Projective Centres of Motion in 3-Space

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Outline

- Review of centres of motion in the plane
- Motion in 3-space
  - The geometry of motion: screws in space
  - Grassmann-Cayley algebra
  - Instantaneous motions & infinitesimal rigidity
- Applications
  - Bar and joint frameworks
  - Supported $n$-gons
  - Hinged panel structures
- Relationship with statics
  - Hinged panel polyhedra
Review of the plane: basic definitions

- Every point $P = (p_1, p_2)$ in the Euclidean plane, $\mathbb{R}^2$ is associated with the projective point $p = (p_1, p_2, 1)$ in $\mathbb{P}^2$.

- Every motion can be viewed as a rotation with centre of motion $c$.

- **The motion triple**
  - The motion of a point is encoded in the motion triple $M(p) = (v_1, v_2, -V \cdot P)$.
  - $V = (v_1, v_2)$ is the velocity of the Euclidean point $P = (p_1, p_2)$.
  - $M(p)$ is a two-extensor in the Grassmann-Cayley algebra: $M(p) = c \lor p$.
  - $p$ is a point and $c$ is the centre of motion.
Review of the plane: first order rigidity

- An **infinitesimal motion** of a planar framework \((V, E, P)\) is an assignment of motion triples \(M(p_i)\) to the projective points \(p = \{ p_1, p_2, \ldots, p_n \}\) such that
  1. for each joint \(i \in V\), \([M(p_i), p_i] = 0\) and
  2. for each bar \(\{i, j\} \in E\), \([M(p_i), p_j] + [M(p_j), p_i] = 0\)

- A framework is **infinitesimally rigid** if every infinitesimal motion of the structure has a single centre \(c\) such that \(c \vee p_i = M(p_i)\) for the points \(p_i\) corresponding to all vertices \(i\) in the framework.
Review of the plane: an example

- the projected triangular prism

- Fix one triangle
  - look for a relative motion of the second triangle

- A relative centre exists
  - $\Leftrightarrow$ the two triangles are perspective from a point
  - $\Leftrightarrow$ the two triangles are perspective from a line
Motion in Space: Rotation

- A rotation in \( \mathbb{E}^3 \) is described by
  1. two points determining the axis of rotation
     - \( Q = (q_1, q_2, q_3) \)
     - \( R = (r_1, r_2, r_3) \)
  2. the angular velocity \( \omega \)

\[
V = \frac{\omega}{||R-Q||} (R - Q) \times (P - S)
\]
Rotation in projective notation

• motion of the point $P$ is given by
  \[ M(p) = (v_1, v_2, v_3, -V \cdot P) \]
  where $V = (v_1, v_2, v_3)$ is the velocity of $P = (p_1, p_2, p_3)$

• $M(p)$ is the quadruple of minors of:
  \[ \omega = \begin{bmatrix} q_1 & r_1 & p_1 \\ q_2 & r_2 & p_2 \\ q_3 & r_3 & p_3 \\ 1 & 1 & 1 \end{bmatrix} \]

$M(p)$ is an oriented section of the plane with area equal to the size of $V$
Centre of rotation

\[ M(p) = \omega \begin{bmatrix} q_1 & r_1 & p_1 \\ q_2 & r_2 & p_2 \\ q_3 & r_3 & p_3 \\ 1 & 1 & 1 \end{bmatrix} \]

• Considering the first two columns:

\[ \omega \begin{bmatrix} q_1 & r_1 \\ q_2 & r_2 \\ q_3 & r_3 \\ 1 & 1 \end{bmatrix} \]

the 6-tuple of 2×2 minors gives the Plücker coordinates for a piece of the axis line

• We call this the **centre** of rotation:

\[ c = \omega(q \lor r) \]

• \( M(p) = c \lor p \)
Translation

• Translations as rotations with centre of rotation at infinity

\[
\mathbf{c} = \omega \begin{bmatrix}
    a_1 & b_1 \\
    a_2 & b_2 \\
    a_3 & b_3 \\
    0 & 0
\end{bmatrix} = (0, 0, 0, t_1, t_2, t_3)
\]

• for any point \( \mathbf{P} \) in \( \mathbb{E}^3 \),

\[
M(\mathbf{p}) = \mathbf{c} \lor \mathbf{p} = \omega \begin{bmatrix}
    a_1 & b_1 & p_1 \\
    a_2 & b_2 & p_2 \\
    a_3 & b_3 & p_3 \\
    0 & 0 & 1
\end{bmatrix}
\]

\[
= (t_1, t_2, t_3, -\mathbf{T} \cdot \mathbf{P})
\]

velocity of any point \( \mathbf{P} \)
Screw motions

- the composition of rotations and translations in space will give a screw motion
- instantaneously, a screw will the addition of the velocity vectors of the constituent motions:
  - representing each motion by its centre $c_i$
    \[
    M(p) = \sum (c_i \lor p) = (\sum c_i) \lor p = S \lor p
    \]
  - $S$ is the screw centre
- Poinsot's central axis theorem:
  Every screw can be written uniquely as the sum of a rotation about an axis and a translation along that axis
Grassmann-Cayley algebra for space

- `'∨' ≡ JOIN ≡ incomplete determinant

<table>
<thead>
<tr>
<th>p</th>
<th>p ∨ q</th>
<th>p ∨ q ∨ r</th>
<th>p ∨ q ∨ r ∨ s</th>
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<tbody>
<tr>
<td>1-extensor</td>
<td>2-extensor</td>
<td>3-extensor</td>
<td>4-extensor</td>
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<tr>
<td>weighted point</td>
<td>directed line segment</td>
<td>oriented area in a plane</td>
<td>oriented volume in space</td>
</tr>
<tr>
<td>e.g. centre of motion</td>
<td>e.g. motion 4-tuple</td>
<td></td>
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- `'∧' ≡ MEET ≡ intersection of 2 subspaces which span the space
  - e.g. line MEET plane \((a_1 ∨ a_2) ∧ (b_1 ∨ b_2 ∨ b_3)\)
    - a weighted point of intersection
  - e.g. plane MEET plane \((a_1 ∨ a_2 ∨ a_3) ∧ (b_1 ∨ b_2 ∨ b_3)\)
    - a weighted segment of the line of intersection
The distance condition

- If two Euclidean points \( P \) and \( Q \) are constrained to maintain a fixed distance
  \[ |P - Q|^2 = c \]
  their velocities \( U \) and \( V \) will satisfy a linear equation
  \[
  (\frac{dP}{dt} - \frac{dQ}{dt}) \cdot (P - Q) = 0 \\
  (U - V) \cdot (P - Q) = 0
  \]

- In projective terms this is equivalent to
  \[
  [M(p), q] + [M(q), p] = 0 \\
  \text{and} \\
  [M(p), p] = 0
  \]
First order rigidity of frameworks in space

• An **infinitesimal motion** of a framework $F = (V, E, \mathbf{p})$ in space is an assignment of motion 4-tuples $M(p_i)$ to the projective points $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$ such that
  1. for each joint $i \in V$, $[M(p_i), p_i] = 0$ and
  2. for each bar $\{i, j\} \in E$, $[M(p_i), p_j] + [M(p_j), p_i] = 0$

• A framework is **infinitesimally rigid** if every infinitesimal motion of the structure has a single centre $c$ such that $c \vee p_i = M(p_i)$ for the points $p_i$ corresponding to all vertices $i$ in the framework.
Basic example: null line of a screw

- the edge condition on \( \{a, b\} \) \( \Rightarrow \)
  
  \[
  [M(a), b] + [M(b), a] = 0 \\
  [c_1, a, b] + [c_2, b, a] = 0 \\
  [c_1 - c_2, a, b] = 0 
  \]

- \( c_1 - c_2 \) is the relative centre of \( A \) with respect to \( B \)

- the line \( ab \) is called a **null line** of the screw
Example: octahedron

- fix one triangle, look for relative motion of the other
- constraints:
  - \( M(a') = \alpha(bc \lor a') \)
  - \( M(b') = \beta(ac \lor b') \)
  - \( M(c') = \gamma(ab \lor c') \)
- \( \triangle a'b'c' \Rightarrow \)
  - \([M(a'), b'] + [M(b'), a'] = 0\)
  - \([M(b'), c'] + [M(c'), b'] = 0\)
  - \([M(c'), a'] + [M(a'), c'] = 0\)

\[ abc \land a'bc' \land a'b'c \land ab'c' = 0 \]
Example: octahedron

\[ abc \land a'bc' \land a'b'c \land ab'c' = 0 \]

- There is an infinitesimal motion of the octahedron whenever
  - the four planes are concurrent in \( P^3 \)
  - or when one of the planes is degenerate
Example: another look at the octahedron

- We can regard the octahedron as a supported triangle

- and generalize to supported $n$-gons
Supported $n$-gons

- take an $n$-gon with vertices $a_1, a_2, \ldots, a_n$ with each vertex supported by 2 legs
- $2n$ legs join the ground in $n$ places
Supported $n$-gons, cont.

- each vertex can move about the line connecting its two legs:
  \[ M(a_i) = \lambda_i(b_1 b_2 a_1) \]
  \[ M(a_n) = \lambda_n(b_n b_1 a_n) \]

- the $n$-gon imposes $n$ conditions between the $a_i$'s:
  \[ [M(a_i), a_{i+1}] + [M(a_{i+1}), a_i] = 0 \]

- the condition for the existence of an infinitesimal motion:
  \[ [b_1 b_2 a_1 a_2] \cdots [b_n b_1 a_n a_1] + (-1)^{n+1} [b_2 b_3 a_2 a_1] \cdots [b_1 b_2 a_1 a_n] = 0 \]
Supported $n$-gons: some classical results

\[ [b_1 b_2 a_1 a_2] \ldots [b_n b_1 a_n a_1] + (-1)^{n+1} [b_2 b_3 a_2 a_1] \ldots [b_1 b_2 a_1 a_n] = 0 \]

Two regular polygons are called \textbf{symmetrically positioned} if they are positioned in two parallel planes, with the centre of one polygon directly above the centre of the other.

(Sang, 1888) If the two polygons are symmetrically positioned with all legs of equal length, the structure has an infinitesimal motion iff $n$ is even.
Supported $n$-gons: some classical results

(Sang, 1888) If the two polygons are symmetrically positioned with the two legs at each point of different lengths, the structure cannot have an infinitesimal motion.

$$[b_1 b_2 a_1 a_2] \cdots [b_n b_1 a_n a_1] + (-1)^{n+1} [b_2 b_3 a_2 a_1] \cdots [b_1 b_2 a_1 a_n] = 0$$
Supported $n$-gons, cont.

- fix $a_2, \ldots, a_n, b_1, \ldots, b_n$ in **general position**
- what are the bad positions for $a_1$?

- $[b_1 b_2 a_1 a_2] \cdots [b_n b_1 a_n a_1] + (-1)^{n+1} [b_2 b_3 a_2 a_1] \cdots [b_1 b_2 a_1 a_n] = 0$
  \[ \Rightarrow \text{the bad positions lie on a quadratic surface} \]
- this surface contains the three lines: $b_1 b_2$, $a_2 b_2$, $b_1 a_n$
- the only quadratic surfaces containing lines are the hyperboloid and a degenerate surface of one or two planes
Articulated panel structures

- structures made of panels hinged together
- a hinge is a line defined by two common points of two panels
  - projectively: $a \vee b$

**Proposition**: if 2 panels $P_1$ and $P_2$ are hinged together along $a \vee b = ab$ then for any instantaneous motion giving centres $S_1$ and $S_2$ to the panels, there is a scalar $\lambda$ such that $S_1 - S_2 = \lambda ab$
Articulated panel structures

- An articulated panel structure is $S = (P, H)$, where
  - $P$ is a finite collection of panels, $P = \{P_1, \ldots, P_n\}$
  - $H$ is an ordered set of 2-extensors, $H = \{\ldots H_{ij} \ldots\}$
    - $H_{ij}$ connects panels $P_i$ and $P_j$
    - $H_{ij} = -H_{ji}$
  - An instantaneous motion of the panel structure is an assignment of a screw centre $S_i$ to each panel $P_i$ such that
    - for each hinge $H_{ij} \in H$:
      $$S_i - S_j = \omega_{ij}H_{ij} \text{ for some scalar } \omega_{ij}$$
  - An articulated panel structure is infinitesimally rigid if each instantaneous motion of the structure has a single centre of motion $D = S_i$ for all panels $P_i$
Articulated panel structures

• a **motion assignment** for a connected panel structure is an assignment of scalars $\omega_{ij}$ to the hinges $H_{ij}$ with $\omega_{ij} = \omega_{ji}$ such that
  • $\sum \omega_{ij} H_{ij} = 0$ for each cycle of panels and hinges in the structure

• **Proposition**: For a connected panel structure with a designated panel $P_1$, there is a one-to-one correspondence between between instantaneous motions of the structure with $S_1 = 0$ and motion assignments.

A motion assignment represents a non-trivial motion iff $\omega_{ij} \neq 0$ for some hinge $H_{ij}$
Cycles of panels

**Proposition:** Given a panel structure which is a single cycle of $k$ panels connected by $k$ hinges, the structure is infinitesimally rigid iff the $k$ lines of the hinges are independent in the Grassmann geometry of lines in space.

**Proof:** consider the cycle $P_1H_{12}P_2 \ldots P_kH_{k1}P_1$

any instantaneous motion with centres $S_i$ for panel $P_i$ gives:

\[
\begin{align*}
S_2 - S_1 &= \lambda_1H_{12} \\
&\vdots \\
S_1 - S_k &= \lambda_kH_{k1}
\end{align*}
\]

adding: $\lambda_1H_{12} + \cdots + \lambda_kH_{k1} = 0$

there is a non-trivial motion

$\iff$ some $\lambda_i \neq 0$

$\iff$ there is a linear dependence among the hinges
Corollary: No cycle of $k > 6$ panels is infinitesimally rigid

Corollary: A cycle of $k = 6$ panels is infinitesimally rigid iff the lines of the hinges do not lie in any linear line complex

- most sets of six 2-extensors in space will be independent, but if they are dependent, the coordinates will satisfy a linear equation, and the lines lie in a linear line complex
- We can view the octahedron as a cycle of 6 panels:

\[\text{rigid}\]
Which sets of up to 6 lines are independent in space?

- 3 lines have rank 2 if they are coplanar and copunctual (they lie in a flat pencil)
- 4 lines have rank 3 in four ways:
Tay bridge
Tay bridge disaster
a **panel polyhedron** is an articulated panel structure that has
- a panel $P_k$ for each face $F_k$ of an **oriented** polyhedron
- an assignment of points $a_i \in \mathbb{R}^3$ for the vertices of the polyhedron such that the hinges are extensors $H_{km} = a_i a_j \leftrightarrow (k, m; i, j)$ is an edge patch of the polyhedron

the bar and joint skeleton of a panel polyhedron is the bar and joint framework formed by the joints at the vertices of the panel polyhedron and edges along the hinges of the panel structure

recall: a **stress** on a bar and joint framework is an assignment of scalars $\lambda_{ij}$ at least one of which is non-zero, such that
- at each vertex $a_i$ of the framework
  \[
  \sum \lambda_{ij} a_i a_j = 0
  \]
  (sum over $j$ with $\{i, j\} \in E$ for fixed $i$)
Equivalence of statics and kinematics

Proposition:
A panel polyhedron has a non-trivial motion assignment
\[ \iff \]
there is a non-trivial stress on the bar and joint skeleton such that for any closed path on the polyhedron not passing through the vertices of the polyhedron, \( \sum \lambda_{ij} a_i a_j = 0 \)
(sum over edges crossed by the closed path)

"Proof":
• Given a motion assignment \( \omega_{km} \) for each hinge \( H_{km} \)
  • define a stress by letting \( \lambda_{ij} = \omega_{km} \) when \( (k, m; i, j) \) is an edge patch
  • \( \lambda_{ij} = \omega_{km} = \omega_{mk} = \lambda_{ji} \)
  • \( \sum \omega_{km} H_{km} = 0 \Rightarrow \sum \lambda_{ij} a_i a_j = 0 \)
Equivalence of statics and kinematics

Corollary:
A spherical panel polyhedron has a non-trivial motion

⇔
the associated bar and joint skeleton has a non-trivial stress
Example: the octahedron again

The octahedron

with an instantaneous motion

with a stress
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