Math 2320M
Instructor: A. Premat

Midterm Exam
March 22nd, 2006

Name: ________________________

ID number: ________________________

NO BOOKS, CALCULATORS OR NOTES ARE PERMITTED. ALSO, NO ANTENNAE TO TALK TO THE ALIENS

Question		Value	Marks
Q1	(a)	3	
	(b)	7	
Q2		8	
Q3	(a)	7	
	(b)	3	
Q4	(a)	4	
	(b)	5	
Q5	(a)	4	
	(b)	4	
	(c)	5	
Total		50	

Question 1.

(a) Define what it means for a set S to be *countable*.

A set S is said to be countable if either it is finite or it has the same cardinality as the set of natural numbers, i.e. there exists a one-to-one correspondence between S and $\mathbb{N}$.

(b) Let S be the set of integers which are divisible by 3; in other words $S = \{3k : k \in \mathbb{Z}\}$. Show that S is countable.

Define $f : \mathbb{N} \to S$ by

$$f(n) = \begin{cases} 3\left(\frac{n}{2}\right) & \text{if } n \text{ is even,} \\ -3\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

The function f is a one-to-one correspondence between $\mathbb{N}$ and S; hence S is countable.

Question 2. Prove that $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ for all $n \geq 1$.

Let $P(n): \ 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Basis step:
$P(1): \ 1^3 = \left[\frac{1(1+1)}{2}\right]^2$ is true.

Inductive step:
Assume $P(k)$ is true, i.e. $1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$.
Then

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= k^2 \frac{(k+1)^2}{2^2} + (k+1)^3 \\
&= \frac{(k+1)^2}{2^2}\left[k^2 + 4(k+1)\right] \\
&= \frac{(k+1)^2}{2^2}\left[k^2 + 4k + 4\right] \\
&= \frac{(k+1)^2}{2^2}(k+2)^2 \\
&= \left[\frac{(k+1)((k+2)}{2}\right]^2
\end{aligned}$$

Hence $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Question 3.

(a) Find the solution to the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ with initial conditions $a_0 = 3$ and $a_1 = 4$.

The characteristic equation is $r^2 - r - 6 = (r-3)(r+2) = 0$, and its roots are 3 and -2. Hence the general solution to the recurrence relation is $a_n = \alpha\, 3^n + \beta\,(-2)^n$ for some constants α and β.

From the initial conditions, we get

$$3 = a_0 = \alpha + \beta$$
$$4 = a_1 = 3\alpha - 2\beta$$

Hence $\alpha = 2$ and $\beta = 1$, and the solution to the recurrence relation with initial conditions is

$$a_n = 2\,(3)^n + (-2)^n$$

(b) If the characteristic equation of a recurrence relation is

$$(r-2)^2(r+3)^3(r+5),$$

what is the general solution of the recurrence relation?

The general solution is $(A+Bn)(2)^n + (C+Dn+En^2)(-3)^n + F(-5)^n$ where A, B, C, D, E and F are constants.

Question 4.

(a) What is the least number of people needed to ensure that at least three of them have the same birthday? (Assume that nobody's birthday occurs on February 29th.)

By the generalized Pigeonhole Principle, we need to find the smallest $x \in \mathbb{N}$ such that $\lceil \frac{x}{365} \rceil = 3$. Hence $x = 2 \times 365 + 1 = 731$.

(b) How many bit strings of length nine either begin with a 1 or end with two 0's?

The number of bit strings of length nine that either begin with a 1 or end with two 0's

= the number of bit strings of length nine that begin with a 1

+ the number of bit strings of length nine that end with two $0's$

− the number of bit strings of length nine that begin with a 1 and end with two $0's$

$= 2^8 + 2^7 - 2^6$.

Question 5. The Martians have taken over the planet and are selecting people to bring back to Mars to serve as snacks for the "big game" this coming Sunday. To the Martians, all citizens of a given country are indistinguishable, but citizens from different countries are different.

If the Martians have 50 countries (one of which is Canada) from which to choose the humans, in how many different ways, can the Martians choose:

(a) five people if no two are citizens of the same country?

We need to find the number of ways of choosing 5 objects from 50 different types of objects if no repetitions are allowed. This is

$$\binom{50}{5}$$

(b) five people if they can choose as many as they want from any country?

We need to find the number of ways of choosing 5 objects from 50 different types of objects if repetitions are allowed. This is

$$\binom{(50-1)+5}{5} = \binom{54}{5}$$

(c) five people if no more than two are Canadians?

We need to find the number of ways of choosing 5 objects from 50 different types of objects if repetitions are allowed but no more than two objects of one fixed type (Canadian) are chosen. We break up the problem into three mutually exclusive cases:

Case 1. No Canadians are chosen.

Case 2. Exactly one Canadian is chosen.

Case 3. Exactly two Canadians are chosen.

So the answer is the number in Case 1 + the number in Case 2 + the number in Case 3. This is

$$\binom{(49-1)+5}{5} + \binom{(49-1)+4}{4} + \binom{(49-1)+3}{3}$$
$$= \binom{53}{5} + \binom{52}{4} + \binom{51}{3}$$