available for the basic payoffs. Analytic tractability is often crucial for the calibration to market data and is helpful for the implementation of numerical algorithms for exotics. Extensions of the Black-Scholes formula have been directed towards three main model classes: (i) state dependent volatility models postulate a deterministic relationship between the underlying state variable, time and the local volatility; (ii) stochastic volatility models assume that the volatility follows a distinct but correlated process; (iii) jump models can be regarded as limits of stochastic volatility models where the volatility can occasionally be singularly large at some points in time, so large as to cause the underlying sample path to have discontinuous jumps.

As Dupire [1] demonstrated, state dependent volatility models are able to reproduce arbitrage-free implied volatility surfaces. Robust estimations however require either regularizations or settling on a parametric form for the local volatility. Parametric extensions of the Black-Scholes model based on geometric Brownian motion led at first to two three-parameter families, namely the models with constant elasticity of variance by Cox and Ross [2] and the quadratic volatility models in Rady [3]. More recently, these models were found to be particular cases of hypergeometric Brownian motions, a class of solvable models with seven adjustable parameters introduced in [4]. On the jump-models front, geometric Brownian motions and the corresponding pricing formulas were extended in various ways; particularly noteworthy is the variance-gamma model by Madan et al. in [5], [6], [7], a class of pure jump processes admitting the Black-Scholes case as a limit which is analytically solvable in the case of European calls and puts. Stochastic volatility models were considered by several authors including Hull-White [8] and Heston [9], [10] and Stein-Stein [11].

The three volatility models are all able to reproduce most of the features of market implied volatility skews for European options at a fixed maturity, although sometimes they require unnatural choices of parameters. Stochastic volatility models encounter difficulties capturing the very short term behavior of the implied volatility skew, which imply very large values of the instantaneous volatility as a way to emulate a jump component. Jump models have an opposite shortcoming as the corresponding implied volatility surface flattens out too rapidly unless jump amplitudes increase as a function of time at a fairly steep rate. State dependent volatility models instead run into dynamic inconsistencies, as the implied volatility of call options of the same strike tend to stay constant as the underlying moves, a phenomenon sometimes called sticky-strike dynamics. Instead, empirical evidence shows that options of the same delta (not the same strike) tend to retain a roughly constant implied volatility as time evolves [12].

Studies by Bates [13], [14] indicate that stochastic volatility and jumps are both features of the real-world process and both effects are reflected in option prices. An important feature of equity price

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processes is also the so-called *leverage effect*, according to which price levels are negatively correlated with spot volatility. From the modeling perspective, this effect can be captured by both local volatility models and stochastic volatility models in which equity returns are negatively correlated to volatility changes. From the hedging viewpoint, the leverage effect allows one to partially hedge the option's vega exposure with the stock. Also jumps are important, since prices of path-dependent options are substantially affected by their presence: one would expect that the better hedge ratios and the most consistent pricing schemes across exotic options result from the more inclusive models, as long as they are properly calibrated.

Motivated by these reasons, some recent derivatives literature seeks to combine the various volatility models. On the front of combining jumps with stochastic volatility, Carr, Madan, Gemain and Yor [15] discuss time-changed Levy processes. The variance-gamma model with binomial volatility presents combinatorial complexities which are tackled in a recent article [16] within the context of the model of lines. Within the credit derivatives domain, the credit barrier models introduced in [17], [18], have as underlying a credit quality parameter following gamma time-changed hypergeometric Brownian motions. In this article, we discuss a novel approach aimed at combining the three volatility models together within an analytically amenable framework. The original motivation was that of improving efficiency for the calibration of the aforementioned credit derivative models in [17], [18] but results are here presented in a broader context as they are of more general applicability.

Our method bears some relation with the recent work by Carr on Fourier methods in option pricing, [19]. When using ordinary (trigonometric) Fourier series one expands payoffs and pricing kernels in terms of a basis of exponential functions with imaginary arguments in the logarithm of the underlying. Exponentials are a suitable basis for models where the underlying doesn't follow a state-dependent process, as for instance a geometric Brownian motion with constant log-normal volatility, a variance-gamma model or a Heston model with stochastic volatility. Carr in [20] also considers a variety of extensions to jump models with stochastic volatility and succeeds to model the leverage effect by correlating the jump process with the time change through which the stochastic volatility is expressed. In our approach instead, we include state-dependency by considering more general Fourier transforms, based on special functions which admit trigonometric functions as a particular case. Although a more general treatment is possible and will be covered in a forthcoming publication, in this article we limit ourselves to the case which appears to be the most relevant for applications where the basis functions are expressed in terms of Laguerre polynomials. The more general expansions allow one to model the leverage effect through a direct state dependency of the local volatility and jump amplitudes, rather than through correlations. In our models, jumps and the stochastic volatility component are also related to stochastic time changes, but we show that in the chosen Laguerre representation they can be explicitly evaluated and recast in terms of simple changes of the time dependent part of the expansion coefficients.

The article is organized as follows. In the next section we recast the formulas for hypergeometric Brownian motions within the framework of Laguerre expansions, which are also reviewed in Appendix A. In section 3, we discuss time changes giving rise to either jumps or stochastic volatility effects and show how to express pricing kernels in terms of Laplace transforms of the time changes. Two examples are worked out in detail, one related to variance-gamma time changes and the second to stochastic volatility models where the squared volatility follows a CIR process. It turns out that arbitrary linear combinations among different time changes can be easily accounted for. Concluding remarks close the paper.

2. Hypergeometric Brownian Motions

In this section, we give a new simplified derivation of the main results about pricing models built upon the hypergeometric Brownian motions in [4]. This article contains a general procedure to construct a driftless process starting from a generic diffusion process X_t with drift $m(x)$ and volatility $\sigma(x)$, i.e. satisfying the equation

$$dX_t = m(X_t)dt + \sigma(X_t)dW. \tag{2.1}$$

To complete the formal definition of the process X_t, we add that X_t is bound to take values on a (possibly infinite) interval $D = (a, b)$ where $\infty \leq a < b \leq \infty$ and the process terminates upon hitting the boundary.

Solutions to the Black-Scholes equation with a generic payoff function as final time condition can be expressed in terms of the so-called *pricing kernel* associated to $\mathcal{L}^X$, i.e. the function $p^X(t, x_0, x_1)$ defined for $t > 0$ such that

$$\frac{d}{dt} p^X(t, x_0, x_1) = \mathcal{L}^X p^X(t, x_0, x_1) \quad \text{and} \quad \lim_{t \to 0} p^X(t, x_0, x_1) = \delta(x_0 - x_1).$$

Here, $\mathcal{L}^X$ is the so-called Markov generator given by

$$\mathcal{L}^X = m(x) \frac{d}{dx} + \frac{\sigma(x)^2}{2} \frac{d^2}{dx^2}. \tag{2.2}$$

and acts on the variable x_0 in the first argument. From the probabilistic point of view the pricing kernel $p^X(t, x_0, x_1)$ as a function of the end-point x_1 is the conditional probability density of the variable X_t given the initial condition $X_0 = x_0$.

A family of driftless processes can be constructed based on the following ingredients:

(i) a parameter $\rho > 0$,
(ii) two linearly independent solutions $f_1(x)$, $f_2(x)$ to the differential equation $\mathcal{L}^X f = \rho f$.
(iii) two constants c_1, c_2 such that the function $g(x) = c_1 f_1(x) + c_2 f_2(x)$ is strictly positive, i.e. $g(x) > 0$ for all $x \in D$.

The process $e^{-\rho t} g(X_t)$ is positive and driftless and can thus be interpreted as a new numeraire and used to define a measure change. Let Q denote the measure such that

$$\frac{dQ_t}{dP_t} = e^{-\rho t} g(X_t).,\tag{2.3}$$

where P is the measure corresponding to equation (2.1).

Consider the function

$$Y(x) = \frac{c_3 f_1(x) + c_4 f_2(x)}{c_1 f_1(x) + c_2 f_2(x)}\tag{2.4}$$

Since the transformed process $e^{-\rho t}(c_3 f_1(X_t) + c_4 f_2(X_t))$ is also driftless under P, the process $Y(X_t)$ is driftless under Q. It turns out that the function $Y(x)$ is invertible for all choices of the constants c_3, c_4 such that $c_1 c_4 - c_2 c_3 \neq 0$. In fact, the derivative $Y'(x)$ can be expressed through the Wronskian between the independent solutions f_1 and f_2, namely

$$Y'(x) = (c_1 c_4 - c_2 c_3) \frac{W_{f_1, f_2}(x)}{g^2(x)} \quad \text{where} \quad W_{f_1, f_2}(x) = \det \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}.$$

Notice that

$$W_{f_1, f_2}(x) = W(x_0) \exp\left(-2 \int_{x_0}^{x} \frac{m(u)}{\sigma(u)^2} du \right)$$

Hence, $Y'(x)$ is nonzero and $Y(x)$ is invertible. Let $X(y)$ denote the inverse of the function $Y(x)$.

The process $Y_t = Y(X_t)$ is a martingale and satisfies the stochastic differential equation $dY_t = \nu(Y_t) dW_t^Q$ with volatility function $\nu(y)$ equals

$$\nu(y) = Y'(X(y)) \sigma(X(y)).\tag{2.5}$$

The probability density function $p^Y(t, y_0, y_1)$ for the process $Y_t = Y(X_t)$ can be expressed in terms of the corresponding probability density function $p^X(t, x_0, x_1)$ for the process X_t as follows:

$$\begin{aligned} p^Y(t, y_0, y_1) &= E_t^Q \Big[\delta(Y(X_t) - Y(x_1)) | Y(X_0) = Y(x_0) \Big] = \\ &= E_t^P \left[\frac{g(X_t)}{g(X_0)} \frac{e^{-\rho t}}{Y'(x_1)} \delta(X_t - x_1) | X_0 = x_0 \right] = \frac{g(x_1)}{g(x_0)} \frac{1}{Y'(x_1)} e^{-\rho t} p^X(t, x_0, x_1) \end{aligned} .$$

Here and throughout the article, we set $y_0 = Y(x_0)$, $y_1 = Y(x_1)$ and

$$H(x_0, x_1) = \frac{g(x_1)}{g(x_0)} \frac{1}{Y'(x_1)}.\tag{2.6}$$

Next, we specialize to the case whereby the Markov generator $\mathcal{L}^X$ has discrete spectrum. This happens when the eigenvalue equation

$$\mathcal{L}^X \psi_n(x) = \lambda_n \psi_n(x) \tag{2.7}$$

admits a complete set of eigenvalues λ_n and eigenfunctions $\psi_n(x), n = 0, 1, ...$ which are orthogonal with respect to some measure $\mu(x)dx$, i.e.

$$\int \psi_n(x)\psi_m(x)\mu(x)dx = \delta_{mn}. \tag{2.8}$$

As we discuss in detail in the implementation section below, this is the case for instance for the generator of the CIR process. If the eigenfunctions are known, the pricing kernel can be represented as the following series:

$$p^X(t, x_0, x_1) = e^{t\mathcal{L}^X}(x_0, x_1) = \sum_{n=0}^{\infty} e^{\lambda_n t}\psi_n(x_0)\psi_n(x_1)\mu(x_1). \tag{2.9}$$

The pricing kernel for the martingale process Y_t is then given by

$$p^Y(t, y_0, y_1) = H(x_0, x_1)\mu(x_1)\left[\sum_{n=0}^{\infty} e^{-(\rho-\lambda_n)t}\psi_n(x_0)\psi_n(x_1)\right].$$

Notice that the time and space variables are separated in each term of the representation, a useful property that - as we discuss below - allows one to readily include jumps and stochastic volatility into the process.

An interesting expansion also applies for European call option prices defined as follows:

$$C^Y(t, y_0, K) = E^Q((Y_t - K)^+ | Y_0 = y_0), \tag{2.10}$$

where $(y)^+ = \max(y, 0)$. If $k = X(K)$ is the strike price in the X-variable, then

$$\begin{aligned} C^Y(t, y_0, K) &= (y_0 - K)^+ + \\ &+ \frac{1}{2}\nu^2(K)H(x_0, k)\left[G(\rho, x_0, k) + \mu(k)\sum_{n=0}^{\infty}\frac{1}{\rho - \lambda_n}e^{-(\rho-\lambda_n)t}\psi_n(x_0)\psi_n(k)\right] \end{aligned} \tag{2.11}$$

where $G(\rho, x_0, x_1)$ is the Green function defined either as $\int_0^\infty e^{-\rho s}p^X(s, x_0, x_1)ds$ or as the integral kernel of the operator $(\rho - \mathcal{L}^X)^{-1}$. This formula is useful in practice as it gives a computationally efficient method for evaluating European call option prices and thus calibrate the model to market data. A proof is sketched in Appendix A.

3. Jumps and Stochastic Volatility

In this section we start with a Markov martingale Y_t constructed in the previous section and show how to introduce the jumps and stochastic volatility. To achieve this task we are going to use the stochastic time change which can be conveniently introduced due to separation of variables in formulas (2.11) and (2.10).

A *stochastic time change process* T_t is defined as a right-continuous non-decreasing process started from 0 and with values in $[0, \infty)$. The time changed version of the process Y_t is the process $\tilde{Y}_t = Y_{T_t}$. We shall refer to t as to the *calendar time* and call $s = T_t$ the *financial time* coordinate. As discussed in the introduction, we make the simplifying assumption that the time change process T_t is independent of the underlying process Y_t. In this case, if $\rho_t(ds)$ is the distribution of T_t, we have that

$$E[f(\tilde{Y}_t)] = \int E[f(Y_s)]\rho_t(ds). \tag{3.1}$$

for all continuous functions $f(x)$. A stochastic time change is also characterized by the Laplace transform:

$$L(t, \lambda) = E\left[e^{-\lambda T_t}\right]. \tag{3.2}$$

Formula (3.1) allows one to express as follows the probability kernel for the time-changed martingale process $\tilde{Y}_t$:

$$p^{\tilde{Y}}(t, y_0, y_1) = \int p^Y(s, y_0, y_1)\rho_t(ds).$$

In conjunction with equation (2.10) and in virtue of the fact that in (2.9) the time variable and space variables are separated, we find that

$$p^{\tilde{Y}}(t, y_0, y_1) = H(x_0, x_1)\mu(x_1)\left[\sum_{n=0}^{\infty} L(t, \rho - \lambda_n)\psi_n(x_0)\psi_n(x_1)\right]. \tag{3.3}$$

Similarly, European option prices under the time-changed process are computed as follows:

$$\begin{aligned} C^{\tilde{Y}}(t, y_0, K) &= (y_0 - K)^+ + \\ &+ \frac{1}{2}\nu^2(K)H(x_0, k)\left[G(\rho, x_0, k) + \mu(k)\sum_{n=0}^{\infty}\frac{L(t, \rho - \lambda_n)}{\rho - \lambda_n}\psi_n(x_0)\psi_n(k)\right] \end{aligned} \tag{3.4}$$

A rich variety of time changes can be constructed based on three building blocks. The first building block is represented by deterministic time change process of the form $T_t^d = f(t)$, where $f(t)$ is an increasing right continuous function from $[0, +\infty)$ to $[0, +\infty)$. In this case the Laplace transform is simply

$$L(t, \lambda) = e^{-\lambda f(t)}. \tag{3.5}$$

The second example of the time change is given by nondecreasing jump process T_t^j. If the increments $T_{t_1}^j - T_{t_0}^j$ are independent of $\mathcal{F}_{t_0}$ and time-homogeneous, the stochastic time change process is a nondecreasing Levy process and is called *Bochner subordinator*. It turns out that a Bochner subordinators are characterized by a Bernstein function $\phi(\lambda)$, such that

$$L(t, \lambda) = e^{-t\phi(\lambda)}. \tag{3.6}$$

Perhaps the most popular example of a Bochner subordinator in pricing theory is given by the gamma process Γ_t of density:

$$\rho_t^\Gamma(ds) = \frac{(\frac{\mu}{\nu})^{\frac{\mu^2 t}{\nu}}}{\Gamma(\frac{\mu^2 t}{\nu})} s^{\frac{\mu^2 t}{\nu} - 1} e^{-\frac{\mu}{\nu}s} ds. \tag{3.7}$$

In this case, the Laplace transform is given by

$$L^\Gamma(t, \lambda) = \left(1 + \frac{\lambda\mu}{\nu}\right)^{-\frac{\mu^2}{\nu}t}, \tag{3.8}$$

thus the Bernstein function is $\phi(\lambda) = \frac{\mu^2}{\nu}\ln(1 + \frac{\lambda\mu}{\nu})$.

The third example of stochastic time change is given by process T_t^c with continuous paths which can be represented as follows: $T_t^c = \int_0^t u_s ds$ where u_s is a positive stochastic process. This is the kind of stochastic time change that arises in stochastic volatility models. The process u_s can be considered as a square root of the volatility of the process $\tilde{Y}_t$. The CIR process from the fixed-income literature give example of positive process for which the Laplace transform of T_t^c can be evaluated in analytically closed form. More precisely, if u_t is a CIR process with $du_t = (c - du_t)dt + \zeta dW$, then the Laplace transform has the form

$$L(t, \lambda) = e^{m(t)u_0 + n(t)}. \tag{3.9}$$

where

$$n(t) = \frac{2c}{\zeta_0^2}\log\left(\frac{\gamma\exp(\frac{1}{2}dt)}{\gamma\cosh(\gamma t) + \frac{1}{2}d\sinh(\gamma t)}\right), \qquad m(t) = -\frac{\lambda\sinh(\gamma t)}{\gamma\cosh(\gamma t) + \frac{1}{2}d\sinh(\gamma t)} \tag{3.10}$$

where $\gamma = \frac{1}{2}\sqrt{d^2 - 2\lambda\zeta^2}$. Note that in this case the Laplace transform T_t^c depends on the initial value of u_0, which is intuitively clear since $T_t^c = \int_0^t u_s ds$ clearly depends on u_0. Strictly speaking because of the fact that in this case Laplace transform is a random variable depending on u_0 we should define the Laplace transform as $L_{t_0}(t, \lambda) = E\left[e^{-\lambda T_t} | \mathcal{F}_{t_0}\right]$, but we decided to keep the initial definition to simplify the notations.

The three classes of time changes above can also be combined, which lets us construct a martingale process with diffusion,jumps and stochastic volatility properties at the same time. One can consider a time change of the form

$$T_t = T_t^d + T_t^j + T_t^c$$

where T_t^d is a deterministic function of time, T_t^j is a Bochner subordinator and the third process has the form $T_t^c = \int_0^t u_s ds$ where u_s follows a CIR process. Assuming that the latter two processes are independent, the Laplace transform can be computed in analytically closed form and is given by the product

$$L^T(t, \lambda) = L^{T^d}(t, \lambda) L^{T^j}(t, \lambda) L^{T^c}(t, \lambda) \tag{3.11}$$

Barrier options and exotics are somewhat more difficult to price in the presence of jumps and stochastic volatility. In this case, it is possible to resort to the lattice approximations in (**??**). The analytical framework we presented is however still quite powerful in some cases, such as for instance barriers options struck at constant forward. As a working example, consider a double barrier option, with an up-and-out barrier at the forward $Y = U$ and a down-and-out barrier at $Y = L$. Stochastic volatility is fairly easy to incorporate by directly subordinating pricing formulas. This is possible because the time change process in this case has continuous paths. In case there are jumps instead, the Markov generator in the X-representation for the underlying diffusion process with jumps and absorption outside the interval $B = [X(L), X(U)]$ has the integral kernel

$$\mathcal{L}^{XBJ}(x_0, x_1) = 1_B(x_0) 1_B(x_1) \sum_{n=0}^{\infty} \lambda_n^J \psi_n(x_0) \psi_n(x_1) \mu(x_1). \tag{3.12}$$

Here $1_B(F)$ is the indicator function for B, $\lambda_n^J = -\phi(-\lambda_n)$ and ϕ is the Bernstein function that characterizes the jump process. To find the pricing kernel, we need to exponentiate the operator $\mathcal{L}^{XBJ}$ and to do that one has to diagonalize it. A possible strategy is to carry out the diagonalization in the basis $\psi_n^B(x)$ of the eigenfunctions of the operator $\mathcal{L}^X$ with Dirichlet boundary conditions $\psi_n^B(L) = \psi_n^B(U) = 0$. Let λ_n^B be the corresponding eigenvalues. The matrix elements of the operator $\mathcal{L}^{XBJ}$ in this basis are

$$\mathcal{L}^{XBJ}(m, n) = \sum_{j=0}^{\infty} \lambda_j^J c_{mj} c_{nj}, \quad \text{where} \quad c_{mn} = \int_B \psi_m^B(x) \psi_n(x) \mu(dx). \tag{3.13}$$

Let $a_n^{BJ}(j)$ be the eigenvectors of $\mathcal{L}^{BJ}(m, n)$ and let λ_n^{XBJ} be the corresponding eigenvalues. Furthermore, let $\psi_n^{BJ}(x) = \sum_j a_n^{BJ}(j) \psi_j^B(x)$ be the eigenvectors in the X-representation. Then the pricing kernel in the Y-representation is given by

$$p^{\tilde{Y}BJ}(t, y_0, y_1) = H(x_0, x_1) \sum_{n=0}^{\infty} e^{t\lambda_n^{BJ}} \psi_n^{BJ}(x_0) \psi_n^{BJ}(x_1) \mu(x_1). \tag{3.14}$$

4. Implementation Example

In this section we give a step-by-step construction to illustrate the theory in the paper. As for X_t, we choose the CIR process of equation $dX = (a - bX)dt + \sigma dW$. The Markov generator $\mathcal{L}^X$ in the new variable $\xi = \theta x$ is given by

$$\mathcal{L}^X = \frac{1}{b}\left((\alpha + 1 - \xi) \frac{d}{d\xi} + \xi \frac{d^2}{d\xi^2} \right). \tag{4.1}$$

where $\theta = \frac{2b}{\sigma^2}$ and $\alpha = \frac{2a}{\sigma^2} - 1$. Two linearly independent solutions to the equation $\mathcal{L}^X f = \rho f$ are given by f_1 and f_2 are given by the following formulas:

$$f_1(x) = {}_1F_1\left(\frac{\rho}{b}, \alpha+1, \theta x\right), \qquad f_2(x) = (\theta x)^{-\alpha} {}_1F_1\left(\frac{\rho}{b} - \alpha, 1-\alpha, \theta x\right). \tag{4.2}$$

Here ${}_1F_1(A, B, z)$ is the confluent hypergeometric functions, which can be computed by means of its Taylor expansion:

$${}_1F_1(A, B, z) = 1 + \frac{A}{B} z + \frac{A(A+1)}{B(B+1)} \frac{z^2}{2!} + \dots .$$

The eigenfunctions ψ_n can be expressed in terms of Laguerre polynomials:

$$\psi_n(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}} L_n^{(\alpha)}(\theta x), \tag{4.3}$$

where $L_n^{(\alpha)}$ are Laguerre polynomials of order α. The corresponding eigenvalues are $\lambda_n = -bn$ and the orthogonality measure $\mu(x)dx$ is equal to $\theta^{\alpha+1} x^\alpha e^{-\theta x} dx$. Laguerre polynomials can be computed efficiently by means of the recurrence relation:

$$(n+1) L_{n+1}^{(\alpha)}(z) = (2n + \alpha + 1 - z) L_n^{(\alpha)}(z) - (n+\alpha) L_n^{(\alpha)}(z) \tag{4.4}$$

with initial conditions $L_{-1}^{(\alpha)} = 0$, $L_0^{(\alpha)} = 1$.

Once the constants c_1, c_2, c_3, c_4 are specified, we can construct the diffeomorphism $Y(x)$ and invert it to find the function $X(y)$. The derivative $Y'(x)$ can be expressed explicitly in terms of confluent hypergeometric functions using (4.2) and the identity

$$\frac{d}{dz} {}_1F_1(A, B, z) = \frac{A}{B} {}_1F_1(A+1, B+1, z).$$

This yields the local volatility function $\nu(y)$ in (2.5). The function $H(x_0, x_1)$ in (2.6) can also be expressed in terms of the derivative $Y'(x)$.

To compute the Green's function defined after (2.11), it is convenient to start from the following equation:

$$(\mathcal{L}^X - \rho) G(\rho, x_0, x_1) = \delta(x_1 - x_0).$$

and can be computed as follows: find the two linearly independent solutions of differential equation $\mathcal{L}^X f = \rho f$. The solution can be expressed as follows:

$$G(\rho, x_0, x_1) = \begin{cases} \dfrac{\Gamma(\frac{\rho}{b})}{b\Gamma(\alpha+1)} M(x_0) U(x_1) \mu(x_1) \text{ in case } x_0 < x_1 \\ \dfrac{\Gamma(\frac{\rho}{b})}{b\Gamma(\alpha+1)} M(x_1) U(x_0) \mu(x_1) \text{ otherwise} \end{cases}$$

Here $M(x) = f_1(x)$ and

$$U(x) = \frac{\pi}{\sin(\pi(\alpha+1))} \left(\frac{f_1(x)}{\Gamma(\frac{\rho}{b} - \alpha)} - \frac{f_2(x)}{\Gamma(\frac{\rho}{b}) \Gamma(1-\alpha)} \right).$$

A meaningful choice of parameters we considered is the following: $a = 0.1, \quad b = 0.06, \quad \sigma = 0.42, \quad X_0 = 1, \rho = 0.05, \quad c1 = 5, \ c2 = -2, \ c3 = -2$ while $c4$ is determined out of the condition $g(X_0) = 1$. The gamma process for the Bochner subordinator was chosen with parameters $\mu = 1$ and $\nu = 0.1$ while the stochastic volatility portion is given by $c = 0.1, \quad d = 0.1, \quad \zeta = 0.3$ with initial condition $u_0 = 1$. To assess the speed of convergence, we computed the number N of terms in the series expansion for the price of European calls that one has to retain to achieve *penny accuracy*, i.e. errors smaller than 10^{-4}. We find that six-month maturities require only 100 terms, one-month maturities require about 500 terms while one-week maturities require 2000 terms.

Finally, let's mention that to price barrier options, the coefficients c^{B}_{mn} in (3.13) can be evaluated as follows using integration by parts:

$$c^{\mathrm{B}}_{mn}=\frac{1}{(\lambda_m-\lambda^{\mathrm{B}}_n)}\int_B\left[(\mathcal{L}\psi_m)\psi^{\mathrm{B}}_n-\psi_m(\mathcal{L}\psi^{\mathrm{B}}_m)\right]\mu(dx)=$$
$$=\frac{\nu_0^2}{2(\lambda_m-\lambda^{\mathrm{B}}_n)}\left\{X(L)\psi_m(X(L))\frac{d\psi^{\mathrm{B}}_n}{dx}(X(L))-X(U)\psi_m(X(U))\frac{d\psi^{\mathrm{B}}_n}{dx}(X(U))\right\}.$$

5. Concluding Remarks

In this paper we present an analytical framework to price options subject to a combination of state-dependent volatility, stochastic volatility and jumps. Although the method lends itself quite naturally to lattice discretization, in this paper we focus the attention to continuous methods. The key idea in the paper is to make use of processes whose Markov generator has discrete spectrum, as this simplifies the Fourier analysis considerably. We show that pricing kernels and pricing formulas for European and barrier options can be expressed as series expansions.

6. Appendix A.

Here we prove the pricing formula for European options in **??**. The starting point is the Meyer-Tanaka formula

$$(Y_t-K)^+=(Y_0-K)^++\int_0^t I\{Y_s>K\}dY_s+\frac{1}{2}\int_0^t\delta(Y_s-K)\nu^2(Y_s)ds.$$

The integral $\int_0^t\delta(Y_s-K)\nu^2(Y_s)ds$ can be rewritten as $\int_0^t\delta(Y_s-K)\nu^2(K)ds$, thus taking expectations from both sides of equation (6) we get the following equality:

$$E^Q((Y_t-K)^+|Y_0=y_0)=(y_0-K)^++E^Q\int_0^t I\{Y_s>K\}dY_s+\frac{1}{2}\nu^2(K)E^Q\int_0^t\delta(Y_s-K)ds=$$
$$=(y_0-K)^++\frac{1}{2}\nu^2(K)\int_0^t p^Y(s,y_0,K)ds.$$

Here we used the fact that $\int_0^t I\{Y_s>K\}dY_s$ is a Q-martingale and thus its expectation is zero, and the fact that $E^Q(\delta(Y_s-K)|Y_0=y_0)=p^Y(s,y_0,K)$. Using the fact that the space and time variables in the expression for the pricing kernel $p^Y(s,y_0,K)$ are separated, we find that

$$\int_0^t p^Y(s,y_0,K)ds=\int_0^\infty p^Y(s,y_0,K)ds-\int_t^\infty p^Y(s,y_0,K)ds=$$
$$=H(x_0,k)\int_0^\infty e^{-\rho s}p^X(s,x_0,k)ds-H(x_0,k)\mu(k)\int_t^\infty\sum_{n=0}^\infty e^{-(\rho-\lambda_n)s}\psi_n(x_0)\psi_n(k)ds=$$
$$=H(x_0,k)\left[G(\rho,x_0,k)+\mu(k)\sum_{n=0}^\infty\left(\int_t^\infty e^{-(\rho-\lambda_n)s}ds\right)\psi_n(x_0)\psi_n(k)\right]$$

where we used the equation (2.10).

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