Weighted risk capital allocations in the presence of systematic risk

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Abstract. Determining aggregate risk capital is a fundamental problem of modern Enterprise Risk Management, and the determination process has been fairly well studied. The allocation problem, on the other hand, is generally much more involved even when a specific risk measure inducing the allocation rule is assumed, let alone the case when a class of risk measures is considered. In this paper we put forward arguments showing that the problems of determining and allocating the aggregate risk capital can often be viewed as being of similar complexity. In particular, we show that this is the case for the entire class of weighted risk capital allocations, as well as for risk portfolios that are exposed to systematic and specific risk factors. We provide detailed analyses of the Weighted Insurance Pricing Model (WIPM) under multiplicative and additive systematic-risk frameworks. Also, a Gini-type WIPM, which is related to the WIPM in a similar way as the dual (i.e., rank dependent) utility theory is related to the classical utility theory, is proposed.

Key words and phrases: weighted risk measure, weighted risk capital allocation, weighted insurance pricing model, Gini measure of variability, systematic risk.

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1 Introduction

We call real-valued random variables (r.v.'s) risks, and we let $\mathcal{X}$ denote a collection of such risks: routinely, they are regarded as losses when positive, and as profits when negative. Risk measures are maps $H : \mathcal{X} \to [0, \infty]$ that assign finite or infinite values to the risks in $\mathcal{X}$. In modern risk management, the risk capital (RC), which for the sake of our discussion equals to the value of the risk measure, is of fundamental importance for actuaries, regulators, and shareholders. Indeed, actuaries and other insurance professionals employ RC's for pricing, and also for general management purposes. Regulators build on RC's distinct capital adequacy requirements. Shareholders view RC's as a riskiness gauge of their investments.

Given two risks $X, Y \in \mathcal{X}$, the determination of the RC for the aggregate risk $S = X + Y$ is nowadays mandatory in the insurance and banking sectors (e.g., Solvency II/Swiss Solvency Test, and Basel III). That being said, the assessment of the aggregate RC $H[S]$ is commonly an initial step in the encompassing and intricate framework of the Enterprise Risk Management (e.g., McNeil et al., 2015). Indeed, assuming that $H[S]$ has been determined, the next step would often be to assess how much riskiness each of the two r.v.'s X and Y contributes to the aggregate risk. One goal of this - usually quite involved - exercise is profitability testing. Other goals are cost sharing and pricing (e.g., Venter, 2004). Formally, RC allocation rules are maps $A : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ that assign finite or infinite values to random pairs (X, S). They are said to be induced by risk measures H when the equation $A[X, X] = H[X]$ holds for all risks $X \in \mathcal{X}$ (e.g., Denault, 2001). Consequently, we denote by $A[X, S]$ the RC contribution of the risk r.v. X to the overall risk S due to the allocation rule A.

There are numerous ways for allocating aggregate RC, and the literature on the allocation rules is vast and growing rapidly (e.g., Dhaene et al., 2012; references therein). In this paper, we restrict ourselves to the class of weighted RC allocation rules (Furman and Zitikis, 2008b)

$$A_w[X, S] = \frac{\mathbf{E}[Xw(S)]}{\mathbf{E}[w(S)]}, \tag{1.1}$$

which are induced by the class of weighted risk measures (Furman and Zitikis, 2008a)

$$H_w[X] = \frac{\mathbf{E}[Xw(X)]}{\mathbf{E}[w(X)]} \tag{1.2}$$

with $w : (-\infty, \infty) \to (-\infty, \infty)$ denoting 'weight' functions that are, usually, non-decreasing and non-negative. Besides the obvious generality, the weighted allocation rules are additive, consistent, satisfy the no-unjustified loading property and, under certain conditions, the (consistent) no-undercut (Furman and Zitikis, 2008b). In addition, the weighted allocation rules are optimal in the sense of Dhaene et al. (2012). Functional (1.1) provides a common theoretical ground for many RC allocation rules that are found in the literature. We refer to, e.g., Bühlmann (1980, 1984) and Kamps (1998) for the Esscher's and Kamp's allocations, respectively; Overbeck (2000), Panjer and Jing (2001), and Tsanakas and Barnett (2003), Tsanakas (2008) for the Conditional Tail Expectation (CTE) and, more generally, distorted allocations; Choo and de Jong (2015) for the trade-off allocations. These are, of course, only a few illustrative references.

Numerous research papers have been devoted to deriving expressions for the functional A_w under various weight functions w and joint cumulative distribution functions (c.d.f.'s) $F_{X,Y}$ of (X, Y). For illustrative examples, we refer to Dhaene et al. (2008) for elliptical distributions; Cai and Li (2005) for phase-type distributions; Furman and Landsman (2010) for Tweedie distributions; Vernic (2006, 2011) for skew-normal and Pareto distributions. We note that all of the aforementioned papers consider the weighted RC allocation rule induced by the CTE risk measure, only, that is equation (1.1) with $w(x) = \mathbf{1}\{x > \mathrm{VaR}_q[X]\}$, where $\mathbf{1}$ denotes the indicator function, and $\mathrm{VaR}_q[X],\ q \in (0, 1)$ is the value-at-risk, and even in this very special case, the list of references that we have provided above is incomplete. In general, deriving expressions for the functional A_w even with special weight functions in mind is a rather complicated task, and thus obtaining a 'good looking' formula is rarely feasible. The complexity stems from the dependence between the risk r.v.'s X and Y, and numerous complications are hard to circumvent when the assumption of stochastic independence is unsatisfactory. One of the main contributions of this paper is that we characterize the risk portfolios (X, Y) for which the allocation exercise $A_w[X, S]$ is of the same complexity as the task of determining the aggregate RC $H_w[S]$.

We have arranged the rest of this paper as follows. We set off in Section 2 by formulating an example of $F_{X,Y}$ for which the weighted RC allocation rules can be derived in closed form for every weight function of interest, and then we formulate our main question that we aim to answer in the present paper. In Sections 3 and 4 we state and prove our main results and elucidate them with examples. Specifically, we show that there exists an encompassing class of joint c.d.f.'s with, e.g., gamma, inverse Gaussian, and Pareto margins, and also meaningful dependencies, for which the allocation $A_w[X, S]$ is proportional to the aggregate RC $H_w[S]$, and, remarkably, the coefficient of proportionality does not depend on the choice of the weight function, and, therefore, on the choice of the allocation rule. Our findings are thus akin to the classical Capital Asset Pricing Model (CAPM)

(Levy, 2011; references therein), but unlike the CAPM, we do not require the finiteness of second moments of the involved risk r.v's. In more detail, we discuss this contribution in Section 5. Section 6 contains concluding remarks.

2 A simple example and a question arising from it

Let the c.d.f. $F_{X,Y}$ be bivariate elliptical, succinctly $E(\boldsymbol{\mu}, B, g_2)$, where $\boldsymbol{\mu} = (\mu_X, \mu_Y)$ is a two-dimensional vector of means, B is a positive-definite matrix with diagonal entries b_X^2 and b_Y^2 and the (identical) off-diagonal entries $\gamma_{X,Y}$, and $g_2 : [0, \infty) \to [0, \infty)$ is a density generator, such that $\int_0^\infty g_2(y)dy < \infty$ (e.g., Fang et al., 1990). Further let $r_{X|Y}(y) = \mathbf{E}[X - \mu_X |\ Y = y]$. It is well-known (e.g., Fang et al., 1990) that

$$r_{X|Y}(y) = \frac{\gamma_{X,Y}}{b_Y^2}(y - \mu_Y) \quad \text{for all} \quad y \in (-\infty, \infty),$$

and so the regression function $r_{X|Y}$ is linear.

Proposition 2.1. *Let $(X, Y) \sim E(\boldsymbol{\mu}, B, g_2)$, and set $\mu_S = \mu_X + \mu_Y, b_S^2 = b_X^2 + 2\gamma_{X,Y} + b_Y^2$ and $\gamma_{X,S} = b_X^2 + \gamma_{X,Y}$. Then, if $H_w[S]$ is well-defined and finite, we have the equation*

$$A_w[X, S] = \mu_X + \frac{\gamma_{X,S}}{b_S^2}\left(H_w[S] - \mu_S\right) \tag{2.1}$$

for all legitimate weight functions w.

Proof. Since the distribution of (X, S) is elliptical (Fang et al., 1990), the assertion follows from Proposition 4.1 in Furman and Zitikis (2008b). □

Equation (2.1) is of the kind we are looking for. Namely, it reduces the determination of $A_w[X, S]$ to calculating the aggregate RC $H_w[S]$. It is also simple and easy to convey to senior management. Moreover, the equation recovers, e.g., the CTE-type results of Panjer and Jing (2001) and, more generally, Dhaene et al. (2008) when we set $w(x) = \mathbf{1}\{x > \mathrm{VaR}_q[X]\}$. Of course, equation (2.1) is valid for other than the just-mentioned weight functions w, and so, in turn, it is valid for a great variety of RC allocation rules.

The drawback of Proposition 2.1 is that it assumes ellipticity of the joint c.d.f. $F_{X,Y}$, which makes equation (2.1) unattractive for actuaries because the class of elliptical distributions is hardly a good candidate for modelling insurance risks. Hence, the problem that engages us throughout the rest of the present paper is this:

Problem 2.1. *Let X and Y be (possibly dependent) risk r.v.'s, and let $S = X + Y$. Are there risk pairs (X, Y) as well as constants α and β such that the equation*

$$A_w[X, S] = \alpha + \beta \times H_w[S] \tag{2.2}$$

holds for all legitimate weight functions w?

We conclude this section with a connection of Problem 2.1 to pricing. Recall that the risk measure H is called an actuarial premium calculation principle (p.c.p.) if, first, $H[X]$ depends solely on the c.d.f. of the loss r.v. X, a property which is known as 'conditional state independence', and, second, the safety-load bound $H[X] \geq \mathbf{E}[X]$ holds for all X that have finite means. Hence, subject to the condition that the weight function w is non-decreasing, weighted risk measure (1.2) can be used as an actuarial p.c.p. If we want to relax the conditional state independence and therefore allow the pricing functional to depend on some exogenous random factors, such as the aggregate risk r.v. S, then allocation rule (1.1) can be interpreted as an economic p.c.p. (Bühlmann, 1980, 1984; Furman and Zitikis, 2009). In the latter case, and coming back to the set-up in Proposition 2.1, if we assume that the matrix B is the variance-covariance matrix and so $\gamma_{X,S}/b_S^2 = \mathbf{Cov}[X, S]/\mathbf{Var}[S]$, then equation (2.1) in a way rediscovers the classical CAPM equation (Levy, 2011; and references therein).

Intuitively, the CAPM relates the expected return on an asset to the expected return on the market portfolio of all assets in the economy. More precisely, let the r.v.'s X and S denote, respectively, the return on the risky asset and the return on the market portfolio of all assets in the economy. Then the CAPM implies that the expected return on the risky asset of interest is the risk-free rate of interest r_f plus a risk premium, that is,

$$\mathbf{E}[X] = r_f + \frac{\mathbf{Cov}[X, S]}{\mathbf{Var}[S]}(\mathbf{E}[S] - r_f). \tag{2.3}$$

The similarity of the CAPM equation and equation (2.1) is of course evident. Namely, setting $w(x) \equiv const$ in (2.1), we see that the difference between the two equations is mainly terminological and is stipulated by the nature of financial and insurance pricing: for example, the CAPM 'loads' the risk-free interest rate that serves as a reasonable starting point in general finance, whereas equation (2.1) substitutes the r_f with the so-called net premium of the risk r.v. $X \in \mathcal{X}$.

Equation (2.1) is a special case of another CAPM-like equation (2.2), In the following we demonstrate that the answer to the question posed in Problem 2.1 is in affirmative, and we call equation (2.2) the Weighted Insurance Pricing Model (Furman and Zitikis, 2009; see also Furman and Zitkis, 2017, for additional details).

3 Multiplicative systematic risk and the WIPM

Apart from the specific expressions of the two coefficients, equation (2.2) holds in the elliptical case (see equation (2.1)) not because of the elliptical distribution per se but due to the linearity of the centred regression function $r_{X|S}$. This observation is the basis for our research in the current section, which we divide into two parts: general considerations in Section 3.1 and illustrative examples in Section 3.2.

3.1 General considerations

Theorem 3.1. *Let X and S be two non-negative r.v.'s with finite means. Furthermore, let the conditional expectation $\mathbf{E}[X \mid S < \epsilon]$ converge to 0 when $\epsilon \downarrow 0$. If there exists a constant β such that*

$$\mathbf{E}[X \mid S] = \beta S \quad a.s. \tag{3.1}$$

then we have

$$\mathbf{Cov}[X, w(S)] = \beta\ \mathbf{Cov}[S, w(S)] \tag{3.2}$$

for every weight function $w : [0, \infty) \to (-\infty, \infty)$ for which the covariances above are well-defined and finite. In the opposite direction: if there exists a constant β such that equation (3.2) holds for all the specified weight functions, then equation (3.1) holds. Also, we necessarily have the equation $\beta = \mathbf{E}[X]/\mathbf{E}[S]$.

Note 3.1. In the theorem, we impose a condition on the expectation $\mathbf{E}[X \mid S < \epsilon]$, for which to be well defined, we implicitly assume $\mathbf{P}[0 < S < \epsilon] > 0$ for all $\epsilon > 0$.

Proof of Theorem 3.1. To prove the first part of the theorem, we assume equation (3.1) and have, for every legitimate weight function w, that

$$\begin{aligned}\mathbf{E}[Xw(S)] &= \mathbf{E}[\mathbf{E}[X \mid S]w(S)] \\ &= \beta\ \mathbf{E}[Sw(S)],\end{aligned}$$

which also implies $\mathbf{E}[X] = \beta\ \mathbf{E}[S]$ for the choice $w(s) \equiv 1$. Equation (3.2) follows.

To prove the second part of the theorem, we assume that equation (3.2) holds for all legitimate weight functions w for which the two covariances are well-defined and finite. Therefore equation (3.2) reduces to

$$\mathbf{E}[v(S)u(S)] = 0 \tag{3.3}$$

with the notation $v(S) = \mathbf{E}[X \mid S] - \beta S$ and $u(S) = w(S) - \mathbf{E}[w(S)]$. Since equation (3.3) holds for every function u for which the expectation is well-defined, we must have $v(S) = \alpha$ almost surely, for some constant α. This implies $\mathbf{E}[X \mid S] = \alpha + \beta S$ almost surely, and thus

$$\begin{aligned} \mathbf{E}[X \mid S < \epsilon] &= \mathbf{E}[\mathbf{E}[X \mid S] \mid S < \epsilon] \\ &= \mathbf{E}[\alpha + \beta S \mid S < \epsilon] \\ &= \alpha + \beta \mathbf{E}[S \mid S < \epsilon]. \end{aligned} \tag{3.4}$$

Since the expectation on the left-hand side of equation (3.4) converges to 0 when $\epsilon \downarrow 0$ by assumption, and since $\mathbf{E}[S \mid S < \epsilon]$ obviously converges to 0, equation (3.4) therefore implies that we must have $\alpha = 0$ irrespective of the value of β. This completes the proof of Theorem 3.1. □

Insurance risks take non-negative values and are positively skewed, which immediately rules out the class of elliptical distributions. In the next theorem we establish necessary and sufficient conditions on the joint c.d.f. of X and Y such that condition (3.1) is satisfied and, therefore, equation (2.2) holds.

Theorem 3.2. *Let X and Y be two non-negative r.v.'s with finite and positive means $\mathbf{E}[X] > 0$ and $\mathbf{E}[Y] > 0$, and let $S = X + Y$. There exists a constant $\beta > 0$ such that equation (3.1) holds if and only if the function*

$$t \mapsto \frac{(d/du)\mathcal{L}(u,v)}{(d/dv)\mathcal{L}(u,v)}\Big|_{(u,v)=(t,t)} \tag{3.5}$$

is constant, say $\gamma > 0$, on the real half-line $[0, \infty)$, where $\mathcal{L}(u,v) = \mathbf{E}[\exp\{-uX - vY\}]$ is the joint Laplace transform of (X, Y). The aforementioned constant β must be equal to $\mathbf{E}[X]/\mathbf{E}[S]$, and the constant γ must be equal to $\mathbf{E}[X]/\mathbf{E}[Y]$.

Proof. Assuming the validity of equation (3.1), we have

$$\begin{aligned} \mathbf{E}[Xe^{-tS}] &= \mathbf{E}[\mathbf{E}[X \mid S]e^{-tS}] \\ &= \beta\ \mathbf{E}[Se^{-tS}] \\ &= \beta\ \mathbf{E}[Xe^{-tS}] + \beta\ \mathbf{E}[Ye^{-tS}], \end{aligned}$$

which is equivalent to

$$\mathbf{E}[Xe^{-tS}] = \frac{\beta}{1-\beta}\mathbf{E}[Ye^{-tS}]. \tag{3.6}$$

Hence, function (3.5) is constant with γ equal to $\beta/(1-\beta)$, which is equal to $\mathbf{E}[X]/\mathbf{E}[Y]$ because $\beta = \mathbf{E}[X]/\mathbf{E}[S]$ as readily follows by setting $t = 0$ in (3.6), or from Theorem 3.1.

Suppose now that function (3.5) is constant, γ, which is equivalent to saying that the equation

$$\mathbf{E}[Xe^{-tS}] = \gamma\ \mathbf{E}[Ye^{-tS}] \tag{3.7}$$

holds for all $t \geq 0$. Note at the outset that by setting $t = 0$, we get $\gamma = \mathbf{E}[X]/\mathbf{E}[Y]$. With $g(S) := \mathbf{E}[X \mid S]$ and $h(S) := \mathbf{E}[Y \mid S]$, which are of course non-negative, equation (3.7) becomes

$$\mathbf{E}[g(S)e^{-tS}] = \gamma \mathbf{E}[h(S)e^{-tS}], \tag{3.8}$$

which holds for all $t \geq 0$. Since $g(S) + h(S) = S$ almost surely, and because of the uniqueness of the Laplace transform, we must have the equation

$$\frac{g(S)}{\mathbf{E}[X]} = \frac{S - g(S)}{\mathbf{E}[Y]} \quad \text{a.s.},$$

which is equivalent to

$$g(S) = \frac{\mathbf{E}[X]}{\mathbf{E}[X] + \mathbf{E}[Y]} S \quad \text{a.s.}.$$

Using the definition of $g(S)$, the latter equation can equivalently be rewritten as $\mathbf{E}[X \mid S] = \beta S$ almost surely, with $\beta = \mathbf{E}[X]/\mathbf{E}[S]$, which is what we claim in equation (3.1). The proof of Theorem 3.2 is finished. □

3.2 Illustrative examples

We illustrate Theorem 3.2 with the help of the Multiplicative Background Risk Model (MBRM) that has been quite popular in general economic theory (e.g., Franke et al., 2006; references therein) and actuarial science (e.g., Su, 2016; references therein).

Example 3.1. Let X_1, Y_1 and Z be independent non-negative r.v.'s interpreted as two specific and one systematic (background) risk factors (r.f.'s), respectively. Furthermore, assume that X_1 and Y_1 are identically distributed, whereas the distribution of Z is arbitrary. Set $X = ZX_1$ and $Y = ZY_1$, which means that the pair (X, Y) admits the multiplicative background construction. As in this case $(X, S) =_d (Y, S)$, function (3.5) is constant (i.e., equal to 1) and therefore according to Theorem 3.2, condition (3.2) holds with $\beta = 1/2$, and in turn WIPM equation (2.2) reduces to

$$A_w[X, S] = \frac{1}{2} H_w[S] \tag{3.9}$$

for every legitimate weight function w. Speaking generally, equation (2.2) holds for all identically distributed and exchangeable (not necessarily independent) r.v.'s $X,\ Y$.

Specifically, let X_1 and Y_1 be two independent copies of the standard exponential r.v. $\mathcal{E}$, succinctly $\mathcal{E} \sim Exp(1)$, and let Z be distributed inverse Gamma with the shape and rate parameters equal to $\gamma > 0$ and 1, respectively. The distributions of X and Y are then Pareto (e.g., Arnold, 1983, for specification; Seal, 1980, for an extensive list of empirical applications in the general and life insurance). Moreover, the joint distribution of X and Y is then the classical bivariate Pareto (e.g., Arnold, 1983, for specification; e.g., Albrecher et al., 2011, for applications in ruin theory) with the probability density function (p.d.f.)

$$f(x, y;\ \gamma) = \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)}(1+x+y)^{-(\gamma+2)}, \tag{3.10}$$

where $x \geq 0$ and $y \geq 0$. The distribution of $S = X + Y$ is therefore the so-called generalized Pareto, and hence

$$A_w[X, S] = \frac{1}{2}\int_0^{\infty} \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)} \frac{w(s)s^2}{\mathbf{E}[w(S)](1+s)^{\gamma+2}} ds, \tag{3.11}$$

which can be computed explicitly for many choices of the weight function w. Sarabia et al. (2016) have recently considered an application of the MBRM discussed in this example in the context of automobile insurance claims, and remarkably formula (3.11) unifies many of their results. This concludes Example 3.1.

The Pareto distribution has regularly provided adequate fits to heavy-tailed insurance data (Seal, 1980). If lighter tails are of interest, then we recommend to model the systematic r.f. Z in Example 3.1 using the p.d.f. (Albrecher et al., 2011)

$$f(z;\ \gamma) = \frac{\sin(\gamma\pi)}{\pi z}\left(\frac{z}{\alpha} - 1\right)^{-\gamma}, \quad z > \alpha > 0, \quad \gamma \in (0,\ 1)$$

and leave the rest of the set-up in the example unchanged. The distributions of the r.v.'s X and Y are then gamma with shape parameter $\gamma \in (0, 1)$ and rate parameter $\alpha > 0$. There are numerous examples of applying gamma distributions for modeling insurance risks (e.g., Hossack et al., 1983; as well as more recent Hürlimann, 2001; Alai et al., 2013).

We next sketch an example in which the random pair (X, Y) has an MBRM-type dependence with non-identically distributed margins, and yet equation (2.2) holds.

Example 3.2. Let the systematic r.f. Z have any distribution on $[0, \infty)$, and let the specific r.f.'s X_1 and Y_1 be jointly bivariate Dirichlet (Fang et al., 1990). We note in passing that, unlike in Example 3.1, the r.v.'s X_1 and Y_1 are stochastically dependent and not identically distributed. However, as before, we assume that they are independent of the r.v. Z. The r.v.'s X and Y, where $X = ZX_1$ and $Y = ZY_1$, are then jointly bivariate Liouville (Fang et al., 1990). We observe that function (3.5) is $\mathbf{E}[X_1]/\mathbf{E}[Y_1]$ and therefore according to Theorem 3.2, we have condition (3.1) with

$$\beta = \mathbf{E}[X_1]/(\mathbf{E}[X_1 + Y_1]) = \mathbf{E}[X_1],$$

where the rightmost equality holds because $X_1 + Y_1 = 1$ almost surely. In turn, WIPM equation (2.2) reduces to

$$A_w[X, S] = \mathbf{E}[X_1] \; H_w[Z]$$

for every legitimate weight function w.

More specifically, let the systematic r.f. Z be distributed generalized Pareto, succinctly $Z \sim GP(\gamma, \xi, \theta)$, where γ, ξ and θ are all positive parameters. Then the distribution of the r.v.'s X and Y is generalized Beta of the 2nd kind (e.g., Cummins et al., 1990, for applications in the context of fire losses experienced by a major university). Furthermore, the joint p.d.f. of the just-mentioned pair of r.v.'s is

$$f(x, y;\; \gamma, \xi_1, \xi_2, \theta) = \theta^{\gamma} \frac{\Gamma(\gamma + \xi)}{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\gamma)} x^{\xi_1 - 1} y^{\xi_2 - 1} \left(\theta + x + y\right)^{-(\gamma+\xi)}, \tag{3.12}$$

where $x \geq 0, y \geq 0$, and ξ_1, ξ_2 are Dirichlet parameters such that $\xi_1 + \xi_2 = \xi$ (Yang et al., 2011, for applications of the underlying copula when modelling multivariate heavy-tailed data). Comparing p.d.f.'s (3.12) and (3.10), we easily observe that this particular member of the Liouville class of distributions can be viewed as a non-exchangeable extension of the Arnold's classical bivariate Pareto distribution discussed in Example 3.1. Nevertheless, even in this significantly more general case, we readily have that

$$A_w[X, S] = \frac{\xi_1}{\xi} \int_0^{\infty} \theta^{\gamma} \frac{\Gamma(\gamma + \xi)}{\Gamma(\gamma)\Gamma(\xi)} \frac{w(s) s^{\xi}}{\mathbf{E}[w(S)](\theta + s)^{\gamma+\xi}} ds,$$

which can be computed explicitly for many choices of the weight function w. This concludes Example 3.2.

4 Additive systematic risk and the WIPM

Just like the previous section, this section is also divided into two parts: general considerations in Section 4.1 and illustrative examples in Section 4.2.

4.1 General considerations

The following theorem deals with the linearity of $\mathbf{E}[X \mid S]$ with respect to S when the r.v.'s X and Y are independent. We find this theorem particularly useful when exploring the WIPM equation in the context of the additive exposure of X and Y to the systematic and specific risk factors.

Theorem 4.1. *Let X and Y be two non-negative and independent r.v.'s with finite and positive means $\mathbf{E}[X] > 0$ and $\mathbf{E}[Y] > 0$, and let $S = X + Y$. Denote by F and G the c.d.f.'s of X and Y, respectively. The following two statements are equivalent to the constancy of function (3.5):*

(1) *The function*

$$t \mapsto \frac{\log \mathcal{L}_X(t)}{\log \mathcal{L}_Y(t)} \tag{4.1}$$

is constant, say $\gamma > 0$, on the real half-line $[0, \infty)$, where $\mathcal{L}_X(t) = \mathbf{E}[\exp\{-tX\}]$ and $\mathcal{L}_Y(t) = \mathbf{E}[\exp\{-tY\}]$ are the Laplace transforms of X and Y, respectively. In this case, γ must be equal to $\mathbf{E}[X]/\mathbf{E}[Y]$.

(2) *The equation*

$$X^* + Y =_d X + Y^* \tag{4.2}$$

holds, where X^* and Y^* are size-biased random variables whose c.d.f.'s F^* and G^* are defined by $dF^*(x) = xdF(x)/\mathbf{E}[X]$ and $dG^*(y) = ydG(x)/\mathbf{E}[Y]$, respectively.

Proof. To show that the constancy of functions (3.5) and (4.1) are equivalent under the independence of X and Y, we first rewrite function (3.5) as

$$t \mapsto \frac{\mathcal{L}'_X(t)\mathcal{L}_Y(t)}{\mathcal{L}_X(t)\mathcal{L}'_Y(t)} = \frac{(d/dt)\log \mathcal{L}_X(t)}{(d/dt)\log \mathcal{L}_Y(t)}. \tag{4.3}$$

Function (4.3) is constant if and only if (Pinelis, 2002, Corollary 1.3 and Remark 1.5) the function

$$t \mapsto \frac{\log \mathcal{L}_X(t)}{\log \mathcal{L}_Y(t)} \tag{4.4}$$

is constant, because $\log \mathcal{L}_X(0) = 0$ and $\log \mathcal{L}_Y(0) = 0$. This completes the proof of the equivalence of the constancy of functions (3.5) and (4.1), and so the proof of part (1).

To establish the equivalence of parts (1) and (2), we first recall that part (1) is equivalent to

$$\frac{\mathcal{L}'_X(t)\mathcal{L}_Y(t)}{\mathcal{L}_X(t)\mathcal{L}'_Y(t)} = \gamma \tag{4.5}$$

for all $t \geq 0$, where $\gamma = \mathbf{E}[X]/\mathbf{E}[Y]$. Equation (4.5) can of course be rewritten as

$$\mathbf{E}\left[\frac{X}{\mathbf{E}[X]}e^{-t(X+Y)}\right] = \mathbf{E}\left[\frac{Y}{\mathbf{E}[Y]}e^{-t(X+Y)}\right]. \tag{4.6}$$

Since X and Y are independent, equation (4.6) can be further rewritten as

$$\mathbf{E}[e^{-t(X^*+Y)}] = \mathbf{E}[e^{-t(X+Y^*)}], \tag{4.7}$$

and since it holds for every $t \geq 0$, this means that the Laplace transforms of the r.v.'s $X^* + Y$ and $X + Y^*$ are identical. This, in turn, implies that the r.v.'s $X^* + Y$ and $X + Y^*$ have identical c.d.f.'s, which establishes the equivalence of equation (4.2) and the constancy of function (4.1). This concludes the proof of Theorem 4.1. □

4.2 Illustrative examples

We exemplify Theorem 4.1 using the Additive Background Risk Model (ABRM) that has been discussed extensively in general economic theory (Gollier and Pratt, 1996; references therein) and actuarial science (Tsanakas, 2008; Furman and Landsman, 2010; Ostaszewski and Xu, 2012; Alai et al., 2013; Avanzi et al., 2016; references therein). We start with a simple case, which complements Example 3.1 by relaxing the assumption of identical distributions on the r.v.'s therein, and in this way provides a basis for our next Example 4.2, where the risks X and Y are dependent.

Example 4.1. Let X and Y be two non-negative and stochastically independent r.v.'s having infinitely divisible c.d.f.'s and such that the equation $\mathcal{L}_X(t) = (L_Y(t))^\gamma$ holds for all $t \geq 0$ and $\gamma = \mathbf{E}[X]/\mathbf{E}[Y]$. This implies (Pakes et al., 1996) that $X^* =_d X + W_X$ and $Y^* =_d Y + W_Y$, where W_X and W_Y are two r.v.'s independent of X and Y, respectively. Then condition (4.2) holds if $W_X =_d W_Y$. In this case, condition (3.2) holds with $\beta = \mathbf{E}[X]/\mathbf{E}[S]$, and the WIPM equation becomes

$$A_w[X, S] = \frac{\mathbf{E}[X]}{\mathbf{E}[S]} H_w[S] \tag{4.8}$$

for every legitimate weight function w.

To specialize, let X and Y be independent and gamma distributed r.v.'s with positive shape parameters γ_X and γ_Y, respectively, and the rate parameters 1. Condition (4.2) holds because gamma is an infinitely divisible distribution, and $W_X =_d W_Y \sim \mathcal{E}$. Therefore, the WIPM equation holds and, with the notation $\gamma_+ = \gamma_X + \gamma_Y$, is given by

$$A_w[X, S] = \frac{\gamma_X}{\gamma_+} \int_0^\infty e^{-s} \frac{w(s)s^{\gamma_+}}{\mathbf{E}[w(S)]\Gamma(\gamma_+)} ds.$$

The expression for the A_w can be further simplified for special weight functions w. This completes Example 4.1.

Expression (4.8) also holds for other classes of r.v.'s, such as Poisson, negative binomial (discrete), and compound Poisson with gamma secondary (severity) distribution (mixed). All of these distributions are popular when modeling risks in general insurance (e.g., Hossack et al., 1983).

Example 4.2. Let X_1, Y_1, and Z be independent (but not necessarily identically distributed) r.v.'s having the support $[0, \infty)$ and interpreted as two specific and one systematic r.f.'s, and let the r.v.'s admit the stochastic representations $X = Z + X_1$ and $Y = Z + Y_1$. We refer to, e.g., Cardinale et al. (2006) and Avanzi et al. (2016) for an extensive actuarial discussion. We are interested in the linearity of $\mathbf{E}[X \mid S]$ with respect to $S = X + Y$. To derive conditions under which the aforementioned linearity holds, we start with the equation

$$\mathbf{E}[X \mid S = s] = \frac{1}{2}\mathbf{E}[2Z \mid 2Z + X_1 + Y_1 = s] + \mathbf{E}[X_1 \mid X_1 + 2Z + Y_1 = s]. \tag{4.9}$$

According to Theorem 4.1, the two expectations on the right-hand side of equation (4.9) are linear functions of $s \geq 0$ if and only if, respectively, the functions

$$t \mapsto \frac{\log \mathcal{L}_{2Z}(t)}{\log \mathcal{L}_{X_1}(t) + \log \mathcal{L}_{Y_1}(t)} \quad \text{and} \quad t \mapsto \frac{\log \mathcal{L}_{X_1}(t)}{\log \mathcal{L}_{2Z}(t) + \log \mathcal{L}_{Y_1}(t)}$$

are constant. This happens whenever the functions

$$t \mapsto \frac{\log \mathcal{L}_{X_1}(t)}{\log \mathcal{L}_{2Z}(t)} \quad \text{and} \quad t \mapsto \frac{\log \mathcal{L}_{Y_1}(t)}{\log \mathcal{L}_{2Z}(t)} \tag{4.10}$$

are constant. According to Theorem 4.1, the latter two functions are constant if and only if there are constants β_1 and β_2 such that

$$\mathbf{E}[X_1 \mid X_1 + 2Z = s] = \beta_1 s \quad \text{for all} \quad s \geq 0$$

and

$$\mathbf{E}[Y_1 \mid Y_1 + 2Z = s] = \beta_2 s \quad \text{for all} \quad s \geq 0,$$

where

$$\beta_1 = \frac{\mathbf{E}[X_1]}{\mathbf{E}[X_1] + 2\mathbf{E}[Z]} \quad \text{and} \quad \beta_2 = \frac{\mathbf{E}[Y_1]}{\mathbf{E}[Y_1] + 2\mathbf{E}[Z]}.$$

Thus

$$\mathbf{E}[X \mid S = s] = \beta s \quad \text{for all} \quad s \geq 0 \tag{4.11}$$

with

$$\beta = \frac{\dfrac{1+\beta_1}{1-\beta_1}}{\dfrac{1+\beta_1}{1-\beta_1}+\dfrac{1+\beta_2}{1-\beta_2}} = \frac{\mathbf{E}[X_1]+\mathbf{E}[Z]}{2\mathbf{E}[Z]+\mathbf{E}[X_1]+\mathbf{E}[Y_1]}. \tag{4.12}$$

In summary, if both functions (4.10) are constant, then statement (4.11) holds with the constant β given by formula (4.12), and the WIPM equation reduces to

$$A_w[X,S] = \frac{\mathbf{E}[X_1]+\mathbf{E}[Z]}{2\mathbf{E}[Z]+\mathbf{E}[X_1]+\mathbf{E}[Y_1]} H_w[S].$$

As a special case, let X_1, Y_1 and Z be independent and gamma distributed r.v.'s such that both functions in (4.10) are constant. Namely, we let X_1 and Y_1 have shape parameters $\gamma_1 > 0$ and $\gamma_2 > 0$, respectively, and the rate parameter $\alpha > 0$, and we also choose the systematic r.f. as $Z \sim Ga(\gamma_0, 2\alpha)$. Consequently, the (dependent) r.v.'s X and Y are gamma (e.g., Hossack et al., 1983, for actuarial applications). Then, with the notation $\gamma_+ = \gamma_0 + \gamma_1 + \gamma_2$, condition (3.2) holds with

$$\beta = \frac{\gamma_0 + 2\gamma_1}{\gamma_+},$$

and the WIPM equation reduces to

$$A_w[X,S] = \frac{\gamma_0 + 2\gamma_1}{2\gamma_+} \int_0^\infty e^{-\alpha s} \frac{w(s)(\alpha s)^{\gamma_+}}{\mathbf{E}[w(S)]\Gamma(\gamma_+)} ds$$

for every legitimate weight function w. This can further be rewritten as

$$A_w[X,S] = \frac{\gamma_0 + 2\gamma_1}{2\gamma_+} \mathbf{E}[v(S)], \tag{4.13}$$

where $v(s) = sw(s)/\mathbf{E}[w(S)]$ and $S \sim Ga(\gamma_+, \alpha)$.

An interesting generalization of the idea above is to consider the compound Poisson (CP) distribution with gamma secondary (severity) distribution and the Laplace transform

$$\mathcal{L}(t) = \exp\left\{\lambda(1+t/\alpha)^{-\gamma} - 1)\right\} \quad \text{for all} \quad t \geq 0,$$

with positive parameters λ, γ and α. Namely, assume that X_1 and Y_1 are $CP(\lambda_1, \gamma, \alpha)$ and $CP(\lambda_2, \gamma, \alpha)$, respectively, and let Z be $CP(\lambda_0, \gamma, 2\alpha)$. Then the r.v.'s X and Y are CP with the gamma severity-distribution (Smyth and Jorgensen, 2002; Wüthrich, 2003; Boucher and Davidov, 2012, for applications in claim reserving and estimating run-off triangles), and, with the notation $\lambda_+ = \lambda_0 + \lambda_1 + \lambda_2$, we have

$$\begin{aligned} A_w[X,S] &= \frac{\lambda_0 + 2\lambda_1}{2\lambda_+} \sum_{k=1}^{\infty} \frac{e^{-\lambda_+} k^{\lambda_+}}{k!} \int_0^\infty \frac{w(s)}{\mathbf{E}[w(S)]} \frac{e^{-\alpha s}(s\alpha)^{k\gamma}}{\Gamma(k\gamma)} ds \\ &= \frac{\lambda_0 + 2\lambda_1}{2\lambda_+} \sum_{k=1}^{\infty} \frac{e^{-\lambda_+} k^{\lambda_+}}{k!} \mathbf{E}[v(S_k)], \end{aligned}$$

where $v(s) = sw(s)/\mathbf{E}[w(S)]$ and S_k is the k-fold convolution of independent and identically distributed gamma r.v.'s. This concludes Example 4.2.

Equation (4.13) also holds more generally for the r.f.'s that are Tweedie exponential dispersion models. We recall in this respect that all Tweedie distributions are infinitely divisible and scale invariant, which are key factors for functions (4.10) to be constant. Besides the gamma and CP with gamma severities, the class of Tweedie distributions contains, e.g., the Poisson, inverse Gaussian and positive stable distributions, all of which are appropriate candidates for modelling insurance risks (e.g., Avanzi et al., 2016; references therein).

5 Variability without variance: Gini-type WIPM

In CAPM equation (2.3), the variance acts as a measure of dispersion, and as a result, the CAPM cannot be applied on risks that do not have finite second moments. As such risks are quite common in finance (e.g., Rachev et al., 2005), this has been considered a serious drawback of the CAPM. Owen and Rabinowitch (1983) circumvent this problem by employing the scale parameters of the elliptical distribution as measures of dispersion. Unfortunately, their solution only works when the joint c.d.f. of X and Y is jointly elliptical, which is very rarely the case in the context of insurance risks. That being said, ample empirical evidence for the insurance risks with infinite second moments have been reported (e.g., in Seal, 1980).

An alternative measure of dispersion, which neither requires finiteness of the variance nor is related to the elliptical family of distributions, is the Gini Mean Difference (GMD). Specifically, the GMD functional is given by

$$\mathrm{GMD}[X] = \mathbf{E}[|X^* - X^{**}|\,], \tag{5.1}$$

where X^* and X^{**} are two independent copies of $X \in \mathcal{X}$. Under the assumption that X has a continuous c.d.f. F_X, the GMD admits the following form

$$\mathrm{GMD}[X] = 4\ \mathbf{Cov}[X, F_X(X)] \tag{5.2}$$

(e.g., Yitzhaki and Schechtman, 2013; references therein).

In view of equation (5.2), we call the earlier discussed weighted functional with $w_X(x) = w\left(F_X(x)\right)$ the Gini weighted functional. According to Theorem 3.1, in this case and when the regression function $r_{X|S}$ is linear, equation (2.2) is given by

$$A_{w,\mathrm{Gini}}[X,S] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, F_S(S)]}{\mathbf{Cov}[S, F_S(S)]}\left(H_{w,\mathrm{Gini}}[S] - \mathbf{E}[S]\right), \tag{5.3}$$

where

$$H_{w,\mathrm{Gini}}[S] = \frac{\mathbf{E}[Sw(F_S(S))]}{\mathbf{E}[w(F_S(S))]}.$$

We next illustrate equation (5.3) with two examples.

Example 5.1. Let X and Y have the jointly elliptical c.d.f. with expectations μ_X, μ_Y, positive-definite matrix B with diagonal entries b_X^2 and b_Y^2 and off-diagonal entries $\gamma_{X,Y} = \gamma_{Y,X}$, and a density generator g_2. From, e.g., Fang et al. (1990), we readily have that $S \sim E(\mu_S, b_S^2, g_1)$, where $\mu_S = \mu_X + \mu_Y$ and $b_S^2 = b_X^2 + 2\gamma_{X,Y} + b_Y^2$, and we also have that equation (3.2) holds with $b = \gamma_{X,S}/b_S^2$. As a result, the Gini WIPM equation reduces to

$$A_{w,\mathrm{Gini}}[X,S] = \mu_X + \frac{\gamma_{X,S}}{b_S^2}\left(\int_{-\infty}^{\infty} \frac{s w_S(s)}{\mathbf{E}[w_S(S)]}\frac{c_1}{b_S}\; g_1\left(\frac{1}{2}\left(\frac{s-\mu_S}{b_S}\right)^2\right) ds - \mathbf{E}[S]\right) \tag{5.4}$$

with the normalizing constant

$$c_1 = \left(\sqrt{2}\int_0^{\infty} s^{-1/2} g_1(s) ds\right)^{-1}$$

and the density generator $g_1 : [0,\infty) \to [0,\infty)$ satisfying

$$\int_0^{\infty} s^{-1/2} g_1(s) ds < \infty.$$

Besides the already mentioned in this paper CTE-type allocation formula, equation (5.4) encompasses the Gini Shartfall allocation (Furman et al., 2017), and, more generally, the distorted allocation-type formulas (Tsanakas and Barnett, 2003), among others, for the jointly elliptical risk r.v.'s X and Y. This concludes Example 5.1.

As we already mentioned, in actuarial science and, in particular, in non-life insurance, risk r.v.'s without finite second moments manifest frequently (e.g., Seal, 1980). Theorems 3.1 and 3.2 imply in this respect that for non-negative X and Y, the Gini WIPM is given by

$$A_{w,\mathrm{Gini}}[X,S] = \frac{\mathbf{E}[X]}{\mathbf{E}[S]} H_{w,\mathrm{Gini}}[S] \tag{5.5}$$

for every legitimate weight function for which $H_{w,\mathrm{Gini}}[S]$ is well-defined and finite.

Example 5.2. As in Example 3.2, let the r.v.'s X and Y be jointly Liouville (Fang et al., 1990), and let the systematic r.f. be generalized Pareto; we refer to Arnold (1983) for the specification, and to Cummins et al. (1990) for applications in the context of fire losses of a major university. The p.d.f. of the pair (X, Y) is then given by

$$f(x, y;\ \gamma, \xi_1, \xi_2, \theta) = \theta^\gamma \frac{\Gamma(\gamma+\xi)}{\Gamma(\xi_1)\Gamma(\xi_2)\Gamma(\gamma)} x^{\xi_1-1} y^{\xi_2-1} (\theta+x+y)^{-(\gamma+\xi)},$$

where $x \geq 0$, $y \geq 0$, and ξ_1, ξ_2 are Dirichlet parameters such that $\xi_1 + \xi_2 = \xi$. From equation (5.5), we then have that

$$A_{w,\mathrm{Gini}}[X, S] = \frac{\xi_1}{\xi} \int_0^\infty \theta^\gamma \frac{\Gamma(\gamma+\xi)}{\Gamma(\gamma)\Gamma(\xi)} \frac{w_S(s)s^\xi}{\mathbf{E}[w_S(S)](\theta+s)^{\gamma+\xi}} ds.$$

The latter equation can be simplified for a variety of weight functions, and it holds for all risk r.v.'s X and Y that have at least first moments finite. This concludes Example 5.2.

6 Summary

In this paper we have characterized those pairs of risks for which the problems of allocating and determining aggregate risk capitals are of a similar computational complexity. We have shown that the class of distributions for which this happens is quite encompassing and contains joint c.d.f.'s whose marginal distributions have non-negative supports, positive skewness, varying thickness of the right tail, and dependence structures that are meaningful to risk professional.

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References

Alai, D.H., Landsman, Z., Sherris, M. 2013. Lifetime dependence modelling using a truncated multivariate gamma distribution. *Insurance: Mathematics and Economics* **52**(3), 542 – 549.

Albrecher, H., Constantinescu, C., Loisel, S. 2011. Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics and Economics* **48**(2), 265 – 270.

Arnold, B.C. 1983. *Pareto Distributions.* International Cooperative Publishing House.

Avanzi, B., Taylor, G., Vu, P.A., Wong, B. 2016. Stochastic loss reserving with dependence: a flexible multivariate Tweedie approach. *Insurance: Mathematics and Economics* **71**, 63 – 78.

Boucher, J.P., Davidov, D. 2012. On the importance of dispersion models for claims reserving: an application with the Tweedie distributions. *Variance* **5**(2), 158 – 172.

Bühlmann, H. 1980. An economic premium principle. *ASTIN Bulletin: The Journal of the International Actuarial Association* **11**(1), 52 – 60.

Bühlmann, H. 1984. The general economic premium principle. *ASTIN Bulletin: The Journal of the International Actuarial Association* **14**(1), 13 – 21.

Cai, J., Li, H. 2005. Conditional tail expectations for multivariate Phase-type distributions. *Journal of Applied Probability* **42**(3), 810 – 825.

Cardinale, M., Katz, G., Kumar, J., Orszag, J.M. 2006. Background risk and pension. *British Actuarial Journal* **12**(1), 79 – 152.

Choo, W., de Jong, P. 2015. The trade-off insurance premium as a two-sided generalisation of the distortion premium. *Insurance: Mathematics and Economics* **65**, 238 – 246.

Cummins, J.D., Dionne, G., McDonald, J.B., Pritchett, B.M. 1990. Applications of the GB2 family of distributions in modelling insurance loss processes. *Insurance: Mathematics and Economics* **9**, 257 – 272.

Denault, M. 2001. Coherent allocation of risk capital. *Journal of Risk* **4**(1), 7 – 21.

Dhaene, J., Henrard, L., Landsman, Z., Vandendorpe, A., Vanduffel, S. 2008. Some results on the CTE based capital allocation rule. *Insurance: Mathematics and Economics* **42**(2), 855 – 863.

Dhaene, J., Tsanakas, A., Valdez, E.A., Vanduffel, S. 2012. Optimal capital allocation principles. *Journal of Risk and Insurance* **79**(1), 1 – 28.

Fang, K.T., Kotz, S., Ng, K.W. 1990. *Symmetric Multivariate and Related Distri-*

butions. Chapman & Hall.

Franke, G., Schlesinger, H., Stapleton, R.C. 2006. Multiplicative background risk. *Management Science* **52**, 146 – 153.

Furman, E., Landsman, Z. 2010. Multivariate Tweedie distributions and some related capital-at-risk analysis. *Insurance: Mathematics and Economics* **46**(2), 351 – 361.

Furman, E., Zitikis, R. 2008a. Weighted premium calculation principles. *Insurance: Mathematics and Economics* **42**(1), 459 – 465.

Furman, E., Zitikis, R. 2008b. Weighted risk capital allocations. *Insurance: Mathematics and Economics* **43**(2), 263 – 269.

Furman, E., Zitikis, R. 2009. Weighted pricing functionals with applications to insurance: an overview. *North American Actuarial Journal* **13**, 483 – 496.

Furman, E., Zitkis, R. 2017. An adaptation of the classical CAPM to insurance: the weighted insurance pricing model. *Casualty Actuarial Society E-Forum*, **2017**, 1–12.

Furman, E., Wang, R., Zitkis, R. 2017. Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks. *Journal of Banking and Finance*, **83**, 70–84.

Gollier, C., Pratt, J.W. 1996. Risk vulnerability and the tempering effect of background risk. *Econometrica* **64**(5), 1109 – 1123.

Hossack, I., Polard, J., Zehnwirth, B. 1983. *Introductory Statistics with Applications in General Insurance.* Cambridge University Press, Cambridge.

Hürlimann, W. 2001. Analytical evaluation of economic risk capital for portfolio of gamma risks. *ASTIN Bulletin: The Journal of the International Actuarial Association* **31**(1), 107 – 122.

Kamps, U. 1998. On a class of premium principles including the Esscher premium. *Scandinavian Actuarial Journal* **1**, 75 – 80.

Levy, H. 2011. *The Capital Asset Pricing Model in the 21st Century: Analytical, Empirical, and Behavioral Perspectives.* Cambridge University Press, Cambridge.

McNeil, A.J., Frey, R., Embrechts, P. 2005. *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton.

Ostaszewski, K., Xu, M. 2012. *Optimal Capital Allocation: Mean-Variance Models.* Technical Report. Illinois State University, Normal, IL.

Overbeck, L. 2000. Allocation of economic capital in loan portfolios. *Measuring risk in complex systems*, Franke, J., Haerdle, W., Stahl, G. (eds), Springer.

Owen, J., Rabinovitch, R., 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice. *Journal of Finance* **38**(3), 745 – 752.

Pakes, A.G., Sapatinas, T., Fosam, E.B. 1996. Characterization, length-biasing and infinite divisibility. *Statistical Papers* **37**, 53 – 69.

Panjer, H., Jing, J. 2001. *Solvency and capital allocation.* University of Waterloo, Institute of Insurance and Pension Research, Research Report 01-14, 1 – 8.

Pinelis, I. 2002. L'Hospital type rules for monotonicity with applications. *Journal of Inequalities in Pure and Applied Mathematics*, **3**(1), Article 5.

Rachev, S., Stoyanov, S., Biglova, A., Fabozzi, F. 2005. An empirical examination of daily stock return distributions for U.S. stocks. In D. Baier, R. Decker, L. Schmidt-Thieme (Eds.), *Springer Series in Studies in Classification, Data Analysis, and Knowledge Organization. Data Analysis and Decision Support.* Berlin: Springer.

Sarabia, J.M., Gómez-Dńiz, E., Prieto, F., Jordá, V. 2016. Risks aggregation in multivariate dependent Pareto distributions. *Insurance: Mathematics and Economics* **71**, 154 – 163.

Schechtman, E., Yitzhaki, S. 1987. A measure of association based on Gini's Mean Difference. *Communications in Statistics Theory and Methods* **A16**, 207 – 231.

Seal, H.L. 1980. Survival probabilities based on Pareto claim distributions. *ASTIN Bulletin: The Journal of the International Actuarial Association* **11**, 61 – 71.

Smyth, G.K., Jorgensen, B. 2002. Fitting Tweedie's compound Poisson model to insurance claims data: dispersion modelling. *ASTIN Bulletin: The Journal of the International Actuarial Association* **32**(01), 143 – 157.

Su, J. 2016. *Multiple Risk Factors Dependence Structures with Applications to Actuarial Risk Management.* Ph.D. Dissertation, York University, Toronto, Canada.

Tsanakas, A. 2008. Risk measurement in the presence of background risk. *Insurance: Mathematics and Economics* **42**(2), 520 – 528.

Tsanakas, A., Barnett, C. 2003. Risk capital allocation and cooperative pricing of insurance liabilities. *Insurance: Mathematics and Economics* **33**(2), 239 – 254.

Venter, G. G. 2004. Capital allocation survey with commentary. *North American Actuarial Journal* **8**(2), 96 – 107.

Vernic, R. 2006. Multivariate skew-normal distributions with applications in insurance. *Insurance: Mathematics and Economics* **38**(2), 413 – 426.

Vernic, R. 2011. Tail conditional expectation for the multivariate Pareto distribution of the second kind: Another approach. *Methodology and Computing in Applied Probability* **13**(1), 121 – 137.

Wüthrich, M. 2003. Claims reserving using Tweedie's compound Poisson model. *ASTIN Bulletin: The Journal of the International Actuarial Association* **33**(2), 331 –

346.

Yang, X., Frees, E.W., Zhang, Z. 2011. A generalized beta copula with applications in modeling multivariate long-tailed data. *Insurance: Mathematics and Economics* **49**(2), 265 – 284.