York University

Faculty of Science and Engineering

Math 1200

Final Examination

NAME (print): _____
(Family) (Given)

SIGNATURE: _____

STUDENT NUMBER: _____

SECTION LETTER: _____
(A: Tuesday) (B: Wednesday) (C: Thursday)

Instructions:

1. Time allowed: 180 minutes.
2. There are 3 questions on 10 pages. Page 10 has no questions on it.
3. Questions can be solved in more than one way.
4. Make sure that you write clearly and carefully using full English sentences.

Question	**Points**	**Marks**
1	48	
2	12	
3	40	
BONUS	3	
Total	$100 + 3$	

IF YOU HAVE COMPLETED THE ONLINE COURSE EVALUATION, SIGN THIS STATEMENT TO RECEIVE YOUR BONUS POINTS.

I have completed the online course evaluation.

SIGNATURE: _____

1. (48 points) Prove or disprove each of the following statements

   (a) (8 points) Let a, b, c, d be integers. If $a - b$ is even and $c - d$ is even, then $ac - bd$ is even.

   (b) (8 points) Let a and b be integers. If a^2 is divisible by 45 and b^2 is divisible by 45, then ab is divisible by 45.

(c) (8 points) For every positive integer n, $n^2 - n + 17$ is a prime number.

(d) (8 points) There exists a smallest positive real number.

(e) (8 points) Let r, s be rational numbers. If $r \leq s$ then $r^2 \leq s^2$.

(f) (8 points) For all integers $n > 2$, $2^n > n^2$.

2. (12 points) Recall that a function $f : A \to B$ is one to one (or injective) if whenever $a \neq a'$ in A, then $f(a) \neq f(a')$ and that f is onto (surjective) if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.

   (a) (6 points) If the set A contains 12 elements and the set B contains 9 elements, is there a one to one (injective) function $f : A \to B$? Explain why your answer is correct.

   (b) (6 points) If the sets A and B are finite with the same number of elements, is there an onto (surjective) function $f : A \to B$ which is not one to one? Explain why your answer is correct.

3. (40 points) Let n be a positive integer.

   (a) (8 points) Use Mathematical Induction to prove that

$$(1+2+3+\cdots+n)^2 \quad = \quad 1^3+2^3+3^3+\cdots+n^3\,.$$

   Recall that $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$.

(b) The remainder of the question gives an alternate proof of the formula,

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3\,.$$

Begin with the special case,

$$(1+2+3+4+5)^2 = 1^3+2^3+3^3+4^3+5^3\,.$$

i. (3 points) Below is the multiplication table for the numbers 1 to 5.

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

Why does the sum of its entries equal the product $(1+2+3+4+5)(1+2+3+4+5)$? Call this sum S.
Hint: Use the distributive law.

ii. (4 points) Consider the partition of the multiplication table into the five "L" shaped sets of entries shown.

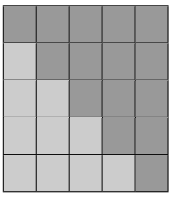
1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

Use this partition to verify that the sum of the entries in the multiplication table is

$$S = 1+2(1+2+1)+3(1+2+3+2+1)+4(1+2+3+4+3+2+1)+5(1+2+3+4+5+4+3+2+1)\,.$$

iii. (3 points) Explain how to count boxes in this picture to obtain the relation,

$$1+2+3+4+5+4+3+2+1=5^2\,.$$



iv. (2 points) Verify that

$$\begin{aligned} 1 &= 1^2 \\ 1+2+1 &= 2^2 \\ 1+2+3+2+1 &= 3^2 \\ 1+2+3+4+3+2+1 &= 4^2 \\ 1+2+3+4+5+4+3+2+1 &= 5^2\,. \end{aligned}$$

Find a formula for the general expression,

$$1+2+3+\cdots+(n-1)+n+(n-1)+\cdots+3+2+1\,.$$

A. (4 points) Prove this formula using a picture as in iii.

B. (4 points) Prove this formula using Mathematical Induction.

v. (4 points) Briefly explain why

$(1+2+3+4+5)^2 = 1+(2+4+2)+(3+6+9+6+3)+(4+8+12+16+12+16+12+8+4)+(5+10+15+20+25+20+15+10+5) = 1^3+2^3+3^3+4^3+5^3$.

vi. (8 points) Generalize what you have done for $n = 5$ to give a derivation of

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3\,.$$

Note: This involves generalizing the statements in parts i., ii., and iv. and explaining how together they prove the identity,

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3\,.$$

The end