Mortality Laws

Mathematical Formulae

From the time of De Moivre (1725), suggestions have been made as to the law of mortality as a mathematical formula, of which the most famous is perhaps that of Gompertz (1825). Since John Graunt (1620–1674) life tables had been constructed empirically. These life tables represented mortality over the whole human lifespan, and it was natural to ask if the mathematical functions defined by the life table could be described by simple laws, as had been so successfully achieved in natural philosophy. The choice of function that should be described by a law of mortality (a mathematical formula depending on age) has varied, as different authors have considered $\mu_x$, $q_x$, or $m_x$ (see International Actuarial Notation) (or something else) to be most suitable; in modern terminology we would model the parameter that is most natural to estimate, given the underlying probabilistic model for the data.

It should be noted that, since $q_x$ and $m_x$ are all $\mu_x + \frac{1}{2}$ to first order, it makes little difference at younger ages whether the formula is used to represent the functions $\mu_x$, $q_x$, or $m_x$ (see below for advanced ages).

Since laws of mortality are attempts to summarize empirical observations, they have been intimately linked with the statistical techniques of analyzing mortality data, and, nowadays, would be described as parametric forms for quantities appearing in statistical models. In some cases, these models may have a physiological basis and may attempt to model the ageing process.

A typical modern pattern of mortality can be divided into three periods. The first period is the mortality of infants, namely, a rapid decrease of mortality during the first few years of life. The second period contains the so-called ‘accident hump’ where the deaths are mainly due to accidents, for example, in cars or motor bicycles. The third period is the almost geometric increase of mortality with age (the rate of increase slowing down after age 80) – senescent mortality.

We will now describe the most important laws of mortality that have been proposed and their features.

De Moivre (1725) [7] used a 1-parameter formula:

$$\mu_x = \frac{1}{\omega - x}; \text{ equivalent to}$$

$$s_0 p_0 = \left(1 - \frac{x}{\omega}\right)$$

De Moivre used age 86 for $\omega$.

Lambert (1776) [17] used a 4-parameter formula:

$$s_0 p_0 = \left\{a - \frac{x}{x}\right\}^2 - b\left(e^{c x} - e^{d x}\right)$$

Babbage (1823) [2] used a 2-parameter formula, assuming that $s_0 p_0$ was quadratic (rather than linear, as De Moivre had assumed)

$$s_0 p_0 = 1 - bx - ax^2$$

for suitable values of $a$, $b$, $c$, see [1].

(First) Gompertz (1825) [11] used a 2-parameter formula:

$$\mu_x = Be^x$$

or equivalently

$$\mu_x = e^{\alpha + \beta x}; \quad s_0 p_0 = e^{-k(\alpha - 1)}$$

The important conclusion of Gompertz was that the force of mortality (practically the same as the rate of mortality for one-half of a year less except for high ages) increased in geometric progression with age. This simple law has proved to be a remarkably good model in different populations and in different epochs, and many subsequent laws are modifications of it, to account for known deviations at very young or very old ages, or for particular features.

Young (1826) proposed a very complex mathematical expression involving $x^{40}$ [1, 35].

Littrow (1832) [19] used a polynomial of degree higher than 2 (as an extension of Babbage).

Moser (1839) [23] used a 5-parameter formula of the form:

$$s_0 p_0 = 1 - ax^\frac{1}{3} + bx^\frac{9}{4} - cx^\frac{17}{8} - dx^\frac{26}{12} + ex^\frac{33}{14}$$

where the values $(a, b, c, d, e)$ for Brune’s Tables, used by the celebrated C. F. Gauss for the Widows’ Fund at Göttingen University are given in [23].

Second Gompertz (1860) [12] was a 10-parameter formula:

$$\ln l_x = -bc^x + gh^x - xdf^x - jk^{m(x-n)}$$
This was intended to represent mortality over the whole span of life, which was not adequately described by the first Gompertz law. Note that the derivative of the function \( \ln(l(x)) \) is equivalent to the force of mortality \( \mu_x \). This formula seemed to have been in advance of its time but was too complex for normal practical use.

Third Gompertz (1862) [13] was a version of the second Gompertz formula above. First Makeham (1867) [20] was a 3-parameter formula:

\[
\mu_x = A + Be^x
\]

or equivalently

\[
\mu_x = \alpha_1 + e^{\alpha_2 + \alpha_3 x}; \quad \mu_0 = e^{-(b+c^2+1)}
\]  

(7)

This extended the first Gompertz law simply by including a term independent of age, representing non-senescent deaths, for example, from accidents.

Double Geometric (unknown author) [16, 18, 24] was a 5-parameter formula:

\[
\mu_x = A + Bc^x + Mn^x
\]

(8)

Oppermann (1870) [25] proposed a 3-parameter formula:

\[
\mu_x = ax^{-\frac{1}{2}} + b + cx^\frac{1}{3},
\]

(9)

which only applies for young ages, \( x \leq 20 \).

Thiele (1871) [31] proposed a 7-parameter formula:

\[
\mu_x = Ae^{-Rx} + Ce^{-D(x-E)^2} + FG^x,
\]

(10)

which covers the whole span of life. Each term in this formula represents one of the recognizable features of human mortality mentioned above. The first term represents infant mortality, declining steeply after birth; the second term represents the ‘accident hump’; the third term is the Gompertz law appropriate at older ages.

Second Makeham (1890) [21] was a 4-parameter formula:

\[
\mu_x = A + Hx + Be^x
\]

or equivalently

\[
\mu_x = \alpha_1 + \alpha_2 x + e^{\alpha_1 + \alpha_2 x}; \quad \mu_0 = e^{-(b+c^2+1)}
\]

(11)

Perks (1932) [26] proposed a 4-parameter formula (this is known as the logistic curve):

\[
\mu_x = \frac{A + Be^x}{1 + De^x} \quad \text{(PER1)}
\]

(12)

PER1 is the logistic curve, equivalently expressed as \( \mu_x = A + \frac{B}{1 + Kc^x} \) (see HP3 below). Perks (1932) [26] also proposed a 5-parameter formula:

\[
\mu_x = \frac{A + Be^x}{Kc^x + 1 + De^x} \quad \text{(PER2)}
\]

(13)

The effect of the denominators is to flatten out the exponential increase of the Gompertz term in the numerator, noticeable at ages above about 80, which is now a well-established feature of the mortality. PER1 results, if each member of a population has mortality that follows Makeham’s Law \( \mu_x = A + Be^x \) but \( B \), which is fixed for an individual member of the population, follows a gamma distribution (see Continuous Parametric Distributions) for the population as a whole. PER1 was used in the graduation of the table of annuitant lives in the United Kingdom known as a (55).

Weibull (1951) [33] proposed a 2-parameter formula:

\[
\mu_x = Ax^b; \quad \mu_0 = e^{-(Ax^b+1)}
\]

(14)

The English Life Tables (ELT) 11 and 12 [27] (deaths 1950–1952 and 1960–1962) were graduated using a 7-parameter curve

\[
m_x \left( x^{\frac{1}{2}} \right) = a + Ce^{-\beta(x-x_1)^2} + \frac{b}{1 + e^{-\alpha(x-x_1)}}
\]

(15)

This mathematical formula was used for English Life Tables 11 and 12 for an age range above a certain age; in the case of ELT12 from age 27 upwards. The method of splines was used for ELT13 and 14 [8, 22].

Beard (1971) [3, 5] proposed a 5-parameter formula:

\[
q_x = A + \frac{Bc^x}{Ee^{-2x} + 1 + De^x}
\]

(16)

This formula was used to graduate the table of assured lives known as A1949–52 in the United Kingdom.
Barnett (1974) [4] proposed a 4-parameter formula:
\[ q_x = \frac{f(x)}{1 + f(x)} \]
where
\[ f(x) = A - Hx + Bc^x \] (17)
This formula was used in the United Kingdom to graduate the experience of insured lives over the period 1967–1970, and this table is known as A 1967–70.

Wilkie (1976) [6, 34] proposed a family, incorporating as many parameters as are significant
\[ q_x = \frac{f(x)}{1 + f(x)} \]
where
\[ f(x) = \exp \left( \sum_{i=1}^{r+s} \alpha_i x^{i-1} \right) \] (18)
graduated ages.

First and Second Heligman–Pollard – HP1 and HP2 – (1980) [14] are two 8-parameter formulae that cover the whole span of life:
\[ q_x = \frac{f(x)}{1 + f(x)} \]
where
\[ f(x) = A(x + B)C + De^{-E \left( \log_e x \right)^3} + \frac{GHx}{1 + KGx^3} \] (HP3) (22)
which covers the whole span of life. As an example, the Figure 1 shows the published rates of mortality \( q_x \) of the English Life Tables No. 15 (Males) (ELT15M) from ages 0 to 109 (this represents the mortality in England and Wales 1990–1992). ELT15 was graduated by splines, but we can see that the third Heligman–Pollard law gives a very close fit, called ELM15M(Formula) on the figure below. The parameters \( A = 0.59340, B = 10.536, C = 9.5294, D = 6.3492, E = 8.9761, F = 21.328, G = 2.6091, H = 1.1101, I = 1.4243 \). This has \( K = 1.4243 \) giving, at higher ages, \( q_x = \frac{GHx}{1 + 2.4243GHx} \) (a Perks/logistic curve with the constant term being zero) and an asymptotic value of 0.7.

Fourth Heligman–Pollard – HP4 – (1980) [14] was a 9-parameter formula:
\[ q_x = \frac{f(x)}{1 + f(x)} \]
where
\[ f(x) = A^{(x+B)^c} + De^{-E \left( \log_e x \right)^3} + \frac{GHx}{1 + 2.4243GHx} \] (HP4) (23)
which covers the whole span of life.

Forfar, McCutcheon, Wilkie – FMW1 and FMW2 (1988) [9] proposed a general family, incorporating as many parameters \( (r + s) \) as are found to be significant:
\[ \mu_x = GM^{r+s}(x) \]

\[ q_x = \frac{GM^{r+s}(x)}{1 + GM^{r+s}(x)} \] (FMW1) (24)

\[ q_x = \frac{GM^{r+s}(x)}{1 + GM^{r+s}(x)} \] (FMW2) (25)
The formula for \( \mu_x \) incorporates (First) Gompertz and First and Second Makeham. A mathematical formula approach was used in the United Kingdom to graduate the 80 series and 92 series of standard tables for life insurance companies [6, 9, 22]. For UK assured lives (two-year select period) and life office pensioners (‘normal’ retirements, no select period) in the 92 series, (life office deaths 1991–1994) the mathematical formula \( \mu_x = GM(2, 3) \) was fitted to \( \mu_x \).
Mortality at the Highest Ages

Whereas from ages 30 to 80 the yearly rate of increase in mortality rates is almost constant (10% – 11.5% a year in round terms), the rate of increase slows down markedly above age 80 [28–30]. The term \( \frac{GW}{GKH} \) in the Third Heligman–Pollard formula is similar to a Perks (see above) term (with \( A = 0 \)); fitting HP3 to ELT15M gives an asymptotic value of \( (1/K) = 0.7 \) for \( q_x \) (see above). The asymptotic value of \( \mu_x \) then approximates to 1.2 \((-\ln(1-0.7))\). At age 110, \( q_x = 0.55 \), so the asymptotic value has nearly (but not quite) been reached.

Since the rate of growth of \( \mu_x \) and \( q_x \) declines above age 80, perhaps the functional form \( \frac{GM^{1.4}(x)}{1+KGH} \) for either \( \mu_x \) or \( q_x \) may be preferable to \( GM^{1.3}(x) \), but this has not been tried.

Mortality Projections for Pensioners and Annuitants [particularly CMI Report No. 17]

Human mortality has changed dramatically during the few centuries in which laws of mortality have been pursued, notably during the twentieth century [6, 10, 15, 32]. Of interest in themselves, such changes are of great financial importance to life insurers (see Life Insurance). Generally we have seen mortality falling and longevity increasing, which makes pensions and annuities more expensive, so actuaries have attempted to project such future trends in order to price and reserve for annuity contracts. This subject cannot be described in detail here, but a study of the laws of mortality that have been found to apply over time, and how their parameters have changed, gives useful information. Projections of annuitant and pensioner mortality have been made in the United Kingdom since the 1920s. The projections have been based on a double entry table where the axes of the table are age and calendar year.

Future Mortality

Ever since Gompertz (1825), an exponential term has appeared in some form in most laws of mortality; it is convenient to focus on the parameter \( H \) (which is around 1.10) in the Third Heligman–Pollard formula. This appears to be significant as it plausibly relates to the natural aging of the human body, as originally argued by Gompertz. Modern medicine does not appear to have made a difference to \( H \). In fact, \( H \) appears to be increasing with time as Table 1 shows.

Eliminating all infant mortality (the first term in the Third Heligman–Pollard formula) and all
accidents, (the second term) reducing \( q_x \) to 1/3 of its ELT15M(Formula) value, and running linearly into 0.9 times, the ELT15M(Formula) value of \( q_x \) over the ages between 60 and 120, but keeping the proportion constant at 0.9 above age 120 will create a mortality table denoted by ELT(limit) but will only put the expectation of life for males at birth up from 76 (on ELT15M) to 83, on ELT(limit). The age of the oldest person to die in the world (currently believed to be Madame Calment of France, who died at age 122 years 164 days) may, however, not increase much above 122 over the foreseeable future. On ELT(limit), out of 100 million births per annum and a stationary population of 8.5 billion there is less than 1 person alive at age 124. This suggests that until we can find some way of influencing the ageing process there is a limit of, say, 85 years to the expectation of life for male lives at birth and a limit to which the oldest person in the world can live of, say, 125. ELT(limit) is shown in Figure 1.

References


**Further Reading**


*(See also Decrement Analysis; Early Mortality Tables; Life Table)*

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